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On near-imperfect numbers with two distinct prime divisors

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Abstract: Let ρ be a multiplicative arithmetic function defined by $\rho(p^{\alpha})=p^{\alpha}-p^{\alpha-1}+p^{\alpha-2}-\cdots+(-1)^{\alpha}$ for every prime power p^{α}. For a positive integer n, n is called a near-imperfect number if $2\rho(n)=n+d$ where d is a proper divisor of n. Here all near-imperfect numbers with two distinct prime divisors were obtained.

Key words: near-imperfect number; prime divisor; multiplicative arithmetic function

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含有两个不同素因子的盈不完全数

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摘要:对于素数方幂 p^{α},设可乘函数 $\rho(p^{\alpha})=p^{\alpha}-p^{\alpha-1}+p^{\alpha-2}-\cdots+(-1)^{\alpha}$. 称满足条件 $2\rho(n)=n+d$ 的正整数 n 为盈不完全数,其中 d 是 n 的真因子. 给出了含有两个不同素因子的所有盈不完全数.

关键词:盈不完全数;素因子;可乘函数

0 Introduction

Let $\sigma(n)$ be the sum of the positive divisors of a positive integer n. Then n is said to be perfect if and only if $\sigma(n)=2n$. In 2012, Pollack and Shevelev[1] introduced the concept of near-perfect number. A positive integer n is called near-perfect if it is the sum of all of its proper divisors except one of them. The missing divisor is called redundant. In 2013, Ren and Chen[2] determined all near-perfect numbers with two distinct prime factors. Tang et al.[3] proved that there are no odd near-perfect numbers with three distinct prime factors. In 2016, Tang et al.[4] showed that the only odd near-perfect numbers with four distinct prime factors are $3^4\cdot7^2\cdot11^2\cdot19^2$. Recently, Li and Liao[5] considered a special class of near-perfect numbers and obtained some results.

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As a variation of the sum-of-divisors function σ, Iannucci[6] defined a multiplicative arithmetic function ρ by $\rho(1)=1$ and

$$\rho(p^\alpha)=p^\alpha-p^{\alpha-1}+p^{\alpha-2}-\cdots+(-1)^\alpha$$

for every prime power $p^\alpha(\alpha\geqslant 1)$. He said that n is imperfect if $2\rho(n)=n$ and that n is k-imperfect if $k\rho(n)=n$ for some integer $k\geqslant 2$. In fact, Martin[7] introduced the function ρ at the 1999 Western Number Theory Conference and raised three questions (see Ref. [8:72]). In 2013, Tóth[9] pointed out the function ρ has a double character. For related research of the function ρ, one can refer to Refs. [10-11].

Let n be a positive integer and d a proper divisor of n. In analogy with the near-perfect numbers, n is said to be near-imperfect and d is said to be redundant if

$$2\rho(n)=n+d \tag{1}$$

In this paper, we consider near-imperfect numbers with two distinct prime divisors and obtain the following result:

Theorem 0.1 If n is a near-imperfect number with two distinct prime divisors, then

$$n\in\{2^2\cdot 3^2,\ 2^2\cdot 3^3,\ 2^5\cdot 3^2,\ 2^7\cdot 3^4,\ 2^8\cdot 3^5,\ 2^2\cdot 5,\ 2^4\cdot 5,\ 2^5\cdot 5,\ 2^7\cdot 5,\ 2^3\cdot 5^2,\ 2^3\cdot 5^3,\ 2^3\cdot 7,\ 2^5\cdot 7,\ 2^4\cdot 11,\ 2^7\cdot 17,\ 3\cdot 5,\ 3\cdot 7,\ 3^2\cdot 7\}.$$

Throughout this paper, we use the following notation: p_1, p_2 always denote primes with $p_1<p_2$; α_1, α_2 always denote positive integers; γ_1, γ_2 denote nonnegative integers; $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.

1 Lemmas

Lemma 1.1 If $n=2^{\alpha_1}p_2^{\alpha_2}$ is a near-imperfect number with $2\nmid\alpha_1$, $2\nmid\alpha_2$, then

$$n\in\{2^5\cdot 5,\ 2^7\cdot 5,\ 2^3\cdot 5^3,\ 2^3\cdot 7,\ 2^5\cdot 7,\ 2^7\cdot 17\}.$$

Proof Let $n=2^{\alpha_1}p_2^{\alpha_2}$ be a near-imperfect number with redundant divisor $d=2^{\gamma_1}p_2^{\gamma_2}$, where $\gamma_1\leqslant\alpha_1$, $\gamma_2\leqslant\alpha_2$ and $\gamma_1+\gamma_2<\alpha_1+\alpha_2$. By (1), we have

$$(2^{\alpha_1+1}-1)(p_2^{\alpha_2+1}-1)=3(p_2+1)(2^{\alpha_1-1}p_2^{\alpha_2}+2^{\gamma_1-1}p_2^{\gamma_2}) \tag{2}$$

Then $\alpha_1\geqslant 3$. Let

$$f(\alpha_1,\alpha_2)=\left(1-\frac{1}{2^{\alpha_1+1}}\right)\left(1-\frac{1}{p_2^{\alpha_2+1}}\right),$$

$$g(\alpha_1,\alpha_2)=\frac{3(p_2+1)}{4p_2}+\frac{3(p_2+1)}{D},$$

where $D=2^{\alpha_1-\gamma_1+2}p_2^{\alpha_2-\gamma_2+1}$. Then $g(\alpha_1,\alpha_2)=f(\alpha_1,\alpha_2)<1$. Thus $p_2>3$ and

$$3+\frac{12}{p_2-3}<2^{\alpha_1-\gamma_1}p_2^{\alpha_2-\gamma_2} \tag{3}$$

We now discuss four cases according to the value of p_2.

Case 1 $p_2=5$. By (3), we have $D\geqslant 2^3\cdot 5^2$. Thus $f(\alpha_1,\alpha_2)=g(\alpha_1,\alpha_2)\leqslant 0.99$. By (2), we have

$$2^{\alpha_1}5^{\alpha_2}-5^{\alpha_2+1}-2^{\alpha_1+1}+1=9\cdot 2^{\gamma_1}5^{\gamma_2} \tag{4}$$

If $\gamma_2\geqslant 2$, then $2^{\alpha_1+1}\equiv 1\pmod{25}$. Thus $20\mid(\alpha_1+1)$. It means that $\alpha_1\geqslant 19$. However

$$0.99\geqslant f(\alpha_1,\alpha_2)\geqslant\left(1-\frac{1}{2^{20}}\right)\left(1-\frac{1}{5^4}\right)=0.998\cdots,$$

a contradiction. Thus $\gamma_2\in\{0,1\}$. If $\alpha_2\geqslant 3$, then $\alpha_1\in\{3,5\}$. By (4), we have $\alpha_1=\gamma_1=\alpha_2=3$. Thus $n=2^3\cdot 5^3$ and $d=2^3\cdot 5$. Now let $\alpha_2=1$. By (4), we have $\gamma_1=3$, $\gamma_2=1$, $\alpha_1=7$ or $\gamma_1=3$, $\gamma_2=0$, $\alpha_1=5$. Thus $n=2^7\cdot 5$, $d=2^3\cdot 5$ or $n=2^5\cdot 5$, $d=2^3$.

Case 2 $p_2=7$. By (3), we have $D\geqslant 2^2\cdot 7^2$ and

$$\frac{3\cdot 8}{4\cdot 7}+\frac{3\cdot 8}{D}=g(\alpha_1,\alpha_2)=f(\alpha_1,\alpha_2)\geqslant\left(1-\frac{1}{2^4}\right)\left(1-\frac{1}{7^2}\right)=\frac{45}{49}.$$

Thus $D\in\{2^5\cdot 7,\ 2^3\cdot 7^2,\ 2^2\cdot 7^2\}$.

If $D=2^5\cdot 7$, then $\alpha_1=\gamma_1+3$ and $\alpha_2=\gamma_2$. By (2), we have

$$2^{\gamma_1+2}\cdot 7^{\gamma_2}-2^{\gamma_1+4}-7^{\gamma_2+1}+1=0.$$

If $\gamma_2=1$, then $\gamma_1=2$. Thus $n=2^5\cdot 7$ and $d=2^2\cdot 7$.

If $\gamma_2>1$, then $\alpha_2\geqslant 3$ and $\alpha_1\geqslant 5$. However

$$0.96\cdots=\frac{3\cdot 8}{4\cdot 7}+\frac{3\cdot 8}{2^5\cdot 7}=g(\alpha_1,\alpha_2)=f(\alpha_1,\alpha_2)\geqslant\left(1-\frac{1}{2^6}\right)\left(1-\frac{1}{7^4}\right)=0.98\cdots,$$

a contradiction.

If $D=2^3\cdot 7^2$, then $\alpha_1=\gamma_1+1$ and $\alpha_2=\gamma_2+1$. By (2), we have

$$2^{\gamma_1+4}\cdot 7^{\gamma_2}-2^{\gamma_1+2}-7^{\gamma_2+2}+1=0.$$

If $\gamma_2=0$, then $\gamma_1=2$. Thus $n=2^3\cdot 7$ and $d=2^2$. If $\gamma_2>0$, then $\alpha_2\geqslant 3$. However

$$0.91\cdots=\frac{3\cdot 8}{4\cdot 7}+\frac{3\cdot 8}{2^3\cdot 7^2}=g(\alpha_1,\alpha_2)=$$
$$f(\alpha_1,\alpha_2)\geqslant(1-\frac{1}{2^4})(1-\frac{1}{7^4})=0.93\cdots,$$

a contradiction.

If $D=2^2\cdot 7^2$, then $\alpha_1=\gamma_1$ and $\alpha_2=\gamma_2+1$. By (2), we have

$$2^{\gamma_1+1}\cdot 7^{\gamma_2}-2^{\gamma_1+1}-7^{\gamma_2+2}+1=0.$$

Then $\gamma_2>1$, $\alpha_2\geqslant 3$ and $\alpha_1\geqslant 5$. However

$$0.97\cdots=\frac{3\cdot 8}{4\cdot 7}+\frac{3\cdot 8}{2^2\cdot 7^2}=g(\alpha_1,\alpha_2)=$$
$$f(\alpha_1,\alpha_2)\geqslant(1-\frac{1}{2^6})(1-\frac{1}{7^4})=0.98\cdots,$$

a contradiction.

Case 3 $p_2\in\{11, 13\}$. By (3), we have $D\geqslant 2^5 p_2$. However

$$f(\alpha_1,\alpha_2)\geqslant(1-\frac{1}{2^4})(1-\frac{1}{p_2^2})=\frac{15(p_2-1)(p_2+1)}{16p_2^2},$$
$$g(\alpha_1,\alpha_2)\leqslant\frac{3(p_2+1)}{4p_2}+\frac{3(p_2+1)}{2^5\ p_2}=\frac{27(p_2+1)}{32p_2},$$

a contradiction.

Case 4 $p_2\geqslant 17$. By (3), we have $D\geqslant 2^4 p_2$. By $f(\alpha_1,\alpha_2)=g(\alpha_1,\alpha_2)$, we have $D=2^4 p_2$ and $\alpha_1\leqslant 7$. Then $\alpha_1=\gamma_1+2$ and $\alpha_2=\gamma_2$. By (2), we have

$$(p_2-15)2^{\gamma_1-1}p_2^{\gamma_2}-2^{\gamma_1+3}-p_2^{\gamma_2+1}+1=0.$$

Then $\alpha_1=7$, $p_2=17$ and $\alpha_2=1$. Further $n=2^7\cdot 17$ and $d=2^5\cdot 17$.

This completes the proof of Lemma 1.1.

Lemma 1.2 If $n=2^{\alpha_1}p_2^{\alpha_2}$ is a near-imperfect number with $2\nmid\alpha_1$, $2|\alpha_2$, then

$$n\in\{2^5\cdot 3^2,\ 2^7\cdot 3^4,\ 2^3\cdot 5^2\}.$$

Proof Let $n=2^{\alpha_1}p_2^{\alpha_2}$ be a near-imperfect number with redundant divisor $d=2^{\gamma_1}p_2^{\gamma_2}$, where $\gamma_1\leqslant\alpha_1$, $\gamma_2\leqslant\alpha_2$ and $\gamma_1+\gamma_2<\alpha_1+\alpha_2$. By (1), we have

$$(2^{\alpha_1+1}-1)(p_2^{\alpha_2+1}+1)=$$
$$3(p_2+1)(2^{\alpha_1-1}p_2^{\alpha_2}+2^{\gamma_1-1}p_2^{\gamma_2})\qquad(5)$$

It is easy to prove that $\alpha_1\geqslant 3$ and $\gamma_1\geqslant 1$.

If $\alpha_1=3$, then

$$p_2^{\alpha_2+1}+5=4p_2^{\alpha_2}+2^{\gamma_1-1}p_2^{\gamma_2+1}+2^{\gamma_1-1}p_2^{\gamma_2}.$$

Thus $\gamma_2\in\{0, 1\}$ and $p_2>3$. Noting that $1\leqslant\gamma_1\leqslant 3$ and $5\equiv 2^{\gamma_1-1}p_2^{\gamma_2}\pmod 5$, we can get $\gamma_2=1$, $p_2=5$, $\gamma_1=1$ and $\alpha_2=2$. Thus $n=2^3\cdot 5^2$ and $d=2\cdot 5$.

Now let $\alpha_1\geqslant 5$. If $p_2=3$, then

$$2^{\gamma_1+1}3^{\gamma_2+1}+3^{\alpha_2+1}-2^{\alpha_1+1}+1=0.$$

Since $3^{\alpha_2+1}\equiv 3\pmod 8$, we have $\gamma_1=1$. Thus

$$2^{\alpha_1+1}-1=3^{\gamma_2+1}(4+3^{\alpha_2-\gamma_2}).$$

If $\gamma_2\geqslant 2$, then $2^{\alpha_1+1}\equiv 1\pmod{27}$. Thus $18|(\alpha_1+1)$ and $(2^{18}-1)|(2^{\alpha_1+1}-1)$. Noting that $(7\cdot 19)|(2^{18}-1)$, we have $(7\cdot 19)|(4+3^{\alpha_2-\gamma_2})$. It follows that $\alpha_2-\gamma_2\equiv 1\pmod 6$ and $\alpha_2-\gamma_2\equiv 5\pmod{18}$, which is clearly false. Thus $\gamma_2\in\{0, 1\}$.

If $\gamma_2=0$, then

$$2^8(2^{\alpha_1-7}-1)=3^5(3^{\alpha_2-4}-1).$$

If $\alpha_1>7$, then $2^{\alpha_1-7}\equiv 1\pmod{3^5}$. Thus $162|(\alpha_1-7)$ and $(2^{162}-1)|(2^{\alpha_1-7}-1)$. Noting that $262657|(2^{162}-1)$, we obtain $262657|(3^{\alpha_2-4}-1)$. Thus $14592|(\alpha_2-4)$ and $(3^{14592}-1)|(3^{\alpha_2-4}-1)$. However, $3^{14592}\equiv 1\pmod{2^{10}}$, a contradiction. Thus $\alpha_1=7$ and $\alpha_2=4$. Further $n=2^7\cdot 3^4$ and $d=2$.

If $\gamma_2=1$, then

$$2^6(2^{\alpha_1-5}-1)=3^3(3^{\alpha_2-2}-1).$$

If $\alpha_1>5$, then $2^{\alpha_1-5}\equiv 1\pmod{3^3}$. Thus $18|(\alpha_1-5)$ and $(2^{18}-1)|(2^{\alpha_1-5}-1)$. Since $19|(2^{18}-1)$, we have $19|(3^{\alpha_2-2}-1)$. Thus $18|(\alpha_2-2)$ and $(3^{18}-1)|(3^{\alpha_2-2}-1)$. Noting that $757|(3^{18}-1)$, we obtain $757|(2^{\alpha_1-5}-1)$. Then $756|(\alpha_1-5)$ and $(2^{756}-1)|(2^{\alpha_1-5}-1)$. However, $2^{756}\equiv 1\pmod{3^4}$, a contradiction. Thus $\alpha_1=5$ and $\alpha_2=2$. Further $n=2^5\cdot 3^2$ and $d=2\cdot 3$.

If $p_2=5$, then

$$2^{\alpha_1}5^{\alpha_2}-5^{\alpha_2+1}+2^{\alpha_1+1}-1=9\cdot 2^{\gamma_1}5^{\gamma_2}.$$

Since $5^{\alpha_2+1}\equiv 1\pmod 4$, we have $\gamma_1=1$. However

$$27\cdot 5^{\alpha_2}<(2^{\alpha_1}-5)(5^{\alpha_2}+2)<18\cdot 5^{\gamma_2}\leqslant 18\cdot 5^{\alpha_2},$$

a contradiction.

If $p_2=7$, then

$$2^{\alpha_1+1}7^{\alpha_2}-7^{\alpha_2+1}+2^{\alpha_1+1}-1=3\cdot 2^{\gamma_1+2}7^{\gamma_2}.$$

Since $7^{\alpha_2+1}\equiv 7 \pmod{16}$, we have $\gamma_1=1$. However

$$57\cdot 7^{\alpha_2}<(2^{\alpha_1+1}-7)(7^{\alpha_2}+1)<24\cdot 7^{\gamma_2}\leqslant 24\cdot 7^{\alpha_2},$$

a contradiction.

Now we assume that $p_2\geqslant 11$. Let

$$f(\alpha_1,\alpha_2)=(1-\frac{1}{2^{\alpha_1+1}})(1+\frac{1}{p_2^{\alpha_2+1}}),$$

$$g(\alpha_1,\alpha_2)=\frac{3(p_2+1)}{4p_2}+\frac{3(p_2+1)}{D},$$

where $D=2^{\alpha_1-\gamma_1+2}p_2^{\alpha_2-\gamma_2+1}$. By

$$\frac{3(p_2+1)}{4p_2}+\frac{3(p_2+1)}{D}=g(\alpha_1,\alpha_2)=f(\alpha_1,\alpha_2)>1-\frac{1}{2^6}=\frac{63}{64},$$

we have

$$2\leqslant 2^{\alpha_1-\gamma_1}p_2^{\alpha_2-\gamma_2}<4+\frac{80}{5p_2-16}<7.$$

Thus $\alpha_1-\gamma_1\in\{1,2\}$ and $\alpha_2=\gamma_2$. By (5), we have $\alpha_1=\gamma_1+2$ and

$$2^{\alpha_1+1}-1=((15-p_2)2^{\alpha_1-3}+p_2)p_2^{\alpha_2}.$$

If $p_2=11$, then $2^{\alpha_1+1}-1=(2^{\alpha_1-1}+11)11^{\alpha_2}$, a contradiction.

If $p_2=13$, then $2^{\alpha_1+1}-1=(2^{\alpha_1-2}+13)13^{\alpha_2}$, a contradiction.

If $p_2\geqslant 17$, then $(15-p_2)2^{\alpha_1-3}+p_2<0$, a contradiction.

This completes the proof of Lemma 1.2.

Lemma 1.3 If $n=2^{\alpha_1}p_2^{\alpha_2}$ is a near-imperfect number with $2|\alpha_1$, $2|\alpha_2$, then $n=2^2\cdot 3^2$.

Proof Let $n=2^{\alpha_1}p_2^{\alpha_2}$ be a near-imperfect number with redundant divisor $d=2^{\gamma_1}p_2^{\gamma_2}$, where $\gamma_1\leqslant\alpha_1$, $\gamma_2\leqslant\alpha_2$ and $\gamma_1+\gamma_2<\alpha_1+\alpha_2$. By (1), we have

$$(2^{\alpha_1+1}+1)(p_2^{\alpha_2+1}+1)=3(p_2+1)(2^{\alpha_1-1}p_2^{\alpha_2}+2^{\gamma_1-1}p_2^{\gamma_2}).$$

If $p_2=3$, then

$$2^{\alpha_1+1}+3^{\alpha_2+1}+1=2^{\gamma_1+1}3^{\gamma_2+1}.$$

Since $3^{\alpha_2+1}\equiv 3 \pmod 8$, we have $\gamma_1=1$ and $2^{\alpha_1+1}+1=3^{\gamma_2+1}(4-3^{\alpha_2-\gamma_2})$. Thus $\alpha_1=2$, $\alpha_2=2$ and $\gamma_2=1$. Hence $n=2^2\cdot 3^2$ and $d=2\cdot 3$.

Now suppose that $p_2\geqslant 5$. Let

$$f(\alpha_1,\alpha_2)=(1+\frac{1}{2^{\alpha_1+1}})(1+\frac{1}{p_2^{\alpha_2+1}}),$$

$$g(\alpha_1,\alpha_2)=\frac{3(p_2+1)}{4p_2}+\frac{3(p_2+1)}{D},$$

where $D=2^{\alpha_1-\gamma_1+2}p_2^{\alpha_2-\gamma_2+1}$. Then

$$1<f(\alpha_1,\alpha_2)=(1+\frac{1}{2^{\alpha_1+1}})(1+\frac{1}{p_2^{\alpha_2+1}})\leqslant(1+\frac{1}{2^3})(1+\frac{1}{5^3})=1.134.$$

It implies that

$$\frac{125p_2+125}{64p_2-125}\leqslant 2^{\alpha_1-\gamma_1}p_2^{\alpha_2-\gamma_2}<3+\frac{12}{p_2-3}.$$

Thus $5\leqslant p_2\leqslant 13$ or $p_2\geqslant 127$.

If $p_2=5$, then

$$2^{\alpha_1}5^{\alpha_2}+2^{\alpha_1+1}+5^{\alpha_2+1}+1=9\cdot 2^{\gamma_1}5^{\gamma_2}.$$

Since $5\nmid(2^{\alpha_1+1}+1)$, we have $\gamma_2=0$. However, $2^{\alpha_1}5^{\alpha_2}>9\cdot 2^{\gamma_1}$, a contradiction.

If $7\leqslant p_2\leqslant 13$, then $\alpha_1=\gamma_1+2$ and $\alpha_2=\gamma_2$. It follows that

$$2^{\alpha_1+1}+p_2^{\alpha_2+1}+1=(15-p_2)2^{\alpha_1-3}p_2^{\alpha_2}.$$

By $p_2\mid(2^{\alpha_1+1}+1)$, we have $p_2=11$. Since $11^{\alpha_2+1}\equiv 3 \pmod 8$, we have $\alpha_1=3$, which contradicts $2|\alpha_1$.

If $p_2\geqslant 127$, then $\alpha_1=\gamma_1+1$ and $\alpha_2=\gamma_2$. It follows that

$$p_2^{\alpha_2+1}+2^{\alpha_1+1}+1=2^{\alpha_1-2}p_2^{\alpha_2+1}+9\cdot 2^{\alpha_1-2}p_2^{\alpha_2},$$

which is clearly false.

This completes the proof of Lemma 1.3.

Lemma 1.4 If $n=2^{\alpha_1}p_2^{\alpha_2}$ is a near-imperfect number with $2|\alpha_1$, $2\nmid\alpha_2$, then

$$n\in\{2^2\cdot 3^3, 2^8\cdot 3^5, 2^2\cdot 5, 2^4\cdot 5, 2^4\cdot 11\}.$$

Proof Let $n=2^{\alpha_1}p_2^{\alpha_2}$ be a near-imperfect number with redundant divisor $d=2^{\gamma_1}p_2^{\gamma_2}$, where $\gamma_1\leqslant\alpha_1$, $\gamma_2\leqslant\alpha_2$ and $\gamma_1+\gamma_2<\alpha_1+\alpha_2$. By (1), we have

$$(2^{\alpha_1+1}+1)(p_2^{\alpha_2+1}-1)=3(p_2+1)(2^{\alpha_1-1}p_2^{\alpha_2}+2^{\gamma_1-1}p_2^{\gamma_2}) \qquad (6)$$

Now we discuss two cases according to the value of p_2.

Case 1 $p_2=3$. Then

$$3^{\alpha_2+1}=2^{\alpha_1+1}+1+2^{\gamma_1+1}3^{\gamma_2+1}.$$

Since $8\mid(3^{\alpha_2+1}-1)$, we have $\gamma_1\geqslant 2$.

First we assume that $\gamma_2=1$. If $\gamma_1=3$, then $3^{\alpha_2+1}=2^{\alpha_1+1}+5\cdot 29$. However

$$1=\left(\frac{3}{5}\right)^{\alpha_2+1}=\left(\frac{2}{5}\right)^{\alpha_1+1}=(-1)^{\alpha_1+1}=-1,$$

a contradiction. If $\gamma_1\geqslant 4$, then $3^{\alpha_2+1}\equiv 1 \pmod{32}$. Thus $8\mid(\alpha_2+1)$ and $(3^8-1)\mid(3^{\alpha_2+1}-1)$. Noting that $(5\cdot 41)\mid(3^8-1)$, we obtain $(5\cdot 41)\mid(2^{\alpha_1-\gamma_1}+9)$. Then $\alpha_1\equiv\gamma_1 \pmod 4$ and $\alpha_1\equiv\gamma_1+5 \pmod{20}$, a contradiction. Thus $\gamma_1=2$ and

$$2^3(2^{\alpha_1-2}-1)=3^4(3^{\alpha_2-3}-1).$$

If $\alpha_1>2$, then $2^{\alpha_1-2}\equiv 1 \pmod{3^4}$. Thus $54\mid(\alpha_1-2)$ and $(2^{54}-1)\mid(2^{\alpha_1-2}-1)$. Noting that $262657\mid(2^{54}-1)$, we obtain $262657\mid(3^{\alpha_2-3}-1)$. Thus $14592\mid(\alpha_2-3)$ and $(3^{14592}-1)\mid(3^{\alpha_2-3}-1)$. However, $3^{14592}\equiv 1 \pmod{2^{10}}$, a contradiction. Thus $\alpha_1=2$ and $\alpha_2=3$. Further $n=2^2\cdot 3^3$ and $d=2^2\cdot 3$.

Now let $\gamma_2\geqslant 2$. Since $27\mid(2^{\alpha_1+1}+1)$, we have $\alpha_1\equiv 8 \pmod{18}$. Since $19\mid(2^9+1)$ and $(2^9+1)\mid(2^{\alpha_1+1}+1)$, we have $19\mid(3^{\alpha_2-\gamma_2}-2^{\gamma_1+1})$ and

$$(-1)^{\alpha_2-\gamma_2}=\left(\frac{3}{19}\right)^{\alpha_2-\gamma_2}=\left(\frac{2}{19}\right)^{\gamma_1+1}=(-1)^{\gamma_1+1}.$$

It means that $\alpha_2-\gamma_2-\gamma_1\equiv 1 \pmod 2$.

If $\gamma_1\geqslant 3$, then $16\mid(3^{\alpha_2+1}-1)$. Thus $\alpha_2\equiv 3 \pmod 4$ and $5\mid(3^{\alpha_2+1}-1)$. It follows that $5\mid(2^{\alpha_1-\gamma_1}+3^{\gamma_2+1})$ and

$$(-1)^{\alpha_1-\gamma_1}=\left(\frac{2}{5}\right)^{\alpha_1-\gamma_1}=\left(\frac{-1}{5}\right)\left(\frac{3}{5}\right)^{\gamma_2+1}=(-1)^{\gamma_2+1}.$$

It implies that $\alpha_1-\gamma_1-\gamma_2\equiv 1 \pmod 2$, which contradicts $2\mid\alpha_1$, $2\nmid\alpha_2$. Thus $\gamma_1=2$ and $2\mid\gamma_2$. If $\gamma_2\geqslant 4$, then $3^5\mid(2^{\alpha_1+1}+1)$. Thus $\alpha_1\equiv 80 \pmod{162}$ and $(2^{81}+1)\mid(2^{\alpha_1+1}+1)$. Noting that $(19\cdot 163)\mid(2^{81}+1)$, we have $(19\cdot 163)\mid(3^{\alpha_2-\gamma_2}-8)$. It implies that $\alpha_2-\gamma_2\equiv 69 \pmod{162}$ and $\alpha_2-\gamma_2\equiv 3 \pmod{18}$, a contradiction. Thus $\gamma_2=2$ and

$$2^9(2^{\alpha_1-8}-1)=3^6(3^{\alpha_2-5}-1).$$

If $\alpha_1>8$, then $2^{\alpha_1-8}\equiv 1 \pmod{3^6}$. Thus $486\mid(\alpha_1-8)$ and $(2^{486}-1)\mid(2^{\alpha_1-8}-1)$. Noting that $262657\mid(2^{486}-1)$, we obtain $262657\mid(3^{\alpha_2-5}-1)$. Thus $14592\mid(\alpha_2-5)$ and $(3^{14592}-1)\mid(3^{\alpha_2-5}-1)$. However, $3^{14592}\equiv 1 \pmod{2^{10}}$, a contradiction. Thus $\alpha_2=5$ and $\alpha_1=8$. Further $n=2^8 3^5$ and $d=2^2 3^2$.

Case 2 $p_2\geqslant 5$. Let

$$f(\alpha_1,\alpha_2)=\left(1+\frac{1}{2^{\alpha_1+1}}\right)\left(1-\frac{1}{p_2^{\alpha_2+1}}\right),$$

$$g(\alpha_1,\alpha_2)=\frac{3(p_2+1)}{4p_2}+\frac{3(p_2+1)}{D},$$

where $D=2^{\alpha_1-\gamma_1+2}p_2^{\alpha_2-\gamma_2+1}$. Since

$$\frac{24}{25}\leqslant 1-\frac{1}{p_2^{\alpha_2+1}}<f(\alpha_1,\alpha_2)<1+\frac{1}{2^{\alpha_1+1}}\leqslant\frac{9}{8},$$

we have

$$\frac{2p_2+2}{p_2-2}<2^{\alpha_1-\gamma_1}p_2^{\alpha_2-\gamma_2}<\frac{25p_2+25}{7p_2-25}.$$

Thus $p_2\in\{5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41\}$.

If $p_2=5$, then

$$2^{\alpha_1}5^{\alpha_2}-2^{\alpha_1+1}+5^{\alpha_2+1}-1=9\cdot 2^{\gamma_1}5^{\gamma_2}.$$

Since $5\nmid(2^{\alpha_1+1}+1)$, we have $\gamma_2=0$. Noting that $(5^{\alpha_2}-2)2^{\alpha_1}<9\cdot 2^{\gamma_1}$, we obtain $\alpha_2=1$. Thus $\alpha_1=4$, $\gamma_1=3$ or $\alpha_1=2$, $\gamma_1=2$. Further $n=2^4\cdot 5$, $d=2^3$ or $n=2^2\cdot 5$, $d=2^2$.

If $p_2=7$, then

$$2^{\alpha_1+1}7^{\alpha_2}-2^{\alpha_1+1}+7^{\alpha_2+1}-1=3\cdot 2^{\gamma_1+2}7^{\gamma_2}.$$

Noting that $7\nmid(2^{\alpha_1+1}+1)$, we obtain $\gamma_2=0$. By $(7^{\alpha_2}-1)2^{\alpha_1+1}\geqslant 3\cdot 2^{\gamma_1+2}$, we deduce that the above equality can not hold.

If $p_2\geqslant 11$, then $\alpha_1=\gamma_1+2$ and $\alpha_2=\gamma_2$. By (6), we have

$$((p_2-15)2^{\alpha_1-3}+p_2)p_2^{\alpha_2}=2^{\alpha_1+1}+1.$$

If $p_2\geqslant 17$, then

$$((p_2-15)2^{\alpha_1-3}+p_2)p_2^{\alpha_2}\geqslant (2^{\alpha_1-2}+17)17^{\alpha_2}>2^{\alpha_1+1}+1,$$

a contradiction. Thus $p_2\in\{11, 13\}$. By $p_2\mid(2^{\alpha_1+1}+1)$, we have $p_2=11$. It implies that $\alpha_1=4$ and $\alpha_2=1$. Further, $n=2^4\cdot 11$ and $d=2^2\cdot 11$.

This completes the proof of Lemma 1.4.

2 Proof

Proof of Theorem 0.1 Let $n=p_1^{\alpha_1}p_2^{\alpha_2}$ be a near-imperfect number with redundant divisor $d=p_1^{\gamma_1}p_2^{\gamma_2}$, where $\gamma_1\leqslant\alpha_1$, $\gamma_2\leqslant\alpha_2$ and $\gamma_1+\gamma_2<\alpha_1+\alpha_2$. By (1), we have

$$2(p_1^{\alpha_1+1}+(-1)^{\alpha_1})(p_2^{\alpha_2+1}+(-1)^{\alpha_2})=$$

$$(p_1+1)(p_2+1)(p_1^{\alpha_1}p_2^{\alpha_2}+p_1^{\gamma_1}p_2^{\gamma_2}) \quad (7)$$

Then

$$1=2\cdot\frac{p_1^{\alpha_1+1}+(-1)^{\alpha_1}}{p_1^{\alpha_1}(p_1+1)}\cdot\frac{p_2^{\alpha_2+1}+(-1)^{\alpha_2}}{p_2^{\alpha_2}(p_2+1)}-\frac{1}{p_1^{\alpha_1-\gamma_1}p_2^{\alpha_2-\gamma_2}}\geqslant 2\cdot\frac{p_1^{\alpha_1+1}-1}{p_1^{\alpha_1}(p_1+1)}\cdot\frac{p_2^{\alpha_2+1}-1}{p_2^{\alpha_2}(p_2+1)}-\frac{1}{p_1^{\alpha_1-\gamma_1}p_2^{\alpha_2-\gamma_2}}\geqslant 2\cdot\frac{p_1-1}{p_1}\cdot\frac{p_2-1}{p_2}-\frac{1}{p_1}.$$

If $p_1\geqslant 5$, then

$$2\cdot\frac{p_1-1}{p_1}\cdot\frac{p_2-1}{p_2}-\frac{1}{p_1}\geqslant 2\cdot\frac{4}{5}\cdot\frac{6}{7}-\frac{1}{5}>1,$$

a contradiction. Thus $p_1\in\{2, 3\}$. Now we divide into the following four cases according to the parity of α_1 and α_2.

Case 1 $2\nmid\alpha_1$ and $2\nmid\alpha_2$. By Lemma 1.1, we can get

$$n\in\{2^5\cdot 5, 2^7\cdot 5, 2^3\cdot 5^3, 2^3\cdot 7, 2^5\cdot 7, 2^7\cdot 17\}$$

when $p_1=2$. Now let $p_1=3$. By (7), we have

$$(3^{\alpha_1+1}-1)(p_2^{\alpha_2+1}-1)=2(p_2+1)(3^{\alpha_1}p_2^{\alpha_2}+3^{\gamma_1}p_2^{\gamma_2}) \quad (8)$$

If $\alpha_1=1$, $\gamma_1=0$, then

$$(p_2-3)p_2^{\alpha_2}-p_2^{\gamma_2+1}-p_2^{\gamma_2}-4=0.$$

Thus $\gamma_2=0$, $p_2=5$ and $\alpha_2=1$. Hence $n=3\cdot 5$ and $d=1$.

If $\alpha_1=1$, $\gamma_1=1$, then

$$(p_2-3)p_2^{\alpha_2}-3p_2^{\gamma_2+1}-3p_2^{\gamma_2}-4=0.$$

Thus $\gamma_2=0$, $p_2=7$ and $\alpha_2=1$. Hence $n=3\cdot 7$ and $d=3$.

Now suppose that $\alpha_1\geqslant 3$. Let

$$f(\alpha_1, \alpha_2)=(1-\frac{1}{3^{\alpha_1+1}})(1-\frac{1}{p_2^{\alpha_2+1}}),$$

$$g(\alpha_1, \alpha_2)=\frac{2(p_2+1)}{3p_2}+\frac{2(p_2+1)}{D},$$

where $D=3^{\alpha_1-\gamma_1+1}p_2^{\alpha_2-\gamma_2+1}$. If $p_2\geqslant 11$, then

$$0.96\cdots=\frac{2\cdot 12}{3\cdot 11}+\frac{2\cdot 12}{9\cdot 11}\geqslant g(\alpha_1, \alpha_2)=f(\alpha_1, \alpha_2)\geqslant(1-\frac{1}{3^4})(1-\frac{1}{11^2})=0.97\cdots,$$

a contradiction. Thus $p_2\in\{5, 7\}$. By

$$\frac{2(p_2+1)}{3p_2}+\frac{2(p_2+1)}{D}=g(\alpha_1, \alpha_2)=f(\alpha_1, \alpha_2)<1,$$

we have $D=3^{\alpha_1-\gamma_1+1}p_2^{\alpha_2-\gamma_2+1}\geqslant 3p_2^2$ and

$$(1-\frac{1}{3^{\alpha_1+1}})(1-\frac{1}{p_2^{\alpha_2+1}})=f(\alpha_1, \alpha_2)=g(\alpha_1, \alpha_2)=\frac{2(p_2+1)}{3p_2}+\frac{2(p_2+1)}{D}\leqslant 0.96.$$

Thus $\alpha_2=1$. By (8), we have $3^{\alpha_1}p_2-3^{\alpha_1+1}-p_2+1=2\cdot 3^{\gamma_1}p_2^{\gamma_2}$. However, it is impossible since $0\leqslant\gamma_2\leqslant 1$ and $\alpha_1\geqslant 3$.

Case 2 $2\nmid\alpha_1$ and $2\mid\alpha_2$. By Lemma 1.2, we can get

$$n\in\{2^5\cdot 3^2, 2^7\cdot 3^4, 2^3\cdot 5^2\}$$

when $p_1=2$. Now let $p_1=3$. By (7), we have

$$(3^{\alpha_1+1}-1)(p_2^{\alpha_2+1}+1)=2(p_2+1)(3^{\alpha_1}p_2^{\alpha_2}+3^{\gamma_1}p_2^{\gamma_2}) \quad (9)$$

If $\alpha_1=1$, then $3^{\gamma_1}p_2^{\gamma_2}\equiv 4 \pmod{p_2}$, which is impossible. Thus $\alpha_1\geqslant 3$. Let

$$f(\alpha_1, \alpha_2)=(1-\frac{1}{3^{\alpha_1+1}})(1+\frac{1}{p_2^{\alpha_2+1}}),$$

$$g(\alpha_1, \alpha_2)=\frac{2(p_2+1)}{3p_2}+\frac{2(p_2+1)}{D},$$

where $D=3^{\alpha_1-\gamma_1+1}p_2^{\alpha_2-\gamma_2+1}$. If $p_2\geqslant 11$, then

$$\frac{32}{33}=\frac{2\cdot 12}{3\cdot 11}+\frac{2\cdot 12}{9\cdot 11}\geqslant g(\alpha_1, \alpha_2)=f(\alpha_1, \alpha_2)>1-\frac{1}{3^4}=\frac{80}{81},$$

a contradiction. Thus $p_2\in\{5, 7\}$. By

$$\frac{2(p_2+1)}{3p_2}+\frac{2(p_2+1)}{D}=g(\alpha_1, \alpha_2)=f(\alpha_1, \alpha_2)>\frac{80}{81},$$

we have $D=3^2p_2$. Then $\alpha_1=\gamma_1+1$ and $\alpha_2=\gamma_2$. By (9), we have

$$3^{\gamma_1}p_2^{\gamma_2+1}+3^{\gamma_1+2}-p_2^{\gamma_2+1}-1=8\cdot 3^{\gamma_1}p_2^{\gamma_2},$$

which contradicts $\alpha_2\geqslant 2$.

Case 3 $2\mid\alpha_1$ and $2\mid\alpha_2$. By Lemma 1.3, we can get $n=2^2\cdot 3^2$ when $p_1=2$. Now let $p_1=3$. If $p_2\geqslant 11$, then

$$1=\frac{3^{\alpha_1+1}+1}{2\cdot 3^{\alpha_1}}\cdot\frac{p_2^{\alpha_2+1}+1}{p_2^{\alpha_2}(p_2+1)}-\frac{1}{3^{\alpha_1-\gamma_1}p_2^{\alpha_2-\gamma_2}}>\frac{3}{2}\cdot\frac{11}{12}-\frac{1}{3}>1,$$

which is clearly false. Thus $p_2\in\{5, 7\}$. If $3^{\alpha_1-\gamma_1}p_2^{\alpha_2-\gamma_2}\geqslant 5$, then

$$1=\frac{3^{\alpha_1+1}+1}{2\cdot 3^{\alpha_1}}\cdot\frac{p_2^{\alpha_2+1}+1}{p_2^{\alpha_2}(p_2+1)}-\frac{1}{3^{\alpha_1-\gamma_1}p_2^{\alpha_2-\gamma_2}}>$$

$$\frac{3}{2}\cdot\frac{5}{6}-\frac{1}{5}>1,$$

which is false. Thus $\alpha_1=\gamma_1+1$ and $\alpha_2=\gamma_2$.

If $p_2=5$, then

$$3^{\alpha_1+1}+5^{\alpha_2+1}+1=3^{\alpha_1}5^{\alpha_2}.$$

By $5\nmid(3^{\alpha_1+1}+1)$, we deduce that the above equality can not hold.

If $p_2=7$, then

$$3^{\alpha_1+1}+7^{\alpha_2+1}+1=3^{\alpha_1-1}7^{\alpha_2}.$$

By $3\nmid(7^{\alpha_2+1}+1)$, we deduce that the above equality can not hold.

Case 4 $2\mid\alpha_1$ and $2\nmid\alpha_2$. By Lemma 1.4, we can get

$$n\in\{2^2\cdot 3^3,\ 2^8\cdot 3^5,\ 2^2\cdot 5,\ 2^4\cdot 5,\ 2^4\cdot 11\}$$

when $p_1=2$. Now let $p_1=3$. If $p_2\geqslant 11$, then

$$1=\frac{3^{\alpha_1+1}+1}{2\cdot 3^{\alpha_1}}\cdot\frac{p_2^{\alpha_2+1}-1}{p_2^{\alpha_2}(p_2+1)}-\frac{1}{3^{\alpha_1-\gamma_1}p_2^{\alpha_2-\gamma_2}}>$$

$$\frac{3}{2}\cdot\frac{10}{11}-\frac{1}{3}>1,$$

which is clearly false. Thus $p_2\in\{5,7\}$. If $3^{\alpha_1-\gamma_1}p_2^{\alpha_2-\gamma_2}\geqslant 5$, then

$$1=\frac{3^{\alpha_1+1}+1}{2\cdot 3^{\alpha_1}}\cdot\frac{p_2^{\alpha_2+1}-1}{p_2^{\alpha_2}(p_2+1)}-\frac{1}{3^{\alpha_1-\gamma_1}p_2^{\alpha_2-\gamma_2}}>$$

$$\frac{3}{2}\cdot\frac{4}{5}-\frac{1}{5}=1,$$

which is false. Thus $\alpha_1=\gamma_1+1$ and $\alpha_2=\gamma_2$.

If $p_2=5$, then

$$3^{\alpha_1}5^{\alpha_2}+3^{\alpha_1+1}-5^{\alpha_2+1}+1=0.$$

By $5\nmid(3^{\alpha_1+1}+1)$, we deduce that the above equality can not hold.

If $p_2=7$, then

$$3^{\alpha_1-1}7^{\alpha_2}+3^{\alpha_1+1}-7^{\alpha_2+1}+1=0.$$

Noting that $3^{\alpha_1-1}<7$, we can get $\alpha_1=2$ and $\alpha_2=1$. Thus $n=3^2\cdot 7$ and $d=3\cdot 7$.

This completes the proof of Theorem 0.1.

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