New criteria of supersolubility of finite groups

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Graphical abstract

Some new criteria of supersolubility of finite groups are obtained by assuming that some given subgroups are σ-embedded.

Public summary

- Some new properties of σ-embedded subgroups are established.
- Some new criteria of supersolubility of finite groups are obtained.
- Some known results in this research field are generalized.

New criteria of supersolubility of finite groups

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Abstract: We study the structure of finite groups in which some given subgroups are σ-embedded. In particular, we obtain some new criteria for the supersolubility of finite groups, which generalize some known results.

Keywords: finite group; p-nilpotent group; supersoluble group; σ-permutable subgroup; σ-embedded subgroup

CLC number: O152.1 Document code: A
2020 Mathematics Subject Classification: 20D10; 20D20; 20D35

1 Introduction

Throughout this paper, all groups are finite, and G always denotes a group. Moreover, n is an integer, and P is the set of all primes. The symbol π(n) denotes the set of all primes dividing n and π(G) = π(|G|), the set of all primes dividing the order of G.

Let G = ∪ i∈I n_iσ_i, and σ_i∩σ_j = ∅ for all i ≠ j. I is always supposed to be a nonempty subset of the set σ and Π = σ\{I\}. We write σ(n) = {σ,σ_i∩π(n) ≠ ∅} and σ(G) = σ(|G|).

Following Refs. [1–4], G is said to be σ-primary if |σ(G)| ≤ 1. An integer n is a σ-Prime if n ∈ σ(G). A subgroup H of G is called a σ-subgroup of G if |σ(G)| = 1. A set H of subgroups of G is said to be a complete Hall σ-set of G if every nonidentity member of H is a Hall σ-subgroup of G for some i ∈ I and H contains exactly one Hall σ-subgroup of G for every σ,∈ σ(G). G is said to be σ-full if G possesses a complete Hall σ-set; a σ-full group of Sylow type if every subgroup of G is a Dσ-group for all σ,∈ σ(G). A subgroup H of G is said to be σ-subnormal in G if there exists a subgroup chain H = H_0 ≤ H_1 ≤ H_2 ≤ ⋯ ≤ H_n = G such that either H_i is normal in H_0 or H_i/(H_0)/H_n is σ-primary for all i = 1,⋯,t.

It is well known that embedded subgroups and supplemented subgroups play an important role in the finite group theory. For example, a subgroup H of G is said to be σ-normal in G if G has a normal subgroup T such that G = HT and H ∩ T ≤ H_o, where H_o denotes the maximal normal subgroup of G contained in H. A subgroup H of G is called n-embedded in G if G has a normal subgroup T such that HT = H^T and H ∩ T ≤ H_o, where H^T is the largest normal subgroup of G containing H and H_o is the subgroup of H generated by all those subgroups of H which are σ-permutable in G (note that subgroup A of G is said to be σ-permutable in G if AP = PA for any Sylow subgroup P of G). A subgroup H of G is called s-embedded in G if G has an s-permutable subgroup T such that HT = H^⁻ and H ∩ T ≤ H_o, where H^⁻ is the intersection of all s-permutable subgroups of G containing H. A subgroup H of G is called σ-n-embedded in G if there exists a normal subgroup T of G such that HT = H^T and H ∩ T ≤ H_o, where H_o is the subgroup of H generated by all those subgroups of H which are σ-permutable in G (note that a subgroup H of G is called σ-permutable in G if G possesses a complete Hall σ-set 'H such that H_A^T = A^T for all A ∈ H and all x ∈ G). A subgroup H of G is called σ-embedded in G if there exists a σ-permutable subgroup T of G such that HT = H^T and H ∩ T ≤ H_o, where H^T is the σ-permutable closure of H, that is, the intersection of all σ-permutable subgroups of G containing H. By using the above embedded subgroups and supplemented subgroups, people have obtained a series of interesting results (see, for example, Refs. [5–13]).

Some properties of σ-embedded subgroups were analyzed in Refs. [7, 8]. In this paper, we continue the study of σ-embedded subgroups and use them to determine the structure of finite groups. In particular, we obtain some new criteria for the supersolubility of finite groups.

Theorem 1.1. Let G be a σ-full group of Sylow type and H = [H_1,⋯,H_t] a complete Hall σ-set of G such that H_i is a nilpotent σ_i-subgroup for all i = 1,⋯,t. If every cyclic subgroup of any non-cyclic H_i of prime order and order 4 (if 2 ∈ π(H_i) and the Sylow 2-subgroup of H_i is nonabelian) is σ-embedded in G, then G is supersoluble.

Theorem 1.2. Let G be a σ-full group of Sylow type and H = [H_1,⋯,H_t] a complete Hall σ-set of G such that H_i is a nilpotent σ_i-subgroup for all i = 1,⋯,t. Suppose that E is a normal subgroup of G with G/E supersoluble. If every cyclic subgroup of any non-cyclic H_i ∩ E of prime order and order 4 (if 2 ∈ π(H_i ∩ E) and the Sylow 2-subgroup of H_i ∩ E is nonabelian) is σ-embedded in G, then G is supersoluble.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries which will be used in this paper. In Section 3, we give the proofs of Theorems 1.1 and 1.2. In Section 4, we give some applications of our results.
All unexplained terminologies and notations are standard. The reader is referred to Refs. [14–16] if necessary.

2 Preliminaries

Let $\mathcal{L}$ be some nonempty set of subgroups of $G$ and $E \leq G$. Following Ref. [2], a subgroup $A$ of $G$ is called $\mathcal{L}$-permutable if $AH = HA$ for all $H \in \mathcal{L}$, $\mathcal{L}^* \leq$-permutable if $AH^* = H^*A$ for all $H \in \mathcal{L}$ and all $x \in E$. In particular, a subgroup $H$ of $G$ is $\sigma$-permutable in $G$ if $G$ possesses a complete Hall $\sigma$-set $\mathcal{H}$ such that $H$ is $\mathcal{H}^* \leq$-permutable.

**Lemma 2.1.** [1, Lemma 2.8] Let $H, K$ and $N$ be subgroups of a $\sigma$-full group $G$. Let $\mathcal{H} = \{H_i, \cdots, H_l \}$ be a complete Hall $\sigma$-set of $G$ and $L_0 = \mathcal{H}^*$. Suppose that $H$ is $\mathcal{L}$-permutable and $N$ is normal in $G$. Then, $HN/N$ is $\mathcal{L}$-permutable, where $L_0 = \{H_i[N/N], \cdots, H_l[N/N]\}$. In particular, if $H$ is $\sigma$-permutable in $G$, then $HN/N$ is $\sigma$-permutable in $G/N$.

Following Refs. [1, 2], we use $O^\sigma(G)$ to denote the subgroup of $G$ generated by all its $\Omega^\sigma$-subgroups of $G$. Instead of $O^{\sigma}(G)$, we write $O^\sigma(G)$.

**Lemma 2.2.** [3, Lemma 3.1] Let $H$ be a $\sigma$-subgroup of a $\sigma$-full group $G$. Then, $H$ is $\sigma$-permutable in $G$ if and only if $O^{\sigma}(G) \leq N_G(H)$.

**Lemma 2.3.** [3, Theorem 1] Let $G$ be a $\sigma$-full group of Sylow type. Then, the set of all $\sigma$-permutable subgroups of $G$ forms a sublattice of the lattice of all $\sigma$-normal subgroups of $G$. In particular, for any subgroup $A$ of $G$, $A^{\sigma}$ and $A_{\sigma}$ are both $\sigma$-permutable in $G$.

**Lemma 2.4.** Let $G$ be a $\sigma$-full group of Sylow type and $H$ a subgroup of $G$. Then, $H$ is $\sigma$-embedded in $G$ if and only if there exists a $\sigma$-permutable subgroup $T$ of $G$ such that $HT$ is $\sigma$-permutable in $G$ and $H \cap T \leq H_{\sigma}$.

**Proof.** The necessity is evident. Now, we prove the sufficiency. Let $T_i = T_i \cap H_{\sigma}$. Then, by Lemma 2.3, $T_i$ is $\sigma$-permutable in $G$. Clearly, $HT_i = H(T_i \cap H_{\sigma}) = HT \cap H_{\sigma} \leq H_{\sigma}$. Then, by the hypothesis and Lemma 2.3, $HT_i$ is $\sigma$-permutable in $G$. Moreover, since $H_{\sigma} \leq H \cap T = T_i$, we have that $HT_i = H_{\sigma}$. It is clear that $H \cap T \leq H \cap T \leq H_{\sigma}$. This shows that $H$ is $\sigma$-embedded in $G$. The lemma is proved.

**Lemma 2.5.** [3, Lemma 3.1] Let $G = P \times Q$, where $P$ is the Sylow 2-subgroup of $G$ and $|Q| = q$ for some prime number $q \neq 2$. If every cyclic subgroup of $P$ of order 2 order 4 (if $P$ is a nonabelian group) is $\sigma$-embedded in $G$, then $G$ is $\sigma$-nilpotent.

**Lemma 2.6.** [3, Lemma 3.7] Let $G$ be a $\sigma$-full group of Sylow type, $N$ a normal subgroup of $G$ and $H \leq \leq K \leq G$.

1. If $K$ is $\sigma$-embedded in $G$, then $K$ is $\sigma$-embedded in $G$.
2. Suppose that $H$ is $\sigma$-embedded in $G$ and $N \leq H$ or $(H[N/N]) \leq 1$. Then, $HN/N$ is $\sigma$-embedded in $G/N$.

**Lemma 2.7.** [1] Let $G$ be a nonsupersoluble group such that all its proper subgroups are supersoluble. Then:

1. $G$ has a unique normal Sylow subgroup, say $G_\sigma$.
2. $G_\sigma/\Phi(G_\sigma)$ is a minimal normal subgroup of $G/\Phi(G_\sigma)$.
3. If $p > 2$, then $G$ is of exponent $p$.

3 Proofs of Theorems 1.1 and 1.2

To prove Theorems 1.1 and 1.2, we first prove the following results.

**Proposition 3.1.** Let $G$ be a $\sigma$-full group of Sylow type and $P$ a Sylow $p$-subgroup of $G$, where $p$ is the smallest prime dividing $|G|$. Suppose that $p \in \sigma$, for some $\sigma \in \sigma(G)$ and a Hall $\sigma$-subgroup $H$ of $G$ is nilpotent. Then, $G$ is $p$-nilpotent if and only if all subgroups of $P$ that have order $p$ and order 4 (if $P$ is a nonabelian 2-group) is $\sigma$-embedded in $G$.

**Proof.** $(\Rightarrow)$ Since $G$ is $p$-nilpotent, there exists a normal Hall $p'$-subgroup $K$ of $G$. Let $H$ be a subgroup of $P$ with order $p$ or order 4 (if $P$ is a nonabelian 2-group). Then, clearly, $HK$ is $\sigma$-permutable in $G$ and $H \cap K = 1$. By Lemma 2.4, we have that $H$ is $\sigma$-embedded in $G$.

$(\Leftarrow)$ Suppose that this is false and let $G$ be a counterexample of minimal order. We now proceed via the following steps:

1. Let $E$ be a proper subgroup of $G$. Then, $E$ is $p$-nilpotent.

If $p \notin \pi(E)$, then, of course, $E$ is $p$-nilpotent. If $p \in \pi(E)$, then $E$ satisfies the hypothesis of the theorem by Lemma 2.6. Hence, $E$ is $p$-nilpotent by the choice of $G$.

2. $G = P \times Q$ is a minimal nonnilpotent group, where $Q$ is a Sylow $q$-subgroup of $G$ with $p < q$, $P/\Phi(P)$ is a chief factor of $G$, and if $p \neq 2$, the exponent of $P$ is $p$.

This directly follows from Claim (1) and Ref. [18, Chapter IV, Theorem 5.4].

3. $(|Q| = q)$

Assume that $|Q| > q$. Let $K$ be a maximal subgroup of $Q$. Then, $K \neq 1$ and $K$ is normal in $G$ by Claim (2). Now, we consider the quotient group $G/K$. Clearly, the hypothesis holds for $G/K$ by Lemma 2.6. Hence, $G/K$ is $p$-nilpotent by the choice of $G$, and so $G$ is $p$-nilpotent, a contradiction. Hence, (3) holds.

4. There exists some $\sigma \in \sigma(G)$ such that $\sigma \cap \pi(G) = \{q\}$, where $j \neq i$.

Assume that this is false. Then, $\sigma \cap \pi(G) = \{p, q\}$, and thus, $G$ is a $\sigma$-group. It follows from the hypothesis that $G$ is nilpotent, a contradiction. Hence, we have (4).

$(p \neq 2)$

Assume that $p = 2$. In view of Lemma 2.5, $G$ is 2-nilpotent by Claims (2)–(4), a contradiction. Hence, (5) holds.

$(6)$ Final contradiction.

We claim that $[P/\Phi(P)] = P$. Assume that there exists a minimal subgroup $X/\Phi(P)$ of $P/\Phi(P)$ such that $X/\Phi(P)$ is not $\sigma$-permutable in $G/\Phi(P)$. Let $x \in X/\Phi(P)$ and $L = (x)$. Then, $X = L\Phi(P)$ and $|L| = p$ by Claims (2) and (5). If $L = L_{\sigma}$, then by Lemmas 2.1 and 2.3, $X/\Phi(P) = L\Phi(P)/\Phi(P)$ is $\sigma$-permutable in $G/\Phi(P)$, contrary to the choice of $X/\Phi(P)$. Hence, $1 = L_{\sigma} < L$. Then, by the hypothesis and Lemma 2.4, there exists a $\sigma$-permutable subgroup $T$ of $G$ such that $LT$ is $\sigma$-permutable in $G$ and $L \cap T \leq L_{\sigma}$. Then, $T = T \cap P$. Then, $T$, and $LT = L(T \cap P) = LT \cap P$ are both $\sigma$-permutable in $G$ by Lemma 2.3. Hence, $O^\sigma(G) \leq N_G(T)$ by Lemma 2.2 since $T$ is a $\sigma$-subgroup. Note that $Q^* \leq O^\sigma(G)$ if $Q^* \leq G$ by Claim (2), so $Q^* \leq G$. In a contradiction, $Q^* = G$, and so $T \leq G$. This shows that $T/\Phi(P)/\Phi(P) \leq G/\Phi(P)$, which implies that $T/\Phi(P)/\Phi(P) = 1$ or $P/\Phi(P)$ by Claim (2). If $T/\Phi(P)/\Phi(P) = 1$, then $T = P$, so that $L = L_{\sigma} \leq L \cap T \leq L_{\sigma}$, a contradiction. Hence, $T \leq \Phi(P)$. In addition, since $LT$ is $\sigma$-permutable in $G$, we have that $X/\Phi(P) = L\Phi(P)/\Phi(P) = LT/\Phi(P)$ is $\sigma$-permutable in $G/\Phi(P)$ by Lemma 2.1, a contradiction. The
contradiction shows that every minimal subgroup of $P/\Phi(P)$ is $\sigma$-permutable in $G/\Phi(P)$. It follows that every minimal subgroup of $P/\Phi(P)$ is $s$-permutable in $G/\Phi(P)$ since $q \notin \sigma$, and $m(G) = \{p, q\}$ by Claim (2). Hence, $[P/\Phi(P)] = p$ by Ref. [19, Lemma 2.11]. It follows that $p$ is cyclic of exponent $p$, which implies that $Q \leq G$ since $p$ is the smallest prime dividing $|G|$, a contradiction. This completes the proof.

**Proposition 3.2.** Let $G$ be a $\sigma$-full group of Sylow type and $H = \{H_i, \cdots, H_r\}$ a complete Hall $\sigma$-set of $G$ such that $H_i$ is a nilpotent $\sigma$-subgroup for all $i = 1, \cdots, r$. Let $P$ be a normal $p$-group of $G$ with $G/P$ supersolvable. If every cyclic subgroup $H$ of $P$ of order $p$ and order 4 (if $P$ is a nonabelian 2-group) is $\sigma$-embedded in $G$, then $G$ is supersolvable.

**Proof.** Assume that this is false and let $(G, P)$ be a counterexample with $|G| + |P|$ minimal. Without loss of generality, we may assume that $P \leq H_i$.

(1) Let $E$ be a proper subgroup of $G$. Then, $E$ is supersolvable.

It is clear that $(E, E \cap P)$ satisfies the hypothesis by Lemma 2.6(3). Hence, $E$ is supersolvable by the choice of $(G, P)$.

(2) $p$ is the largest prime number dividing $|G|$ and $p > 2$.

Assume that this is false and let $q$ be the largest prime number dividing $|G|$. Let $Q$ be a Sylow $q$-subgroup of $G$. Then, $QP/P \leq G/P$ since $G/P$ is supersolvable. It follows that $QP \leq G$. Hence, by Lemma 2.6(2), every cyclic subgroup $H$ of $P$ of order $p$ and order 4 (if $P$ is a nonabelian 2-group) is $\sigma$-embedded in $Q$. In view of Proposition 3.1, we have that $QP$ is $p$-nilpotent, and so $Q \leq G$. Next, we consider the quotient group $G/Q$. It is easy to see that $(G/Q, PQ/Q)$ satisfies the hypothesis by Lemma 2.6(3). Hence, $G/Q$ is supersolvable by the choice of $(G, P)$. Moreover, since $G/P$ is supersolvable, we obtain that $G$ is supersolvable, a contradiction. Hence, $p$ is the largest prime number dividing $|G|$. It is also clear that $p > 2$.

(3) Let $P_r$ be a Sylow $p$-subgroup of $G$. Then, $P_r \leq G$.

Since $G/P$ is supersolvable and $p$ is the largest prime number dividing $|G|$ by Claim (2), we have that $P_r/P \leq G/P$. This implies that $P_r \leq G$.

(4) $P_r$ is the unique normal Sylow subgroup of $G$, $P_r/\Phi(P_r)$ is a minimal normal subgroup of $G/\Phi(P_r)$, and the exponent of $P_r$ is $p$.

This directly follows from Claims (1)–(3) and Lemma 2.7. (5) $\Phi(P_r) \neq 1$.

Assume that $\Phi(P_r) = 1$. Then, $P_r$ is a minimal normal subgroup of $G$ by Claim (4), and so $P = P_r$. Let $H$ be a subgroup of $P$ with order $p$. Then, by the hypothesis and Lemma 2.4, there exists a $\sigma$-permutable subgroup $T$ of $G$ such that $HT$ is $\sigma$-permutable in $G$ and $H \cap T \leq H_{\sigma}$. Let $T = T \cap P$. Then, $T$ and $HT$ = $H(T \cap P) = HT \cap P$ are both $\sigma$-permutable in $G$ by Lemma 2.3. It follows from Lemma 2.2 that $O''(G) \leq N_{H_{\sigma}}(T)$ since $T$ is a $\sigma$-subgroup of $G$. In addition, since $T \leq P$ and the Hall $\sigma$-subgroup $H_i$ of $G$ is nilpotent by the hypothesis, we obtain that $T \leq H_i$ and thus $T \leq H, O''(G) = G$. Hence, $T_i = 1$ or $p$ by Claim (4). In the former case, we have that $H = HT$ is $\sigma$-permutable in $G$. In the later case, we obtain that $H = H \cap T \leq H \cap H_{\sigma}$ and thus, $H = H_{\sigma}$ is $\sigma$-permutable in $G$ by Lemma 2.3. It follows that $O''(G) \leq N_{H_{\sigma}}(H)$ for $H$ is a subgroup of order $p$. Then, by a similar argument as above, we have that $H \leq G$. Therefore, $H = P$ by Claim (4) since $P = P_r$. This shows that $P$ is a group of order $p$. It follows from $G/P$ is supersolvable that $G$ is supersolvable, a contradiction. Hence, we have (5).

(6) $P = P_1$.

It is clear that $P \Phi(P)/(P_1)(P) \leq G/\Phi(P)$. By Claim (4), we have that $P \Phi(P)/(P_1)(P) = 1$ or $P_1/(\Phi(P))$. The former case shows that $P \leq \Phi(P_1)$, so $G/\Phi(P) = (G/P)/(\Phi(P)/P)$ is supersolvable. Ref. [18, Chapter VI, Theorem 8.6(a)] and Claim (5) show that $G$ is supersolvable, a contradiction. Hence, $P \Phi(P)/(P_1)(P) = P_1/(\Phi(P))$, and thus, $P_1 = P \Phi(P)$. This shows that $P_1 = P$, as desired.

(7) Final contradiction.

Claims (6) and (4) show that the exponent of $P$ is $p$. Hence, there exists a subgroup $H$ of $P$ with order $p$ such that $H \not\leq \Phi(P)$ by Claims (5) and (6). By the hypothesis and Lemma 2.4, there exists a $\sigma$-permutable subgroup $T$ of $G$ such that $HT$ is $\sigma$-permutable in $G$ and $H \cap T \leq H_{\sigma}$.

We claim that $H_{\sigma} = 1$. If not, then $H = H_{\sigma}$ is $\sigma$-permutable in $G$ by Lemma 2.3. It follows from Lemma 2.1 that $H \Phi(P)/(\Phi(P))$ is $\sigma$-permutable in $G/\Phi(P)$. Hence, by Lemma 2.2, we have $O''(G/\Phi(P)) \leq N_{H \Phi(P)}(H \Phi(P)/\Phi(P))$. Moreover, since $H \Phi(P)/(\Phi(P)) \leq \Phi(P)$ and the Hall $\sigma$-subgroup of $G/\Phi(P)$ is nilpotent by the hypothesis, we obtain that $H \Phi(P)/(\Phi(P)) \leq G/\Phi(P)$. In view of Claims (4) and (6), we have that $H \Phi(P) = H \cap P$, and so $P = H \cap P$ is cyclic of order $p$. This shows that $G$ is supersolvable since $G/P$ is supersolvable, a contradiction. Hence, $H_{\sigma} = 1$.

Let $T_i = T \cap P$. Then, $T_i$ is $\sigma$-permutable in $G$ by Lemma 2.3. Note that $T_i$ is a $\sigma$-subgroup of $G$. Hence, $O''(G/\Phi(P)) \leq N_{H \Phi(P)}(T_i \Phi(P)/(\Phi(P)))$ by Lemmas 2.1 and 2.2. With a similar argument as above, we have that $T_i \Phi(P)/(\Phi(P)) \leq G/\Phi(P)$. Hence, $T_i \Phi(P)/(\Phi(P)) = P$ by Claims (4) and (6). If $T_i \Phi(P)/(\Phi(P))$ is a subgroup of $G$, then $HT_i = H \cap T \leq H \cap T \leq H_{\sigma} = 1$, a contradiction. Hence, $T_i \Phi(P) = \Phi(P)$. It is easy to see that $HT_i = H \cap T \leq P$ is also $\sigma$-permutable in $G$ and it is a $\sigma$-subgroup of $G$. As arguments above, we obtain that $HT_i \Phi(P)/(\Phi(P)) = HT \Phi(P)/(\Phi(P)) = P$. It follows from $T_i \Phi(P) = \Phi(P)$ that $HT_i \Phi(P)/(\Phi(P)) = HT \Phi(P)/(\Phi(P)) = P$. However, as $H \not\leq \Phi(P)$, we have that $H \Phi(P) = P$, and so $P = H \cap P$ is cyclic of order $p$. In addition, since $G/P$ is supersolvable, we obtain that $G$ is supersolvable, a contradiction. This contradiction completes the proof.

**Proof of Theorem 1.1.** We prove this theorem by induction on $|G|$. By Proposition 3.1 and Ref. [18, Chapter IV, Theorem 2.8], we know that $G$ possesses an ordered Sylow tower of the supersolvable type. Let $p$ be the largest prime dividing $|G|$ and $P$ a Sylow $p$-subgroup of $G$. Then, $P$ is normal in $G$. If $G = P$, then, clearly, $G$ is supersolvable. If $P < G$, then we consider the quotient group $G/P$. It follows from Lemma 2.6(2) that $G/P$ satisfies the hypothesis, and thus, $G/P$ is supersolvable by induction on $|G|$. In view of Proposition 3.2, we have that $G$ is supersolvable. This completes the proof.

**Proof of Theorem 1.2.** Suppose that this theorem is false and let $(G, E)$ be a counterexample with $|G| + |E|$ minimal. Then, $E \neq 1$.

(1) $E$ is supersolvable.

By Lemma 2.6(1), $E$ satisfies the hypothesis of Theorem
Some applications of our results

It is clear that every $\sigma$-$n$-embedded subgroup and every $\sigma$-permutable subgroup of $G$ are $\sigma$-embedded in $G$. Note that if $\sigma$ is the smallest partition of $\mathbb{P}$, then $\sigma$ is a one-element set for any $i \in I$, then every normal subgroup, every permutable subgroup, every $c$-normal subgroup, every $s$-permutable subgroup, every $s$-embedded subgroup and every $n$-embedded subgroup of $G$ are $\sigma$-embedded in $G$. However, the converse is not true in general (see Refs. [6, Example 1.2] and [7, Example 1.2]). Hence, we may directly obtain the following results from Theorems 1.1 and 1.2 and Propositions 3.1 and 3.2.

**Corollary 4.1.** [8, Theorem 3] Let $G$ be an odd order. If all subgroups of $G$ of prime order are normal in $G$, then $G$ is supersoluble.

**Corollary 4.2.** [9, Theorem 3.1] If every subgroup of prime order and every cyclic subgroup of order 4 are permutable in $G$, then $G$ is supersoluble.

**Corollary 4.3.** If every subgroup of prime order and every cyclic subgroup of order 4 are $s$-permutable in $G$, then $G$ is supersoluble.

**Corollary 4.4.** [10, Theorem 3.2] Let $P$ be a Sylow subgroup of $G$, where $p$ is the smallest prime belonging to $\pi(G)$. If every cyclic subgroup of order $p$ and order 4 (if $p$ is a nonabelian 2-group) is $s$-permutable in $G$, then $G$ is $p$-nilpotent.

**Corollary 4.5.** [11, Theorem 3.1] Suppose that $P$ is a $p$-subgroup of $G$ with $G/P$ is normal. Suppose further that every subgroup of order $p$ and order 4 (if $p$ is a nonabelian 2-group) is $s$-permutable in $G$, then $G$ is supersoluble.

**Corollary 4.6.** [12, Theorem 3.4] Let $N$ be a proper normal subgroup of $G$ with $G/N$ is supersoluble. Suppose that every subgroup of $N$ of prime order and order 4 (if $2 \in \pi(N)$ and the Sylow 2-subgroup of $N$ is nonabelian) is $s$-permutable in $G$, then $G$ is supersoluble.

**Corollary 4.7.** [13, Lemma 3.1] Let $P$ be a Sylow $p$-subgroup of $G$, where $p$ is the smallest prime belonging to $\pi(G)$. If every cyclic subgroup of order $p$ and order 4 is $c$-normal in $G$, then $G$ is $p$-nilpotent.

**Corollary 4.8.** [14, Theorem 4.2] Let $G$ be a group. If every cyclic subgroup of $G$ of prime order and order 4 is $c$-normal in $G$, then $G$ is supersoluble.

**Corollary 4.9.** [15, Theorem 3.4] Let $N$ be a normal subgroup of $G$ with $G/N$ being supersoluble. Suppose that every subgroup of $N$ of prime order and order 4 is $c$-normal in $G$, then $G$ is supersoluble.

**Corollary 4.10.** [16, Corollary 3.9] Let $H$ be a subgroup of $G$ with $G \leq H$. Suppose that every subgroup of $H$ of prime order and order 4 (if $2 \in \pi(H)$ and the Sylow 2-subgroup of $H$ is nonabelian) is $s$-permutable in $G$, then $G$ is supersoluble.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (12101339, 12001526) and Natural Science Foundation of Jiangsu Province, China (BK20200626).

Conflict of interest

The authors declare that they have no conflict of interest.

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