Inequalities of warped product submanifolds in a Riemannian manifold of quasi-constant curvature

Jiahui Wang, Lijuan Cheng, and Yecheng Zhu

School of Mathematics and Physics, Anhui University of Technology, Maanshan 243002, China

Correspondence: Yecheng Zhu, E-mail: zhuyc929@mail.ustc.edu.cn

Graphical abstract

The process of establishing the generalized normalized δ-Casorati curvatures inequality.

Public summary

- We establish Chen-like inequalities for generalized normalized δ-Casorati curvatures of warped product submanifolds in a Riemannian manifold of quasi-constant curvature.

- Our inequalities extend the optimal inequalities involving the scalar curvature and the Casorati curvature of a Riemannian submanifold in a real space form.

Citation: Wang J H, Cheng L J, Zhu Y C. Inequalities of warped product submanifolds in a Riemannian manifold of quasi-constant curvature. JUSTC, 2022, 52(3): 6. DOI: 10.52396/JUSTC-2021-0217
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School of Mathematics and Physics, Anhui University of Technology, Maanshan 243002, China
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Abstract: By optimization methods on Riemannian submanifolds, we establish two inequalities between the intrinsic and extrinsic invariants, for generalized normalized δ-Casorati curvatures of warped product submanifolds in a Riemannian manifold of quasi-constant curvature. We generalize the conclusions of the optimal inequalities of submanifolds in real space forms.

Keywords: Casorati curvature; optimization methods; scalar curvature

CLC number: O186.1 Document code: A

2020 Mathematics Subject Classification: 53C42

1 Introduction

In 1993, Chen [1] introduced δ-invariants, and established relationships between intrinsic invariants and extrinsic invariants for minimal submanifolds. In 1995, Chen [2] found Chen-like inequalities for Riemannian submanifolds and gave some applications of δ-invariants. Submanifolds are ideal submanifolds when Chen-like inequalities are equal and they receive the least possible tension at each point from ambient spaces.

The Casorati curvature was originally introduced in 1980 for surfaces in 3-dimensional Euclidean space and is defined as the normalized square of the length of the second fundamental form (see Ref. [3]). In 2007, Decu et al. [4] introduced the normalized δ-Casorati curvatures δn−1 and δn−1 and established two optimal inequalities involving the scalar curvature and the normalized δ-Casorati curvature. In 2008, Decu et al. [4] introduced the generalized normalized δ-Casorati curvatures δ2,r,n−1 and δ3,r,n−1 and proved two sharp inequalities. In 2017, Park [5] obtained two types of optimal inequalities for the real hypersurfaces of complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians. In 2020, Choudhary and Blaga [7] established some sharp inequalities involving generalized normalized δ-Casorati curvatures for invariant, anti-invariant and slant submanifolds in metallic Riemannian space forms and characterized the submanifolds for which the equality holds.

In this study, we establish Chen-like inequalities for generalized normalized δ-Casorati curvatures of warped product submanifolds in a Riemannian manifold of quasi-constant curvature.

Let Nn1 × Nn2 be two Riemannian manifolds with positive dimensions equipped with Riemannian metrics gNn1 and gNn2, respectively. Let f be a positive function on Nn1. Consider the product manifold Nn1 × Nn2, with its projections π : Nn1 × Nn2 → Nn1 and η : Nn1 × Nn2 → Nn2. The warped product manifold M′ = Nn1 × f Nn2 is the product manifold Nn1 × Nn2 equipped with a Riemannian structure such that

||X||^2 = ||f(π(X))||^2 + f^2(π(x))||η(X)||^2

for any tangent vector X ∈ TM′. Thus, we have g = gNn1 + f gNn2. The function f is called the warping function of the warped product manifold.

A Riemannian manifold (M′, g) is called a Riemannian manifold of quasi-constant curvature if the curvature tensor satisfies the following condition (see Ref. [8]):

\[ R(X, Y, Z, W) = a[\tilde{g}(X, Z)\tilde{g}(Y, W) - \tilde{g}(Y, Z)\tilde{g}(X, W)] + b[\tilde{g}(X, Z)T(Y)T(W) - \tilde{g}(W, Y)T(Y)T(Z) + \tilde{g}(W, Y)T(Z)T(Y) - \tilde{g}(Y, Z)T(W)T(Y)] \]

(2)

where a and b are scalar functions, T is a 1-form defined by

\[ T(X) = \tilde{g}(X, P) \]

(3)

and P denotes the unit vector field. We uniquely decompose the vector field P on M′ into its tangent component P and normal component P, that is,

\[ P = P^T + P^N \]

(4)

Theorem 1.1. Let \( \phi : M^r = N^r_1 \times N^r_2 \rightarrow M^r \) be an isometric immersion of an n-dimensional warped product submanifold \( M^r = N^r_1 \times N^r_2 \rightarrow M^r \) into an m-dimensional Riemannian manifold of a quasi-constant curvature \( M^r \). Then

(i) the generalized normalized δ-Casorati curvature \( \delta_n(r, n−1) \) satisfies

\[
\frac{2}{n(n−1)} \left\{ \frac{q \Delta f}{f} + \frac{p(p−1)a}{2} + \frac{b(q−1)|P|^2}{2} \right\} \frac{\delta_n(r, n−1)}{n(n−1)} \frac{npr(n^2−n^2+r^2)}{(n−1)(n^2−n−r^2)^2} + \frac{a^2}{2} + \frac{b^2}{2} \geq \rho
\]

(5)

for any real number \( r \) such that \( 0 < r < n(n−1) \), where \( \|P\|^2 = \sum_{i=1}^{n} \langle P^i, e_i \rangle^2, \|P\|^2 = \sum_{i=1}^{n} \langle P^i, e_i \rangle^2, \rho \) is the normalized scalar curvature, \( \|H\|^2 \) is the squared mean curvature, a and b are scalar functions;

Received: October 09, 2021; Accepted: January 21, 2022

Cite This: JUSTC, 2022, 52(3): 6 (7pp)
\[\begin{align*}
\text{(ii)} & \quad \text{the generalized normalized } \delta \text{-Casorati curvature } \\
& \quad \delta_c(r; n) \text{ satisfies} \\
& \quad \frac{2}{n(n-1)} \times \left( \frac{qM_f}{f} + \frac{p(p-1)a}{2} + \delta_c(r; n-1) \right) + \frac{b(q-1)a}{2} + \\
& \quad \frac{b(q-1)|Pf|^2}{2} + \delta_c(r; n-1) \frac{npr(n^2 - n + qr - r)|H|^2}{n(n-1)|p^2r^2 + p|^2} \geq p \tag{6}
\end{align*}\]

for any real number \( r > n(n-1) \).

Equalities hold in (5) and (6) if and only if the shape operators for the suitable tangent and normal orthonormal frames are given by

\[
\begin{align*}
A_{m+i} &= \\
& \begin{pmatrix}
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & h_{m+i}^{-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\end{pmatrix}
\end{align*}
\tag{7}
\]

where \( f \) is a function on \( M^r \).

Let \( b = 0 \) and \( a = \text{const} \). Then we have

**Corollary 1.1.** Let \( \phi : M^r \rightarrow N^1 \times N_n \rightarrow \tilde{M}^r(a) \) be an isometric immersion of an \( n \)-dimensional warped product submanifold \( M^r \) into an \( m \)-dimensional Riemannian manifold of constant sectional curvature \( a \). Then

(i) the generalized normalized \( \delta \)-Casorati curvature \( \delta_c(r; n) \text{ satisfies} \\
\[
\frac{2}{n(n-1)} \times \left( \frac{qM_f}{f} + \frac{p(p-1)a}{2} + \delta_c(r; n-1) \right) + \frac{b(q-1)a}{2} + \\
\frac{b(q-1)|Pf|^2}{2} + \delta_c(r; n-1) \frac{npr(n^2 - n + qr - r)|H|^2}{n(n-1)|p^2r^2 + p|^2} \geq p
\]

for any real number \( r \) such that \( 0 < r < n(n-1) \);

(ii) the generalized normalized \( \delta \)-Casorati curvature \( \tilde{\delta}_c(r; n-1) \text{ satisfies} \\
\[
\frac{2}{n(n-1)} \times \left( \frac{qM_f}{f} + \frac{p(p-1)a}{2} + \delta_c(r; n-1) \right) + \frac{b(q-1)a}{2} + \\
\frac{b(q-1)|Pf|^2}{2} + \delta_c(r; n-1) \frac{npr(n^2 - n + qr - r)|H|^2}{n(n-1)|p^2r^2 + p|^2} \geq p
\]

for any real number \( r > n(n-1) \).

Equalities hold in (8) and (9) if and only if the shape operators for the suitable tangent and normal orthonormal frames are given by Eq. (7).

Moreover, let \( p = 0, q = n \) and \( f = 1 \). Then we have

**Corollary 1.2.** Let \( \phi : M^r \rightarrow \tilde{M}^r(a) \) be an isometric immersion of an \( n \)-dimensional warped product submanifold into \( \tilde{M}^r(a) \). We have

(i) for any real number \( r \) such that \( 0 < r < n(n-1) \),
\[
\delta_c(r; n-1) + n(n-1)a \geq n(n-1)p
\]

(ii) for any real number \( r \) such that \( r > n(n-1) \),
\[
\tilde{\delta}_c(r; n-1) + n(n-1)a \geq n(n-1)p
\]

Equalities hold in (10) and (11) if and only if \( M^r \) is an invariantly quasi-umbilical submanifold.

**Remark:** Corollary 1.2 is Theorem 2.1, and Corollary 3.1 in Ref. [5].

## 2 Preliminaries

Let \( M^r \) be an \( n \)-dimensional warped product submanifold of an \( m \)-dimensional Riemannian manifold of quasi-constant curvature \( \tilde{M}^r \). Let \( \nabla \) and \( \tilde{\nabla} \) be the Levi–Civita connection on \( M^r \) and \( \tilde{M}^r \), respectively. Then, the Gauss and Weingarten formulas are given respectively by

\[
\begin{align*}
\tilde{\nabla}_x Y &= \nabla_x Y + h(X;Y) \tag{12} \\
\tilde{\nabla}_X N &= -A_x X + \tilde{\nabla}_X N
\end{align*}
\]

for vector fields \( X, Y \) tangent to \( M^r \), and vector field \( N \) normal to \( M^r \). Here \( h \) denotes the second fundamental form, \( \nabla \) is the normal connection and \( A \) is the shape operator. The second fundamental form and shape operator are related by

\[
\tilde{g}(h(X;Y),N) = g(A_x X,Y) \tag{13}
\]

where \( \tilde{g} \) and \( g \) denote the metric on \( \tilde{M}^r \) and \( M^r \) respectively.

If \( R \) and \( \tilde{R} \) are the curvature tensors of \( M^r \) and \( \tilde{M}^r \) respectively, then the Gauss equation is given by

\[
R(X,Y,Z,W) = \tilde{R}(X,Y,Z,W) + g(h(X;Z),h(Y;W)) - g(h(X;W),h(Y;Z)) \tag{14}
\]

for any vector field \( X, Y, Z, \) and \( W \) tangent to \( M^r \).

Let \( \{e_1, \cdots, e_n\} \) be an orthonormal basis of the tangent space \( T_x M^r \) and let \( \{e_{m+1}, \cdots, e_m\} \) be an orthonormal basis of normal space \( T^*_x M^r \). The mean curvature vector \( H \) at \( x \) is

\[
H(x) = \frac{1}{n} \sum_{i=1}^m \left( \sum_{j=1}^n h_{ij}^c \right) e_j \tag{15}
\]

The squared mean curvature of the submanifold \( M^r \) in \( \tilde{M}^r \) is defined as

\[
|H|^2 = \frac{1}{n} \sum_{i=1}^m \left( \sum_{j=1}^n h_{ij}^c \right)^2 \tag{16}
\]

Also, we set

\[
|\tilde{H}|^2 = \sum_{i=1}^m \sum_{j=1}^n \tilde{g}(h(e_i;e_j),e_j)^2 \tag{17}
\]

Let \( K(e_i;e_j) \) be the sectional curvature of the plane section spanning \( e_i \) and \( e_j \), at \( x \in \tilde{M}^r \). Subsequently, the scalar curvature \( \tau(x) \) of \( M^r \) is given by

\[
\tau(x) = \sum_{i,j=1}^m K(e_i;e_j) \tag{18}
\]

\[\text{DOI: 10.52396/JUSTC-2021-0217} \]

JUSTC, 2022, 52(3): 6
and the normalized scalar curvature $\rho$ of $M^r$ at $x$ is defined as
\[
\rho(x) = \frac{2\tau(x)}{n(n-1)}
\]  
(19)

The Casorati curvature $C$ of the submanifold $M^r$ is the squared norm of the second fundamental form $h$ over dimension $n$ and is given by
\[
C = \frac{1}{r} \sum_{i \neq j \leq r} (h_{ij})^2
\]  
(20)

If $L$ is an $l$-dimensional subspace of $T_xM^r$, where $l \geq 2$ and $\{e_1, \ldots, e_l\}$ is an orthonormal basis of $L$, the scalar curvature $\tau(L)$ of the $l$-plane section $L$ is defined as
\[
\tau(L) = \sum_{1 \leq i < j \leq l} K(e_i \wedge e_j)
\]  
(21)

and the Casorati curvature of the subspace $L$, denoted by $C(L)$, is given by
\[
C(L) = \frac{1}{l} \sum_{i \neq j \leq l} (h_{ij})^2
\]  
(22)

The generalized normalized $\delta$–Casorati curvatures $\delta_r(x; n-1)$ and $\delta_r(x; r-n-1)$ of the submanifold $M^r$ are defined for a positive real number $\rho \neq n(n-1)$ as
\[
[\delta_r(x; n-1)] = rC + \frac{(n-1)(n+r)(n^2 - n - r)}{nr} \inf |C(L)| : L a hyperplane of $T_xM^r$
\]  
(23)

if $0 < r < n^2 - n$; and
\[
[\delta_r(x; r-n-1)] = rC - \frac{(n-1)(n+r)(n^2 + n)}{nr} \sup |C(L)| : L a hyperplane of $T_xM^r$
\]  
(24)

if $r > n^2 - n$.

By Gauss equation, we get
\[
K(e_i \wedge e_j) = \tilde{K}(e_i \wedge e_j) + \sum_{a=1}^k (h_{ij}h_{ij}' - (h_{ij}')(x))
\]  
(25)

where $K(e_i \wedge e_j)$ and $\tilde{K}(e_i \wedge e_j)$ denote the sectional curvatures of the plane section spanned by $e_i$ and $e_j$ at $x$ in the submanifold $M^r$ and in the ambient manifold $\tilde{M}^r$, respectively. By Eqs. (2) and (25), we have
\[
\tau(N^r) = \sum_{a=1}^k \sum_{1 \leq i < j \leq r} (h_{ij}h_{ij}' - (h_{ij}')(x)) + \tau(N^r) = \frac{p(p-1)a}{2} + b(p-1)||P^r||_{ei}^2 + \sum_{a=1}^k \sum_{1 \leq i < j \leq r} (h_{ij}h_{ij}' - (h_{ij}')(x))
\]  
(26)

\[
\tau(N^r) = \sum_{a=1}^k \sum_{1 \leq i < j \leq r} (h_{ij}h_{ij}' - (h_{ij}')(x)) + \tau(N^r) = \frac{q(q-1)a}{2} + b(q-1)||P^r||_{ei}^2 + \sum_{a=1}^k \sum_{1 \leq i < j \leq r} (h_{ij}h_{ij}' - (h_{ij}')(x))
\]  
(27)

where $||P^r||_{ei} = \sum_{a=1}^k g(P^r, e_i)'$, $||P^r||_{ei}' = \sum_{a=1}^k g(P^r, e_i)'$.

**Definition 2.1.** For the differential function $f$ on $M^r$, the Laplacian $g$ and gradient $\nabla f$ of $f$ are defined by
\[
g(\nabla f, X) = X(f)
\]  
(28)

\[
\Delta f = \sum_{i=1}^k ((\nabla_{e_i} e_i)f - e_i e_i f)
\]  
(29)

for any vector field $\chi$ that is tangent to $M^r$.

**Lemma 2.1.** Let $M^r = N^l_x \times N^r_x$ be a warped product submanifold of $M^r$. The relation between the sectional curvature and Laplacian $\Delta f$ of $f$ is
\[
\sum_{1 \leq i < j \leq l} K(e_i \wedge e_j) = \frac{q\Delta f}{f} = q(\Delta \ln f - \|\nabla \ln f\|)^2
\]  
(30)

**Lemma 2.2.** Let $N_i$ be a Riemannian submanifold of a Riemannian manifold $(N_i, g)$, $\varphi : N_i \rightarrow \mathbb{R}$ be a differentiable function and consider the constrained extremum problem
\[
\min_{x \in \phi} \varphi(x)
\]  
(31)

If $x_i \in N_i$ is a solution of the problem (31), then
(i) $(\nabla \varphi)(x_i) \in T_{x_i}N_i$;
(ii) the bilinear form $\Lambda : T_xN_i \times T_xN_i \rightarrow \mathbb{R}$ defined by
\[
\Lambda(X, Y) = Hess_{X} 
\]  
(32)

is positive semidefinite, where $h_i$ is the second fundamental form of $N_i$ in $N_i$ and $\varphi$ is the gradient of $\varphi$.

### 3 Proof of the theorem

**Proof of Theorem 1.1** From Eqs. (26), (27), (30), and the Gauss equation, we obtain
\[
\tau(x) = \sum_{i=1}^k \sum_{1 \leq j \leq l} K(e_i \wedge e_j) + \sum_{j=1}^k K(e_i \wedge e_j) + \sum_{i=1}^k K(e_i \wedge e_j) = \frac{q\Delta f}{f} + \frac{p(p-1)a}{2} + b(p-1)||P^r||_{ei}^2 + \frac{q(q-1)a}{2} + b(q-1)||P^r||_{ei}'^2 + \sum_{a=1}^k \sum_{1 \leq i < j \leq r} (h_{ij}h_{ij}' - (h_{ij}')(x)) + \sum_{a=1}^k \sum_{1 \leq i < j \leq r} (h_{ij}h_{ij}' - (h_{ij}')(x))
\]  
(33)

We define the following quadratic polynomial $P$ in the components of the second fundamental form as
\[
P = \alpha + \frac{(n+1)(n+r)(n^2 - n - r)}{nr} C(L) = 2\tau + 2 \times \left[ \frac{q\Delta f}{f} + \frac{p(p-1)a}{2} + b(p-1)||P^r||_{ei}^2 + \frac{q(q-1)a}{2} + b(q-1)||P^r||_{ei}'^2 \right]
\]  
(34)

where $L$ denotes the hyperplane of $T_xM^r$. Without loss of generality, we can suppose that $L$ is spanned by $e_1, e_2, \ldots, e_{n-r}$. From Eqs. (33) and (34), we have
\[
P = \frac{2}{n} \sum_{a=1}^k \sum_{1 \leq i < j \leq r} (h_{ij}h_{ij}' - (h_{ij}')(x)) - \sum_{a=1}^k \sum_{1 \leq i < j \leq r} (h_{ij}h_{ij}' - (h_{ij}')(x))
\]  
(35)

We consider the quadratic forms
\[
\varphi_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}, \quad \alpha = n + n + 1, \ldots, m
\]  
(36)
defined by

\[ \varphi_\alpha(h_1^{\alpha_1}, \ldots, h_n^{\alpha_n}) = \frac{n^2 - n + nr - 2r}{r} \sum_{i=1}^{\alpha_n} (h_i^{\alpha_1})^2 + \frac{r}{n} (h_n^{\alpha_n})^2 - 2 \sum_{i=1}^{\alpha_n} h_i^{\alpha_1} h_n^{\alpha_n} \]

Then by Eqs. (35) and (37), we derive

\[ \mathcal{P}_\alpha = \sum_{n=1}^{\alpha_n} \varphi_\alpha \]

Next, for \( \alpha \), we consider the extremum problem

\[ \min \varphi_\alpha, \quad \text{subject to} \quad \Gamma : h_1^{\alpha_1} + h_2^{\alpha_2} + \cdots + h_n^{\alpha_n} = K^\alpha \quad (39) \]

where \( K^\alpha \) is a real constant (see Ref. [11]). The partial derivatives of function \( \varphi_\alpha \) are

\[
\begin{align*}
\frac{\partial \varphi_\alpha}{\partial h_i^{\alpha_1}} &= \frac{2(n+r)(n-1)}{r} h_i^{\alpha_1} - 2 \sum_{j=1}^{\alpha_n} h_j^{\alpha_1} \\
&\vdots \\
\frac{\partial \varphi_\alpha}{\partial h_{p+1}^{\alpha_{p+1}}} &= \frac{2(n+r)(n-1)}{r} h_{p+1}^{\alpha_{p+1}} - 2 \sum_{j=p+2}^{\alpha_n} h_j^{\alpha_1} \\
\frac{\partial \varphi_\alpha}{\partial h_n^{\alpha_n}} &= \frac{2(n+r)(n-1)}{n} h_n^{\alpha_n} - 2 \sum_{j=1}^{\alpha_n-1} h_j^{\alpha_n} \\
\frac{\partial \varphi_\alpha}{\partial h_n^{\alpha_n}} &= \frac{2(n+r)(n-1)}{n} h_n^{\alpha_n} - 2 \sum_{j=1}^{\alpha_n-1} h_j^{\alpha_n}
\end{align*}
\]

Applying Lemma 2.2, for an optimal solution \((h_1^{\alpha_1}, h_2^{\alpha_2}, \ldots, h_n^{\alpha_n})\) of the minimum problem, vector \( \text{grad} \varphi_\alpha \) is normal at \( \Gamma \) and collinear with the vector \((1, 1, \ldots, 1)\).

From Eq. (40) and Lemma 2.2, we derive that a critical point of the problem has the following form:

\[
\begin{align*}
K^\alpha &= \sum_{i=1}^{\alpha_n} h_i^{\alpha_1} \\
K^\alpha &= \sum_{i=p+2}^{\alpha_n} h_{i-1}^{\alpha_{i-1}} \\
K^\alpha &= \sum_{i=1}^{\alpha_n} h_i^{\alpha_n}
\end{align*}
\]

We fixed an arbitrary point \( x \in \Gamma \). According to Lemma 2.2, we deduce that the corresponding bilinear form

\[ \Lambda : T \Gamma \times T \Gamma \to \mathbb{R} \]

is given by

\[ \Lambda(X, Y) = \text{Hess}_{\varphi_\alpha}(X, Y) + g(h'(X, Y), (\text{grad} \varphi_\alpha)(x)) \]

where \( h' \) is the second fundamental form of \( \Gamma \) in \( \mathbb{R}^n \) and \( g \) is the inner product on \( \mathbb{R}^n \).

By Eq. (40), for \( i, j \in \{1, \ldots, p\} \), \( i \neq j \) and \( s, t \in \{p+1, \ldots, n\} \), \( s \neq t \), we get

\[
\begin{align*}
\frac{\partial^2 \varphi_\alpha}{\partial h_i^{\alpha_1} \partial h_j^{\alpha_1}} &= \frac{2(n+r)(n-1)-2r}{r} \\
\frac{\partial^2 \varphi_\alpha}{\partial h_i^{\alpha_1} \partial h_j^{\alpha_1}} &= -2 \\
\frac{\partial^2 \varphi_\alpha}{\partial h_i^{\alpha_1} \partial h_j^{\alpha_1}} &= -2 \\
\frac{\partial^2 \varphi_\alpha}{\partial h_i^{\alpha_1} \partial h_j^{\alpha_1}} &= \frac{2r}{n}
\end{align*}
\]

Note that

\[ (\text{Hess}_{\varphi_\alpha})_{ij} = (\varphi_\alpha)_{ij} = \frac{\partial^2 \varphi_\alpha}{\partial h_i^{\alpha_1} \partial h_j^{\alpha_1}} - \frac{\partial \varphi_\alpha}{\partial h_i^{\alpha_1}} \Gamma_i^j \]

where

\[ \Gamma_i^j = \frac{1}{2} K^\alpha \left( \frac{\partial \varphi_\alpha}{\partial h_i^{\alpha_1}} + \frac{\partial \varphi_\alpha}{\partial h_j^{\alpha_1}} - \frac{\partial \varphi_\alpha}{\partial h_i^{\alpha_1}} \frac{\partial \varphi_\alpha}{\partial h_j^{\alpha_1}} \right) \]

Since \( g \) is the inner product on \( \mathbb{R}^n \), \( g_{ij} \) is constant, and \( \Gamma_{ij} \) is 0. Then, we have

\[ (\text{Hess}_{\varphi_\alpha})_{ij} = (\varphi_\alpha)_{ij} = \frac{\partial^2 \varphi_\alpha}{\partial h_i^{\alpha_1} \partial h_j^{\alpha_1}} \]

The Hessian matrix of \( \varphi_\alpha \) is

\[
\text{Hess}_{\varphi_\alpha} = \begin{pmatrix}
2(n+r)(n-1)-2r & \cdots & -2 & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-2 & \cdots & 2(n+r)(n-1)-2r & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 2(n+r)(n-1)-2r & -2 & \cdots & 0 \\
0 & \cdots & 0 & -2 & \cdots & -2 & 2r \\
0 & \cdots & 0 & 0 & \cdots & 0 & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0
\end{pmatrix}
\]
As $\Gamma$ is totally geodesic in $\mathbb{R}^n$, we consider a vector $X = (X_1, X_2, \ldots, X_n)$ tangent to $\Gamma$ at an arbitrary point $x$ on $\Gamma$, that is, we verify the relation $\sum_{i=1}^n X_i = 0$ (see Ref. [11]). Next, we prove $\Lambda(XX) \geq 0$.

(i) For $n = 1$, we have two possibilities

(a) $\text{Hess}_{\phi} = \begin{pmatrix} 2r \\ 0 \end{pmatrix}$, (b) $\text{Hess}_{\phi} = \begin{pmatrix} -2 \\ r \end{pmatrix}$

Since $X_i = 0$, in this two cases $\text{Hess}_{\phi}(X; X) = 0$

(ii) For $n = 2$, we have three possibilities

(a) $\text{Hess}_{\phi} = \begin{pmatrix} 4 & -2 \\ -2 & 4 \\ -2 & 4 \\ -2 & 4 \end{pmatrix}$, (b) $\text{Hess}_{\phi} = \begin{pmatrix} 4 & 0 \\ 0 & r \\ -2 & 0 \\ 0 & -2 \end{pmatrix}$, (c) $\text{Hess}_{\phi} = \begin{pmatrix} 4 & -2 & -2 \\ -2 & 2r & 2r \\ -2 & 2r & 2r \end{pmatrix}$

For (a),

$\text{Hess}_{\phi}(X; X) = \begin{pmatrix} 4 & 2n^2 & 2n^2 & \cdots & 2n^2 \\ 2n^2 & 4 & 4 & \cdots & 4 \\ 2n^2 & 4 & 4 & \cdots & 4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2n^2 & 4 & 4 & \cdots & 4 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \\ X_n \end{pmatrix} = 0$

For (b), $\text{Hess}_{\phi}$ is positive definite, i.e., $\text{Hess}_{\phi}(X; X) > 0$. For (c), $\text{Hess}_{\phi}$ is positive semi-definite, i.e., $\text{Hess}_{\phi}(X; X) \geq 0$.

(iii) For $n \geq 3$, when $n = p$, we have

$\text{Hess}_{\phi}(X; X) = \begin{pmatrix} 2(n+r)(n-1) & \cdots & \cdots & \cdots & \cdots \\ \cdots & 2(n+r)(n-1) & \cdots & \cdots & \cdots \\ \cdots & \cdots & 2(n+r)(n-1) & \cdots & \cdots \\ \cdots & \cdots & \cdots & 2(n+r)(n-1) & \cdots \\ \cdots & \cdots & \cdots & \cdots & 2(n+r)(n-1) \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \\ X_n \end{pmatrix} \geq 0$

When $n > p$, let

$A = \begin{pmatrix} 2(n+r)(n-1) & -2 & \cdots & \cdots & \cdots \\ -2 & 2(n+r)(n-1) & -2 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \cdots \\ -2 & -2 & \cdots & (n+r)(n-1) & -2 \\ -2 & -2 & \cdots & -2 & 2r \end{pmatrix}$

$B = \begin{pmatrix} 2(n+r)(n-1) & -2 & \cdots & \cdots & \cdots \\ -2 & 2(n+r)(n-1) & -2 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \cdots \\ -2 & -2 & \cdots & (n+r)(n-1) & -2 \\ -2 & -2 & \cdots & -2 & \frac{2r}{n} \end{pmatrix}$

Note that

$0 \leq \lambda E - A = \begin{pmatrix} \lambda - \frac{2(n+r)(n-1)}{r} \\ \lambda - \frac{2(n+r)(n-1) - 2pr}{r} \end{pmatrix} \begin{pmatrix} \lambda - \frac{2(n+r)(n-1) - 2pr}{r} \end{pmatrix} = \left( \lambda - \frac{2(n+r)(n-1) - 2pr}{r} \right) \left( \lambda - \frac{2(n+r)(n-1) - 2pr}{r} \right)$

Thus, all eigenvalues of $A$ are greater than 0, i.e., $A$ is positive definite.

Since $n > p$, when $n - p = q = 1$, we have

$B = \begin{pmatrix} 2r \\ \frac{2r}{n} \end{pmatrix}$

$B$ is positive definite. When $n - p = q \geq 2$, we have

$0 \leq \lambda E - B = \begin{pmatrix} \lambda - \frac{2(n+r)(n-1)}{r} \\ \lambda - \frac{2(n+r)(n-1) - 2(q-1)r}{r} \end{pmatrix} \begin{pmatrix} \lambda - \frac{2(n+r)(n-1) - 2(q-1)r}{r} \\ \lambda - \frac{2(n+r)(n-1) - 2(q-1)r}{r} \end{pmatrix} = \left( \lambda - \frac{2(n+r)(n-1)}{r} \right) \left( \lambda - \frac{2(n+r)(n-1) - 2(q-1)r}{r} \right)$

$\frac{2(n+r)}{n} \times \frac{2(n+r)(n-1) - 2(q-1)r}{r} = \left( \lambda - \frac{2(n+r)(n-1)}{r} \right) \left( \lambda - \frac{2(n+r)(n-1) - 2(q-1)r}{r} \right)$

Since

$\lambda = \frac{2(n+r)(n-1)}{r}$

$\lambda = \frac{2(n+r)(n-1) - 2(q-1)r}{r}$

$\lambda = \frac{4(n+r) \times rp}{n}$

(56)
Taking the infimum over all tangent hyperplanes \( L \) of \( T,M^r \) in (63), we obtain

\[
\frac{2}{n(n-1)} \left\{ \frac{q\Delta f}{f} + \frac{p(p-1)a}{2} + b(p-1)||P'||_{\mathcal{C}} + \frac{q(q-1)a}{2} + b(q-1)||P'||_{\mathcal{C}} \right\} + \delta_{\lambda}(r; n-1) \frac{npr(n^2 - n^r + qr - r)||H'||^2}{(n-1)\left((n^2 - n^r + qr - r)^2 + p^2r^2 \right)} \geq \rho \tag{64}
\]

**Case 2:** \( r > n(n-1) \). Similarly, using the same method, we obtain

\[
\frac{2}{n(n-1)} \left\{ \frac{q\Delta f}{f} + \frac{p(p-1)a}{2} + b(p-1)||P'||_{\mathcal{C}} + \frac{q(q-1)a}{2} + b(q-1)||P'||_{\mathcal{C}} \right\} + \delta_{\lambda}(r; n-1) \frac{npr(n^2 - n^r + qr - r)||H'||^2}{(n-1)\left((n^2 - n^r + qr - r)^2 + p^2r^2 \right)} \geq \rho \tag{65}
\]

Equalities hold in (64) and (65) at a point \( x \in M^r \) if and only if inequalities (35) and (61) become equalities. Thus, we have

We prove that all eigenvalues of \( B \) are greater than or equal to 0, i.e., \( B \) is positive semi-definite. Thus, we prove that \( \text{Hess}_{\lambda}(X,X) \geq 0 \).

Combining (i), (ii) and (iii), we have

\[
\lambda(X,X) \geq \text{Hess}_{\lambda}(X,X) \geq 0 \tag{60}
\]

Hence, by Eq. (41), the point \((h^r_1, h^r_2, \ldots, h^r_n)\) is a global minimum point. From Eqs. (37) and (41), we have
By choosing an orthonormal basis such that $e_{n+1}$ is in the direction of the mean curvature vector, we have

\[
\begin{aligned}
  h_{i}^{e_{i}} &= \cdots = h_{n}^{e_{n}} = \frac{p r^{2}}{(n^{2} - n - r + q r) + p^{2} r^{2}} K^{e_{i}}, \\
  h_{e_{n+1}}^{e_{n+1}} &= \cdots = h_{e_{n-1}}^{e_{n-1}} = \frac{(n^{2} - n - r + q r - r) r}{(n^{2} - n - r + q r) + p^{2} r^{2}} K^{e_{n+1}}, \\
  h_{n}^{e_{n}} &= \frac{n(n - 1)(n^{2} - n + q r - r)}{(n^{2} - n - r + q r) + p^{2} r^{2}} K^{e_{n}} \\
  h_{i}^{e_{j}} &= 0, \quad i \neq j
\end{aligned}
\]  

(66)

where $f_{i} = \frac{K^{e_{i+1}}}{(n^{2} - n - q r - r) + p^{2} r^{2}}$ is a function on $M^{n}$.

**Acknowledgements**

This work was supported by the National Natural Science Foundation of China (12026262).

**Conflict of interest**

The authors declare that they have no conflict of interest.

**Biographies**

**Jiahui Wang** is currently a graduate student at the Anhui University of Technology. Her research interests mainly focus on warped product submanifolds and isoparametric hypersurfaces.

**Yecheng Zhu** received his PhD from the University of Science and Technology of China. He is currently an associate professor at the Anhui University of Technology. He is mainly engaged in differential geometry.

**References**


