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Second-order stochastic dominance with respect to the rank-dependent utility model

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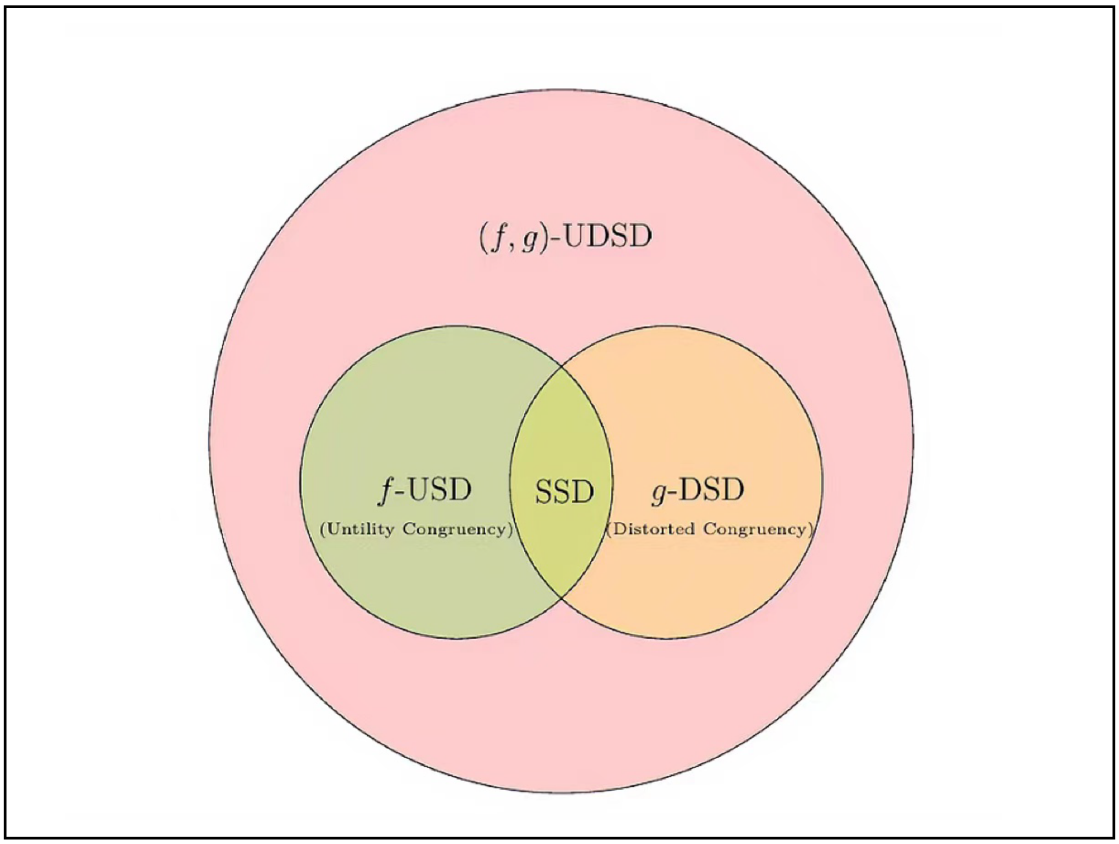
Graphical abstract



A new stochastic dominance based on a fixed utility function and a fixed distortion function.

Public summary

- We introduce a new generalized class of partial orders that contains some rules of stochastic dominance as its special cases.
- We investigate the properties and characterizations of the new partial order through the RDU model.
- The characterization of utility congruency or distortion congruency of the new partial order is presented.
- We provide a general approach to interpolate first-order stochastic dominance and second-order stochastic dominance based on the new partial order.

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Abstract: A generalized family of partial orders is studied, which is semiparametrized by a utility function f and a distortion function g, namely, (f,g)-utility and distorted stochastic dominance ((f,g)-UDSD). Such a family is especially suitable for representing a decision maker's preferences in terms of risk aversion. We characterize the monotonicity of the partial order in the rank-dependent utility model, and the isotonic classes of rank-dependent utility with (f,g)-UDSD are also established. Inspired by the concept of the congruent utility class, we introduce the definition of the congruent distortion class. The characterization of the congruent utility class or distortion class of (f,g)-utility and distorted stochastic dominance is investigated. Based on the main results in this paper, we unify some related results in the existing literature. As an application, we propose a general approach to develop a continuum between first-order stochastic dominance and second-order stochastic dominance based on the partial order.

Keywords: stochastic dominance; utility function; distortion function; rank-dependent utility

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1 Introduction

In decision theory, decision makers measure their preferences regarding uncertainty by assigning different weights to the outcomes of the corresponding random variables. Mathematically, this process may be expressed in terms of integrals of distribution functions or quantiles, expected utilities or distorted expectations (dual theory in Ref. [1]). Applying different transformations in the process yields kinds of partial orders that are tools for representing preferences and risk attitudes. Such partial order is also known as stochastic dominance (SD). The most commonly used method in areas such as economics and finance is the second-order stochastic dominance (SSD), which represents any decision maker who is also risk averse, and it requires increasing and concave utility functions or distortion functions. However, SSD might be limiting for those decision makers who may be mostly risk averse but have some degree of flexibility in their preferences as discussed in Refs. [2−4]. To address this issue, Ref. [5] developed a more general form of stochastic dominance, called SSD with respect to a (utility) function f, which will be denoted by f-USD in this paper. The partial order f-USD is related to the well known expected utility model where the lower bound on the decision maker's measure of risk aversion[6] is specified by the fixed function f. If f is a convex function, then the partial order f-USD is stronger than SSD, and it applied to a class of decision makers who are not strongly risk averse. Similar to the idea of Ref. [5] but from a different perspective that the dual model in Ref. [1] was applied, Ref. [7] developed a class of partial orders called g-distorted stochastic dominance (g-DSD), where g is a given distortion function. Mathematically, these two approaches have different points of view. First, f-USD can be characterized in terms of a class of isotonic expected utilities, while g-DSD is characterized by a class of isotonic distorted expectations. Second, f-USD is based on an integrated distribution function, while g-DSD is related to integrated quantiles.

In this paper, we draw inspiration from these works and develop a more general class of partial orders. In our approach, we refer to the rank-dependent utility model (RDU)[8] where both the utility function and distortion function are involved. Specifically, for a common utility function f and a common distortion function g, the introduced partial order is called (f,g)-utility and distorted stochastic dominance ((f,g)-UDSD). Our main purpose is to study the basic properties of (f,g)-UDSD. An interesting finding is that (f,g)-UDSD can be characterized in terms of a class of RDU functionals but cannot be characterized by a set of expected utilities or distorted expectations for general f and g (see Section 4). Based on our partial order, we provide a general approach to interpolate first-order stochastic dominance (FSD) and SSD, which is a popular issue investigated in various studies[4, 9−12]. In the following, we summarize the main contributions of this paper.

(Ⅰ) We introduce a new generalized class of partial orders that contains some rules of stochastic dominance as its special cases, such as the SSD rule, with respect to a utility function[5] or a distortion function[7].

(Ⅱ) We investigate the properties and characterizations of the new partial order (f,g)-UDSD through the RDU model. Specifically, we study ① the characterization of monotonicity of (f,g)-UDSD in the RDU model (Theorem 3.1); ② the characterization of (f,g)-UDSD in terms of isotonic RDU functionals (Theorem 3.2); ③ the characterization of the strength comparison among (f,g)-UDSD (Theorem 3.3). Based on these findings, we unify some related results in the existing literature. Basically, all three characterizations are related to the shape of the utility function f and distortion function g, especially to its degree of concavity.

(Ⅲ) Similar to the definition of utility congruency proposed in Ref. [9], we introduce distortion congruency. The characterization of utility congruency or distortion congruency of (f,g)-UDSD is presented (Theorem 4.1). The main finding is that (f,g)-UDSD is neither utility congruent nor distortion congruent unless f or g is a linear function, which means that (f,g)-UDSD reduces to f-USD[5] or g-DSD[7].

(Ⅳ) We provide a general approach to interpolate the FSD and SSD based on the partial order (f,g)-UDSD. This approach depends on two functions $\lambda_1:(0,1]\to(0,1]$ and $\lambda_2:(0,1]\to(0,1]$ that are increasing. By choosing different λ_1 and λ_2, various continuums between the FSD and SSD can be developed. In particular, the continuums in Refs. [7, 12] can be seen as special cases under our approach.

The rest of the paper is organized as follows. Section 2 is a preliminary section where we introduce some notation and definitions. In Section 3, we consider the relation between (f,g)-UDSD and the RDU model, and some related results are presented. The utility congruency and distortion congruency of (f,g)-UDSD are investigated in Section 4. In Section 5, we develop a class of continuums between FSD and SSD based on (f,g)-UDSD.

2 Notation and definitions

For simplicity, we use a unified notation $\mathcal{X}$ to denote a set of all random variables with support on an interval $[a,b]$, which encompasses $[a,\infty)$ if $b=\infty$, $(-\infty,b]$ if $a=-\infty$ and $(-\infty,\infty)$ if both. Denote by F_X the distribution function of X. The left inverse function of an increasing function f, denoted by f^{-1}, is defined as

$$f^{-1}(y)=\inf\{x: f(x)\geqslant y\}.$$

The left continuous quantile function of X is denoted by q_X, which is the left inverse function of F_X. For any stochastic order $\leqslant_*$, we will write $X\leqslant_* Y$ or $F\leqslant_* G$ interchangeably. Moreover, a utility function is an increasing real-valued function on $[a,b]$, and $h:[0,1]\to[0,1]$ is a distortion function if it is increasing with $h(0)=0$ and $h(1)=1$. Let $\mathcal{F}$ and $\mathcal{G}$ be the sets of all continuous and strictly increasing utility functions and distortion functions, respectively. The expectation of a random variable X will be denoted by $\mathbb{E}[X]$.

In von Neumann-Morgenstern's expected utility (EU) model, the preference of a decision maker is characterized by $\mathbb{E}[u(\cdot)]$, where u is a utility function, and in Yarri's dual theory (DT)[1], the preference is characterized by a distorted expectation, which is defined by

$$D_h(X)=\int_0^1 q_X(s)\mathrm{d}h(s)=\int_{\mathbb{R}} x\mathrm{d}h(F_X(x))=$$
$$\int_{-\infty}^0\left(\widetilde{h}(1-F_X(x))-1\right)\mathrm{d}x+\int_0^\infty\widetilde{h}(1-F_X(x))\mathrm{d}x,$$

where h is a distortion function and $\widetilde{h}(s)=1-h(1-s)$ is the dual distortion function of h. We recall the definition of SSD, which can be represented equivalently in terms of distribution functions or quantile functions[13].

Definition 2.1. We say that X is dominated by Y with respect to SSD and write $X\leqslant_{\rm SSD} Y$ if and only if

$$\int_a^y F_X(x)\mathrm{d}x\geqslant\int_a^y F_Y(x)\mathrm{d}x,\ \forall y\in[a,b],$$

or equivalently,

$$\int_0^t q_X(s)\mathrm{d}s\leqslant\int_0^t q_Y(s)\mathrm{d}s,\ \forall t\in[0,1].$$

The SSD is also known as increasing concave order in the existing literature of stochastic order. Summering some results (see e.g., Refs. [14, Theorem 4.A.2] and [7, Theorem 4]), $\mathbb{E}[u(\cdot)]$ (D_h, resp.) is isotonic with SSD for every increasing concave utility function (concave distortion function, resp.), i.e., $X\leq_{\rm SSD} Y$ if and only if $\mathbb{E}[u(X)]\leqslant\mathbb{E}[u(Y)]$ ($D_h(X)\leqslant D_h(Y)$, resp.) for all increasing concave utility functions u (concave distortion functions h, resp.). The next two definitions are the SSD rules with respect to utility functions or distortion functions.

Definition 2.2.[5] Let $f\in\mathcal{F}$. We say that X is dominated by Y with respect to f-USD① and write $X\leqslant_f Y$ if and only if

$$\int_a^y F_X(x)\mathrm{d}f(x)\geqslant\int_a^y F_Y(x)\mathrm{d}f(x),\ \forall y\in[a,b],$$

or equivalently,

$$\int_0^t f(q_X(s))\mathrm{d}s\leqslant\int_0^t f(q_Y(s))\mathrm{d}s,\ \forall t\in[0,1].$$

Definition 2.3.[7] Let $g\in\mathcal{G}$. X is dominated by Y with respect to g-DSD and write $X\leqslant^g Y$ if and only if

$$\int_a^y g(F_X(x))\mathrm{d}x\geqslant\int_a^y g(F_Y(x))\mathrm{d}x,\ \forall y\in[a,b],$$

or equivalently,

$$\int_0^t q_X(s)\mathrm{d}g(s)\leqslant\int_0^t q_Y(x)\mathrm{d}g(s),\ \forall t\in[0,1].$$

In the following, we propose a generalized family of stochastic orders derived by comparing via SSD pairs of random variables whose distributions are transformed by a common utility function and a common distortion function.

Definition 2.4. (SSD rule with respect to utility and distortion function) Let $f\in\mathcal{F}$ and $g\in\mathcal{G}$. We say that X is dominated by Y with respect to (f,g)-UDSD and write $X\leqslant_f^g Y$ if and only if

$$\int_a^y g(F_X(x))\mathrm{d}f(x)\geqslant\int_a^y g(F_Y(x))\mathrm{d}f(x),\ \forall y\in[a,b],$$

or equivalently,

$$\int_0^t f(q_X(s))\mathrm{d}g(s)\leqslant\int_0^t f(q_Y(s))\mathrm{d}g(s),\ \forall t\in[0,1].$$

① In Ref. [5], f-USD is called the second degree with respect to f.

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Comparing the above four definitions with the definition of SSD, it is not difficult to observe that for two random variables X, Y,

- $X \leqslant_f Y$ is equivalent to $F_X \circ f^{-1} \leqslant_{\rm SSD} F_Y \circ f^{-1}$.
- $X \leqslant^g Y$ is equivalent to $g \circ F_X \leqslant_{\rm SSD} g \circ F_Y$.
- $X \leqslant_f^g Y$ is equivalent to $g \circ F_X \circ f^{-1} \leqslant_{\rm SSD} g \circ F_Y \circ f^{-1}$.

The mappings $F \mapsto F \circ f^{-1}$ and $F \mapsto g \circ F$ are called shape transforms and probability distortions, respectively, which are two fundamental distributional transforms characterized by Ref. [15]. By the above arguments, the order f-USD (g-DSD) is equivalent to SSD between distributions with shape transforms (probability distortions). While (f,g)-UDSD involves both shape transforms and probability distortions.

The degree of concavity of a real-valued function through the concept of one being more concave than another, which will be frequently used throughout the paper, is defined below.

Definition 2.5. Suppose that f_1 and f_2 have the same support. We say that f_1 is more concave than f_2 and write $f_2 \leqslant_{\rm cv} f_1$ if and only if $f_1 \circ f_2^{-1}$ is concave on the range of f_2.

3 Characterization through the RDU model

In the RDU model[8], each decision maker is characterized by a utility function u and a distortion function h. The RDU of a random variable X is defined by

$$V_{u,h}(X) = \int_{\mathbb{R}} u(x) \mathrm{d}h(F_X(x)).$$

For an RDU-(u,h) decision maker, we say that this decision maker is monotone in a stochastic order $\leqslant_*$ if for any $X, Y \in \mathcal{X}$,

$$X \leqslant_* Y \Longrightarrow V_{u,h}(X) \leqslant V_{u,h}(Y).$$

RDU is a popular behavioral decision model that includes EU and DT models. More specifically, if h is an identity function, then RDU-(u,h) reduces to the EU model, and if u is an identity function, it reduces to the DT model.

As shown in Ref. [16], an RDU-(u,h) decision maker is monotone in SSD if and only if both u and h are concave. In the following proposition, we show a generalization result, which is a characterization of monotonicity in (f,g)-UDSD in the RDU model.

Theorem 3.1. Let $f \in \mathcal{F}$ and $g \in \mathcal{G}$. An RDU-(u,h) decision maker is monotone in (f,g)-UDSD if and only if $f \leqslant_{\rm cv} u$ and $g \leqslant_{\rm cv} h$.

Proof. Let $X_{f,g}$ be a random variable with distribution function $g \circ F_X \circ f^{-1}$. Then we compute $V_{u,h}(X)$ as follows:

$$V_{u,h}(X) = \int_{\mathbb{R}} u(x) \mathrm{d}h(F_X(x)) = \int_{\mathbb{R}} u(x) \mathrm{d}(h \circ g^{-1} \circ F_{X_{f,g}} \circ f(x)) = \int_{\mathbb{R}} u \circ f^{-1}(x) \mathrm{d}(h \circ g^{-1} \circ F_{X_{f,g}}(x)) = V_{u \circ f^{-1}, h \circ g^{-1}}(X_{f,g}).$$

Denote by $\mathcal{Y}$ the set of all random variables on $[f(a), f(b)]$. Since f and g are strictly increasing, the mapping $\mathcal{X} \to \mathcal{Y}: X \mapsto X_{f,g}$ is a bijection. Additionally, note that $X \leqslant_f^g Y$ is equivalent to $X_{f,g} \leqslant_{\rm SSD} Y_{f,g}$. Therefore, an RDU-$(u,h)$ decision maker is monotone in $\leqslant_f^g$ on $\mathcal{X}$ iff an RDU-$(u \circ f^{-1}, h \circ g^{-1})$ decision maker is monotone in $\leqslant_{\rm SSD}$ on $\mathcal{Y}$. Finally, it follows from Ref. [16, Corollary 2] that the desired result holds.

Note that EU and DT are special cases of the RDU model, and the order (f,g)-UDSD generalizes f-USD and g-DSD. The characterization of monotonicity in f-USD, g-DSD and (f,g)-UDSD in the EU, DT or RDU models can be derived from Theorem 3.1, and the related results are shown in Table 1.

Theorem 3.2 below characterizes (f,g)-UDSD in terms of isotonic RDU functionals. A natural question is whether (f,g)-UDSD can be characterized in terms of the isotonic EU model or DT model. We will answer this question and propose some related results in Section 4.

Theorem 3.2. Let $f \in \mathcal{F}$ and $g \in \mathcal{G}$. We have $X \leqslant_f^g Y$ if and only if

$$V_{u,h}(X) \leqslant V_{u,h}(Y)$$

for all utility functions u and distortion functions h such that $f \leqslant_{\rm cv} u$ and $g \leqslant_{\rm cv} h$.

Proof. The necessity can be directly verified by Theorem 3.1. To see sufficiency, suppose that $X, Y \in \mathcal{X}$ satisfy $V_{u,h}(X) \leqslant V_{u,h}(Y)$ for all u, h such that $u \circ f^{-1}$ and $h \circ g^{-1}$ are both concave. For $y \in \mathbb{R}$, let $u_y(x) = -(f(y) - f(x))_+$ and $h = g$. However, one can check that $u_y \circ f^{-1}$ (for all $y \in \mathbb{R}$) and $h \circ g^{-1}$ are both concave. Hence, we have $V_{u_y,h}(X) \leqslant V_{u_y,h}(Y)$ for $y \in \mathbb{R}$. However, we have

$$V_{u_y,h}(X) = \int_{\mathbb{R}} u_y(x) \mathrm{d}h(F_X(x)) = -\int_{-\infty}^{y} (f(y) - f(x)) \mathrm{d}g(F_X(x)) = \int_{-\infty}^{y} g(F_X(x)) \mathrm{d}(f(y) - f(x)) = -\int_{-\infty}^{y} g(F_X(x)) \mathrm{d}f(x),$$

where the third equality holds by using integration by parts. Therefore, we conclude that

$$\int_{-\infty}^{y} g(F_X(x)) \mathrm{d}f(x) \geqslant \int_{-\infty}^{y} g(F_Y(x)) \mathrm{d}f(x), \quad y \in \mathbb{R},$$

which means $X \leqslant_f^g Y$. Hence, we complete the proof.

Theorem 3.2 illustrates that the more concave the utility

Table 1. Characterization of monotonicity in f-USD, g-DSD and (f,g)-UDSD in the EU, DT and RDU models.

SD	EU-(u)	DT-(h)	RDU-(u,h)
f-USD ($f \in \mathcal{F}$)	$f \leqslant_{\rm cv} u$	f convex, h concave	$f \leqslant_{\rm cv} u$, h concave
g-DSD ($g \in \mathcal{G}$)	u concave, g convex	$g \leqslant_{\rm cv} h$	u concave, $g \leqslant_{\rm cv} h$
(f,g)-UDSD ($f \in \mathcal{F}$, $g \in \mathcal{G}$)	$f \leqslant_{\rm cv} u$, g convex	f convex, $g \leqslant_{\rm cv} h$	$f \leqslant_{\rm cv} u$, $g \leqslant_{\rm cv} h$

[Note] The constraint that f or g is convex in the table is necessary, and otherwise, the set of utility functions or distortion functions is empty.

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function and distortion function are, the smaller the set of isotonic RDU functionals is, and hence, the weaker the partial order (f,g)-UDSD is. We are interested in the converse direction: Does a weaker partial order (f,g)-UDSD imply more concave utility function and distortion function? Theorem 3.3, which shows necessary and sufficient conditions on the strength comparison among (f,g)-UDSD, gives a positive answer. Before showing the theorem, we give the specific definition that one partial order is stronger than another.

Definition 3.1. We say $\leqslant_2$ is stronger than $\leqslant_1$ if and only if for all $X,Y\in\mathcal{X}$, $X\leqslant_2 Y$ implies $X\leqslant_1 Y$. When $\leqslant_2$ is strictly stronger than $\leqslant_1$, it means that $\leqslant_2$ is stronger than $\leqslant_1$, and there exist $X,Y\in\mathcal{X}$ such that $X\leqslant_1 Y$, and $X\nleqslant_2 Y$. We say $\leqslant_1$ and $\leqslant_2$ are equivalent if and only if $\leqslant_2$ is stronger than $\leqslant_1$, and $\leqslant_1$ is stronger than $\leqslant_2$.

Theorem 3.3. Suppose $f_1,f_2\in\mathcal{F}$ and $g_1,g_2\in\mathcal{G}$. Then $\leqslant_{f_2}^{g_2}$ is stronger than $\leqslant_{f_1}^{g_1}$ if and only if $f_2\leqslant_{\rm cv} f_1$ and $g_2\leqslant_{\rm cv} g_1$. Moreover, if $f_2\leqslant_{\rm cv} f_1$ and $g_2\leqslant_{\rm cv} g_1$ both hold, and either $f_1\circ f_2^{-1}$ or $g_1\circ g_2^{-1}$ is nonlinear, then $\leqslant_{f_2}^{g_2}$ is strictly stronger than $\leqslant_{f_1}^{g_1}$.

Proof. We first verify the "if" part. By Theorem 3.2, $V_{u,h}$ is isotonic with $\leqslant_{f_i}^{g_i}$, $i=1,2$, for every pair of utility function and distortion function in $A_i:=\{(u,h): f_i\leqslant_{\rm cv} u,\ g_i\leqslant_{\rm cv} h\}$. Since $f_2\leqslant_{\rm cv} f_1$ and $g_2\leqslant_{\rm cv} g_1$ both hold, we have $A_1\subseteq A_2$. This implies $\leqslant_{f_2}^{g_2}$ is stronger than $\leqslant_{f_1}^{g_1}$. To see the "only if" part, we assume by contradiction that $f_2\nleqslant_{\rm cv} f_1$ which means $f_1\circ f_2^{-1}$ is not concave (the proof of the case of $g_2\nleqslant_{\rm cv} g_1$ is similar). By Theorem 3.1 (see also Ref. [16, Corollary 2]), we can construct two random variables X,Y such that $X\leqslant_{\rm SSD} Y$ and $V_{f_1\circ f_2^{-1},g_1\circ g_2^{-1}}(X)>V_{f_1\circ f_2^{-1},g_1\circ g_2^{-1}}(Y)$. Let X' and Y' be such that $F_{X'}=g_2^{-1}\circ F_X\circ f_2$ and $F_{Y'}=g_2^{-1}\circ F_Y\circ f_2$. On the one hand, since $X\leqslant_{\rm SSD} Y$, we have $X'\leqslant_{f_2}^{g_2} Y'$, and hence, $X'\leqslant_{f_1}^{g_1} Y'$ which implies $V_{f_1,g_1}(X')\leqslant V_{f_1,g_1}(Y')$. On the other hand, we have $V_{f_1,g_1}(X')=V_{f_1\circ f_2^{-1},g_1\circ g_2^{-1}}(X)$ and $V_{f_1,g_1}(Y')=V_{f_1\circ f_2^{-1},g_1\circ g_2^{-1}}(Y)$, and hence, $V_{f_1,g_1}(X')>V_{f_1,g_1}(Y')$. This yields a contradiction. Hence, if $\leqslant_{f_2}^{g_2}$ is stronger than $\leqslant_{f_1}^{g_1}$, then $f_2\leqslant_{\rm cv} f_1$ and $g_2\leqslant_{\rm cv} g_1$. The "moreover" part holds because $\leqslant_{f_1}^{g_1}$ is equivalent to $\leqslant_{f_2}^{g_2}$ if and only if $f_2\leqslant_{\rm cv} f_1$, $f_1\leqslant_{\rm cv} f_2$, $g_1\leqslant_{\rm cv} g_2$ and $g_2\leqslant_{\rm cv} g_1$ hold, which implies that $f_1\circ f_2^{-1}$ and $g_1\circ g_2^{-1}$ are both linear functions.

The following corollary, which can be directly obtained by Theorem 3.3, covers the result in Ref. [7, Theorem 1].

Corollary 3.1. For $f\in\mathcal{F}$ and $g\in\mathcal{G}$, we have

(a) $\leqslant_{f_2}$ is stronger than $\leqslant_{f_1}$ if and only if $f_2\leqslant_{\rm cv} f_1$. Moreover, if $f_2\leqslant_{\rm cv} f_1$ and $f_1\circ f_2^{-1}$ is nonlinear, then $\leqslant_{f_2}$ is strictly stronger than $\leqslant_{f_1}$.

(b) $\leqslant^{g_2}$ is stronger than $\leqslant^{g_1}$ if and only if $g_2\leqslant_{\rm cv} g_1$. Moreover, if $g_2\leqslant_{\rm cv} g_1$ and $g_1\circ g_2^{-1}(s)\not\equiv s$, then $\leqslant^{g_2}$ is strictly stronger than $\leqslant^{g_1}$.

4 Utility congruency and distortion congruency

The concept of stochastic dominance arises in decision theory and decision analysis in situations where one gamble can be ranked as superior to another gamble for a broad class of decision makers. For instance, SSD requires increasing and concave utility (distortion) functions for EU (DT) decision makers. Ref. [9] proposed the concept of a congruent utility class with a partial order $\leqslant_*$. Inspired by this, the concept of a congruent distortion class can be similarly established.

Definition 4.1. (Definition of utility congruence[9]). Let U be a nonempty set of utility functions. We say U is congruent with a partial order $\leqslant_*$, if the following statement holds:

$$X\leqslant_* Y \text{ if and only if } \mathbb{E}[u(X)]\leqslant\mathbb{E}[u(Y)] \text{ for all } u\in U.$$

If the set of utility functions exists, we say $\leqslant_*$ is utility congruent. Otherwise, $\leqslant_*$ is called utility incongruent.

Definition 4.2. (Definition of distortion congruence). Let H be a nonempty set of distortion functions. We say H is congruent with a partial order $\leqslant_*$ if the following statement holds:

$$X\leqslant_* Y \text{ if and only if } D_h(X)\leqslant D_h(Y) \text{ for all } h\in H.$$

If the set of distortion functions exists, we say $\leqslant_*$ is distortion congruent. Otherwise, $\leqslant_*$ is called distortion incongruent.

Remark 4.1. Actually, utility (distortion) congruent partial order means that it can be characterized in terms of classes of isotonic utility (distortion) functions. For a utility or distortion congruent partial order, the corresponding congruent set is not unique. For example, SSD is utility congruent with the set of all increasing and concave utility functions and is distortion congruent with the set of all concave distortion functions. However, the utility and distortion congruent sets of SSD can also be chosen as $\{x\mapsto -(x-t)_-: t\in\mathbb{R}\}$ and $\{s\mapsto 1-(s-\alpha)_-/\alpha: \alpha\in(0,1)\}$ where $z_-:=\max(-z,0)$, respectively (see e.g., Ref. [14, Chapter 4]).

Various stochastic orders have been proposed in the literature, and to my knowledge, these rules are either utility congruent or distortion congruent. We give some examples including f-USD and g-DSD in Table 2. Our aim is to

Table 2. Utility congruent or distortion congruent stochastic dominance.

SD	Utility congruent set (u)	Distortion congruent set (h)
f-USD ($f\in\mathcal{F}$)	$f\leqslant_{\rm cv} u$	–
g-DSD ($g\in\mathcal{G}$)	–	$g\leqslant_{\rm cv} h$
nth-SD	$(-1)^{k+1}u^{(k)}\geqslant 0$ for $k=1,\cdots,n-1$, and $(-1)^n u^{(n-1)}$ is increasing	–
Dual nth-SD	–	$(-1)^{k+1}h^{(k)}\geqslant 0$ for $k=1,\cdots,n-1$, and $(-1)^n h^{(n-1)}$ is increasing
(1+γ)-SD ($\gamma\in[0,1]$)	$0\leqslant\gamma u'(y)\leqslant u'(x)$ for all $x\leqslant y$	$0\leqslant\gamma h'(\beta)\leqslant h'(\alpha)$ for all $0\leqslant\alpha\leqslant\beta\leqslant 1$

[Note] One can refer to Refs. [5, 7] for the congruent set of f-USD and g-DSD, respectively; nth-SD and dual nth-SD are proposed in Ref. [17] who focused on nonnegative random variables, and $u^{(k)}$ and $h^{(k)}$ are the kth order derivatives of u and h, respectively; (1+γ)-SD is introduced in Ref. [11] where u' and h' are the derivatives of u and h, respectively. One can refer to Ref. [18, Theorem 3.4] for the distortion congruent set of (1+γ)-SD.

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investigate which utility function f and distortion function g will make (f,g)-UDSD utility congruent or distortion congruent. To this end, we first propose a lemma which is a straightforward result from the definitions of utility and distortion congruence.

Lemma 4.1. Let $\leqslant_1$ and $\leqslant_2$ be two utilities (distortion, resp.) congruent partial orders. Suppose that $\leqslant_1$ is congruent with a set U_1 (H_1, resp.), then $\leqslant_2$ is stronger than $\leqslant_1$ if and only if there exists a set U_2 (H_2, resp.), which is congruent with $\leqslant_2$, satisfies $U_1 \subseteq U_2$ ($H_1 \subseteq H_2$, resp.). Moreover, if $\leqslant_2$ is strictly stronger than $\leqslant_1$, then the existing set satisfies $U_1 \subsetneq U_2$ ($H_1 \subsetneq H_2$, resp.).

Theorem 4.1. Let $f \in \mathcal{F}$ and $g \in \mathcal{G}$. The following statements hold.

(a) $\leqslant_f^g$ is utility congruent if and only if $g(s) \equiv s$. In this case, $\leqslant_f^g$ reduces to $\leqslant_f$.

(b) $\leqslant_f^g$ is distortion congruent if and only if $f(x) = kx + d$ for some $k > 0$ and $d \in \mathbb{R}$. In this case, $\leqslant_f^g$ reduces to $\leqslant^g$.

In particular, if f is nonlinear and $g(s) \not\equiv s$, then $\leqslant_f^g$ is neither utility congruent nor distortion congruent.

Proof. (a) The "if" part follows from Ref. [5, Theorem 2], and the congruent utility class can be chosen as $\{u : f \leqslant_{\mathrm{cv}} u\}$. To see the "only if" part, we assume by contradiction that g is not an identity function and consider the following two cases.

(i) Suppose that g is not convex, and hence, g^{-1} is not concave. By Theorem 3.1, we have for any utility function u, there exist X, Y such that $X \leqslant_f^g Y$ and $\mathbb{E}[u(X)] > \mathbb{E}[u(Y)]$. Therefore, $\leqslant_f^g$ is utility incongruent which yields a contradiction.

(ii) Suppose that g is convex and $g(s) \not\equiv s$. It follows from Theorem 3.3 that $\leqslant_f^g$ is strictly stronger than $\leqslant_f$. Note that $\leqslant_f$ is utility congruent with $U_f := \{u : f \leqslant_{\mathrm{cv}} u\}$. Additionally, note that $\leqslant_f^g$ is utility congruent. By Lemma 4.1, $\leqslant_f^g$ can be characterized by a set of utility functions U such that $U_f \subsetneq U$, that is,

$$X \leqslant_f^g Y \text{ if and only if } \mathbb{E}[u(X)] \leqslant \mathbb{E}[u(Y)] \text{ for all } u \in U. \quad (1)$$

Since $U_f \subsetneq U$, we can find $u_0 \in U$ such that $u_0 \circ f^{-1}$ is nonconcave. Applying Theorem 3.1, there exist X, Y such that $X \leqslant_f^g Y$ and $\mathbb{E}[u_0(X)] > \mathbb{E}[u_0(Y)]$, which contradicts (1).

Combining cases (i) and (ii), we complete the proof of (a).

(b) The "if" part follows from Ref. [7, Theorem 5], and the congruent distortion set is $\{h : g \leqslant_{\mathrm{cv}} h\}$. The proof of the "only if" part is similar to (a). Assume by contradiction that f is nonlinear and consider the following two cases.

① Suppose that f is not convex, and hence, f^{-1} is not concave. By Theorem 3.1, we have for any distortion function h, there exist X, Y such that $X \leqslant_f^g Y$ and $D_h(X) > D_h(Y)$. Therefore, $\leqslant_f^g$ is distortion incongruent which yields a contradiction.

② Suppose that f is convex and nonlinear. By Theorem 3.3, we have $\leqslant_f^g$ is strictly stronger than $\leqslant^g$. Since $\leqslant^g$ is distortion congruent with $H_g := \{u : g \leqslant_{\mathrm{cv}} h\}$, and noting that $\leqslant_f^g$ is utility congruent, it follows from Lemma 4.1 that $\leqslant_f^g$ can be characterized by a set of distortion functions H such that $H_g \subsetneq H$, that is,

$$X \leqslant_f^g Y \text{ if and only if } D_h(X) \leqslant D_h(Y) \text{ for all } h \in H. \quad (2)$$

Since $H_g \subsetneq H$, we can find $h_0 \in H$ such that $h_0 \circ g^{-1}$ is nonconcave. Applying Theorem 3.1, there exist X, Y such that $X \leqslant_f^g Y$ and $D_{h_0}(X) > D_{h_0}(Y)$ which contradicts (2).

Hence, we complete the whole proof.

Theorem 4.1 demonstrates that the class of utility and distortion congruent (f,g)-UDSD is the SSD rule in Ref. [5] and [7], respectively. Moreover, we can generate a new class of stochastic dominance, that is, neither utility congruent nor distortion congruent, by changing f and g in (f,g)-UDSD. Based on Theorem 4.1, we propose some corollaries below.

Corollary 4.1. Let $f \in \mathcal{F}$ and $g \in \mathcal{G}$. The partial order $\leqslant_f$ is equivalent to $\leqslant^g$ if and only if $f(x) = kx + d$ for some $k > 0$ and $d \in \mathbb{R}$, and $g(s) \equiv s$. In other words, $\leqslant_f$ and $\leqslant^g$ are both SSD.

Proof. Since $\leqslant_f$ is utility congruent and $\leqslant^g$ is distortion congruent, the equivalence between $\leqslant_f$ and $\leqslant^g$ implies that $\leqslant_f$ and $\leqslant^g$ are both utility and distortion congruent. By Theorem 4.1, we verify the desired result.

Corollary 4.2. Let $f \in \mathcal{F}$ and $g \in \mathcal{G}$. The partial order $\leqslant_f^g$ is both utility and distortion congruent if and only if $f(x) = kx + d$ for some $k > 0$ and $d \in \mathbb{R}$, and $g(s) \equiv s$. In other words, $\leqslant_f^g$ is SSD.

Corollary 4.1 illustrates that the intersection of the classes of stochastic dominance introduced in Refs. [5, 7] is SSD. The SD rule between FSD and SSD proposed by Ref. [11] is both utility and distortion congruent (as shown in Table 2). Applying Corollary 4.2, we obtain that the SD rule in Ref. [11] cannot be represented in the form $\leqslant_f^g$. This result is also proposed in Ref. [19], where the technical tool is quite different from ours.

5 Fractional degree stochastic dominance with (*f*, *g*)-UDSD

In this section, we search for a family of utility functions and distortion functions that makes (f,g)-UDSD a class of partial orders that covers the preferences of decision makers from first-order stochastic dominance (FSD) to SSD. Ref. [12] proposed such a family of partial orders by choosing a class of exponential utility functions with f-USD, while Ref. [7] chose power distortion functions with g-DSD. In our construction, these two classes of functions will both be involved.

We first recall the definition of FSD as follows.

Definition 5.1. We say that X is dominated by Y with respect to FSD and write $X \leqslant_{\mathrm{FSD}} Y$ if and only if

$$F_X(x) \geqslant F_Y(x),\ \forall y \in [a,b],$$

or equivalently,

$$q_X(s) \leqslant q_Y(s),\ \forall t \in [0,1].$$

It is well known that FSD is both utility congruent and distortion congruent with the set of all utility functions and distortion functions, respectively (see e.g., Refs. [20, 21]). The class of exponential utility functions is defined as

$$f_c(x) = \begin{cases} \mathrm{e}^{(1/c-1)x}, & \text{if } 0 < c < 1; \\ x, & \text{if } c = 1. \end{cases}$$

The power distortion function is defined as $g_c(s) = s^{1/c}$ for $c \in (0,1]$. Moreover, we define

$$\Lambda = \{\lambda : (0,1] \to (0,1] :\ \lambda \text{ increasing, and } \lambda(1) = 1\}.$$

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Definition 5.2. Let $c \in (0,1]$ and $\lambda_1, \lambda_2 \in \Lambda$ such that either $\lim_{x\to 0}\lambda_1(x) = 0$ or $\lim_{x\to 0}\lambda_2(x) = 0$ holds. We say that Y dominates X with respect to $(1+c)_{\lambda_1,\lambda_2}$-SD, denoted by $X \leqslant_{(1+c)_{\lambda_1,\lambda_2}\text{-SD}} Y$, if $X \leqslant^{g_{\lambda_2(c)}}_{f_{\lambda_1(c)}} Y$ where $f_{\lambda_1(c)}$ and $g_{\lambda_2(c)}$ are exponential utility function and power distortion function, respectively. In particular, we write $X \leqslant_{1_{\lambda_1,\lambda_2}\text{-SD}} Y$, if $X \leqslant_{(1+c)_{\lambda_1,\lambda_2}\text{-SD}} Y$ for all $c \in (0,1]$.

The partial order $(1+c)_{\lambda_1,\lambda_2}$-SD depends on two parameters of functions λ_1 and λ_2. More precisely, it is (f,g)-UDSD with utility function $f = f_{\lambda_1(c)}$ and distortion function $g = g_{\lambda_2(c)}$. In particular, if λ_1 is an identity function and $\lambda_2 \equiv 1$, then $(1+c)_{\lambda_1,\lambda_2}$-SD reduces to the order in Ref. [12], and if $\lambda_1 \equiv 1$ and λ_2 is an identity function, then it reduces to the one in Ref. [7]. The next theorem illustrates that $(1+c)_{\lambda_1,\lambda_2}$-SD for $c \in [0,1]$ is a continuum from FSD to SSD for any λ_1 and λ_2 that satisfies the conditions in Definition 5.2.

Theorem 5.1. Let $\lambda_1, \lambda_2 \in \Lambda$ such that either $\lim_{x\to 0}\lambda_1(x) = 0$ or $\lim_{x\to 0}\lambda_2(x) = 0$ holds. Then we have

(a) $X \leqslant_{1_{\lambda_1,\lambda_2}\text{-SD}} Y$ if and only if $X \leqslant_{\text{FSD}} Y$.

(b) $X \leqslant_{2_{\lambda_1,\lambda_2}\text{-SD}} Y$ if and only if $X \leqslant_{\text{SSD}} Y$.

(c) If $c_1 < c_2$, then $(1+c_1)_{\lambda_1,\lambda_2}$-SD is stronger than $(1+c_2)_{\lambda_1,\lambda_2}$-SD. Moreover, if λ_1 or λ_2 is strictly increasing, then $(1+c_1)_{\lambda_1,\lambda_2}$-SD is strictly stronger than $(1+c_2)_{\lambda_1,\lambda_2}$-SD.

Proof. (a) The sufficiency is trivial. To see necessity, we assume $\lim_{x\to 0}\lambda_1(x) = 0$ as the proof of the other case is similar. We first define a class of utility functions as

$$U_c = \{u : f_{\lambda_1(c)} \leqslant_{\text{cv}} u\},\ \ c \in (0,1].$$

Note that $g_{\lambda_2(c)}$ is convex for all $c \in (0,1]$. It follows from Theorem 3.1 (see also Table 1) that $X \leqslant_{1_{\lambda_1,\lambda_2}\text{-SD}} Y$ implies $\mathbb{E}[u(X)] \leqslant \mathbb{E}[u(Y)]$ for all $u \in \bigcup_{c\in(0,1]} U_c$. Next, define a class of strictly increasing and twice differentiable utility functions as

$$\widetilde{U}_c = \left\{u : -\frac{u''}{u'} \geqslant -\frac{f''_{\lambda_1(c)}}{f'_{\lambda_1(c)}} = 1 - \frac{1}{\lambda_1(c)}\right\},\ \ c \in (0,1].$$

By Ref. [6, Theorem 1], we have $\widetilde{U}_c \in U_c$ for all $c \in (0,1]$. Hence, we obtain $X \leqslant_{1_{\lambda_1,\lambda_2}\text{-SD}} Y$ implies $\mathbb{E}[u(X)] \leqslant \mathbb{E}[u(Y)]$ for all $u \in \bigcup_{c\in(0,1]} \widetilde{U}_c$ that is the set of all strictly increasing and twice differentiable utility functions. This means that $X \leqslant_{1_{\lambda_1,\lambda_2}\text{-SD}} Y$ implies $X \leqslant_{\text{FSD}} Y$.

The proof of (b) is trivial because $\lambda_1(1) = \lambda_2(1) = 1$, and f_1 and g_1 are both identity functions.

(c) One can easily check that $f_a \leqslant_{\text{cv}} f_b$ and $g_a \leqslant_{\text{cv}} g_b$, and $f_a \circ f_b^{-1}$ and $g_a \circ g_b^{-1}$ are nonlinear for all $0 < a < b \leqslant 1$. Hence, the desired result holds by applying Theorem 3.3.

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Conflict of interest

The author declares that he has no conflict of interest.

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