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The shock wave solution to nonlinear nonlocal singularly perturbed fractional order equation Cauchy problem

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Abstract: A class of Cauchy problem for the nonlinear nonlocal singular perturbation fractional order equation was considered. First, the outer solution to the original Cauchy problem was obtained. Then, using the stretched variables and the composing expansion method the shock wave layer and initial layer were constructed. Finally, using the theory of differential inequality the asymptotic behavior of the solution to the original Cauchy problem of nonlinear nonlocal singular perturbation fractional order equation was studied and its uniformly valid asymptotic estimation was proved.

Key words: nonlinear; fractional order differential equation; shock wave
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非线性非局部奇摄动分数阶方程Cauchy问题的激波解

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摘要:研究了一类非线性非局部奇摄动分数阶方程Cauchy问题.首先求出了原Cauchy问题的外部解.然后利用伸长变量、合成展开法构造出解的激波层和初始层校正项.最后利用微分不等式理论,研究了原非线性非局部奇摄动分数阶方程Cauchy问题解的渐进性态并证明了它的一致有效的渐近估计式.

关键词:非线性;分数阶微分方程;激波

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0 Introduction

Many physical problems can be solved using the fractional order derivative, such as the complicated seepage flow, heat conduction phenomena, etc. They cannot be described using the idea of normative derivative, but can be solved using the idea of fractional order derivative[1]. The fractional order derivative is extended to the derivative of integer order. The fractional order derivative have practical use, such as the numerical inversions in fractional order diffusion equation[2-9], etc. In this paper, we construct asymptotic solution for a class of nonlinear nonlocal fractional order differential equation using the singular perturbation theory, and obtain its uniformly valid estimation for asymptotic behavior.

The nonlinear problem is an attractive subject in the mathematical circles. Many approximate methods have been studied, including the boundary layer method, the method of averaging, the multiple scales method and the method of matched expansion[10-11]. A great deal of work has been done in this area such as the singularly perturbed degenerated parabolic equations and application to morpho-dynamics in tided environment[12], the asymptotic integration of degenerate singularly perturbed systems of parabolic partial differential equations[13], the meshless method based on moving kriging interpolation for a time-fractional order diffusion equation[14].

Using the singular perturbation method the authors also considered a class of nonlinear problems, such as the homotopic mapping solving method for gain fluency of a laser pulse amplifier[15], the gain fluency of a laser pulse amplifier[16], the inter-decadal sea-air oscillator model[17] and the anti-periodic solutions for a kind of nonlinear differential equation with multiple deviating arguments[18], etc. In this paper, we constructed an asymptotic solution for a class of nonlinear nonlocal fractional order differential equation.

We study the following singular perturbation Cauchy problem for the fractional order differential equation:

$$\left.\begin{aligned}&\sum_{r=1}^{n}\varepsilon^{\alpha r}a_r(D_t^\alpha)^r u+u=f(t,Tu,\varepsilon),\\&0<t<\infty\end{aligned}\right\}\qquad(1)$$

$$(D_t^\alpha)^j u(0)=A_j,\ j=0,1,\cdots,n-1\qquad(2)$$

where ε is a small positive parameter, n is an even number, $a_r\,(r=1,2,\cdots,n)$ and $A_j\,(j=0,1,\cdots,n-1)$ are constants and α is a positive fraction less than 1, and α-th fractional order derivative $D_t^\alpha u$ of $u(t)$ is defined by

$$D_t^\alpha u\equiv\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t(t-s)^{-\alpha}u(s)\mathrm{d}s,$$

where Γ is a Gamma function and integral Tu denoted by $Tu=\int_0^\infty u(s)\mathrm{d}s$.

We assume that

[H_1] $f(t,Tu,\varepsilon)\,(t\neq t_0)$ is a sufficiently smooth function in corresponding domains;

[H_2] $f_\varepsilon\neq 0, f_{Tu}\geqslant ct^{-\alpha}>0$, where c is a positive constant;

[H_3] there is a sufficiently smooth solution $U_0(t)$ $(t\neq t_0)$ for the nonlocal equation $u-f(t,Tu,0)=0$.

1 Outer solution

We construct the outer solution $U(t,\varepsilon)$ of the nonlinear nonlocal problem (1)～(2). Let

$$U(t,\varepsilon)=\sum_{i=0}^{\infty}U_i(t)\varepsilon^{\alpha i}\qquad(3)$$

Substituting Eq. (3) into Eq. (1), developing the nonlinear term $f(t,Tu,\varepsilon)$ in ε^α, equating coefficients of the same powers for ε^α to zero, and from the solution $U_0(t)$ for the reduced equation, we obtain $U_i(t)\ (i=1,2,\cdots)$ successively:

$$U_i(t)=F_i(t)-G_i(t),\ i=1,2,\cdots\qquad(4)$$

where

$$F_i(t)=\frac{1}{i!}\Big[\frac{\partial^i}{\partial\varepsilon^{\alpha i}}f(t,T[\sum_{j=0}^{\infty}U_j(t)\varepsilon^{\alpha j}],\varepsilon)\Big]_{\varepsilon=0},$$

$$G_i(t)=\frac{1}{i!}\Big[\frac{\partial^i}{\partial\varepsilon^{\alpha i}}(\varepsilon^{-\alpha i}\sum_{r=1}^{n}\varepsilon^{\alpha r}a_r(D_t^\alpha)^r\sum_{j=0}^{\infty}U_j(t)\varepsilon^{\alpha j})\Big]_{\varepsilon=0}.$$

Substituting $U_0(t)$ and $U_i(t)(i=1,2,\cdots)$ into Eq. (3), we obtain the outer solution $U(t,\varepsilon)$ for the singular perturbation problem (1)～(2). But the outer solution (3) may not satisfy the point t_0 and the initial condition (2), so we need to construct shock wave layer correction V near point $t=t_0$ and the initial layer correction W near $t=0$.

2 Shock wave layer correction

Let the solution for the nonlinear nonlocal problem (1)～(2) be of the form

$$u=U(t,\varepsilon)+V(\sigma,\varepsilon) \tag{5}$$

with

$$V(\sigma,\varepsilon)=\sum_{i=0}^{\infty}V_i(\sigma)\varepsilon^{\alpha i} \tag{6}$$

where $\sigma=|t-t_0|/\varepsilon$ is a stretched variable[10-11].

Substituting Eqs. (5), (6) into Eqs. (1)～(2), developing the nonlinear term $f(t,T(U+V),\varepsilon)$ in ε^α, and equating coefficients of the same powers of ε^α to zero, we have

$$\sum_{r=1}^{n}\varepsilon^{\alpha r}a_r(D_\sigma^\alpha)^r V_0+V_0= f(t_0,T(U_0+V_0),0)-f(t_0,TU_0,0) \tag{7}$$

$$\left.\begin{aligned}&(D_\sigma^\alpha)^j V_0(0)=(D_\sigma^\alpha)^j U_0(0),\\ &j=0,1,\cdots,n-1\end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned}&\sum_{r=1}^{n}\varepsilon^{\alpha r}a_r(D_\sigma^\alpha)^r V_i+V_i=\\ &f_{Tu}(t_0,T(U_0+V_0,0)T(U_i+V_i))+\overline{F}_i,\\ &0<t<\infty,i=1,2,\cdots\end{aligned}\right\} \tag{9}$$

$$\left.\begin{aligned}&(D_\sigma^\alpha)^j V_i(0)=(D_\sigma^\alpha)^j U_i(0),\\ &j=0,1,\cdots,n-1\end{aligned}\right\} \tag{10}$$

where $\overline{F}_i(i=1,2,\cdots)$ are inductively known functions, whose constructions are omitted.

From the hypotheses and theory for characteristic equation of corresponding fractional order differential equations with constant coefficients, the fractional order differential equations initial value problems (7)～(10), we can obtain solutions $V_i(\sigma),i=0,1,2,\cdots$, successively which have shock wave layer behavior near $t=t_0$[10-11]:

$$\left.\begin{aligned}&V_i(\sigma)=O(\exp(-k_i\sigma))=O(\exp(-k_i\frac{|t-t_0|}{\varepsilon})),\\ &i=0,1,2,\cdots,0<\varepsilon\ll 1\end{aligned}\right\} \tag{11}$$

where $k_i,i=1,2,\cdots$, are positive constants.

Substituting $V_i(\sigma)(i=0,1,2,\cdots)$ into Eq. (6), we obtain the shock wave layer correction $V(\sigma,\varepsilon)$ near $t=t_0$ of the solution $u(x,\varepsilon)$ to the problem (1)～(2).

3 Initial layer correction

Let the solution to the initial value problem (1)～(2) be

$$u=U(t,\varepsilon)+W(\tau,\varepsilon) \tag{12}$$

with

$$W(\tau,\varepsilon)=\sum_{i=0}^{\infty}W_i(\tau)\varepsilon^{\alpha i} \tag{13}$$

where $\tau=t/\varepsilon$ is a stretched variable[10-11].

Substituting Eqs. (12) and (13) into initial value problem (1)～(2), developing the nonlinear term $f(\varepsilon\tau,T(U+W),\varepsilon)$ in ε^α and equating coefficients of the same powers of ε^α to zero, we obtain

$$\left.\begin{aligned}&\sum_{r=1}^{n}\varepsilon^{\alpha r}a_r(D_\tau^\alpha)^r W_0+W_0=\\ &f(0,T(U_0+W_0),0)-f(t,TU_0,0),\\ &0<t<\infty\end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned}&(D_t^\alpha)^j W_0(0)=A_j-(D_t^\alpha)^j U_0(0),\\ &j=0,1,\cdots,n-1\end{aligned}\right\} \tag{15}$$

$$\left.\begin{aligned}&\sum_{r=1}^{n}\varepsilon^{\alpha r}a_r(D_\tau^\alpha)^r W_i+W_i=\\ &f_{Tu}(0,T(U_0+W_0,0)T(U_i+W_i))+\overline{G}_i,\\ &0<t<\infty,i=1,2,\cdots\end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned}&(D_t^\alpha)^j W_i(0)=A_j-(D_t^\alpha)^j U_i(0),\\ &j=0,1,\cdots,n-1,i=1,2,\cdots\end{aligned}\right\} \tag{17}$$

where $A_i=\frac{1}{i!}[\frac{\partial^i A}{\partial\varepsilon^{\alpha i}}]_{\varepsilon=0}$ and $\overline{G}_i,i=1,2,\cdots$, are inductively known functions, whose constructions are omitted too.

From the fractional order differential equations initial value problems (14)～(17), we can obtain solutions $W_i(\tau), i = 0,1,2,\cdots$, successively which possess initial layer behavior near $t=0$:

$$\left.\begin{aligned} &W_i(\tau)=O(\exp(-\tilde{k}_i\tau)=O(\exp(-\tilde{k}_i\frac{t}{\varepsilon}),\\ &i=0,1,2,\cdots,0<\varepsilon\ll 1\end{aligned}\right\} \tag{18}$$

where $\tilde{k}_i, i=1,2,\cdots$, are positive constants.

Substituting $W_i(\tau), i=0,1,2,\cdots$, into Eq. (13), we can obtain the initial layer correction $W(\tau,\varepsilon)$ near $t=0$ of the solution $u(t,\varepsilon)$ for the initial value problem (1)～(2).

From Eqs. (3),(6),(13), then we obtain the asymptotic solution $u_{\text{asy}}(t,\varepsilon)$ to the initial value problem (1)～(2) be

$$\left.\begin{aligned} &u_{\text{asy}}(t,\varepsilon)=\sum_{i=1}^{\infty}[U_i(t)+V_i(\sigma)+W_i(\tau)]\varepsilon^{\alpha i},\\ &0\leqslant t<\infty,0<\varepsilon\ll 1.\end{aligned}\right\} \tag{19}$$

4 Uniform validity of asymptotic solution

Now we prove the asymptotic solution (19) is an uniformly valid asymptotic expansion in ε.

Theorem 4.1 Under the hypotheses $[H_1]$～$[H_3]$, there is a solution $u(t,\varepsilon)$ to the nonlinear nonlocal initial value problem (1)～(2) for the singular perturbation fractional order differential equation, which possesses the following uniformly valid asymptotic expansion in ε on $t\in[0,\infty)$

$$\left.\begin{aligned} &u(t,\varepsilon)=\sum_{i=0}^{m}[U_i(t)+V_i(\frac{|t-t_0|}{\varepsilon})+\\ &\qquad W_{1i}(\frac{t}{\varepsilon})]\varepsilon^{\alpha i}+O(\varepsilon^{\alpha(m+1)}),\\ &t\in[0,\infty),0<\varepsilon\ll 1\end{aligned}\right\} \tag{20}$$

Proof We construct two auxiliary functions $\bar{\alpha}(t,\varepsilon), \bar{\beta}(t,\varepsilon)$:

$$\bar{\alpha}(t,\varepsilon)=\sum_{i=0}^{m}[U_i(t)+V_i(\frac{|t-t_0|}{\varepsilon})+ W_{1i}(\frac{t}{\varepsilon})]\varepsilon^{\alpha i}-\delta\varepsilon^{\alpha(m+1)} \tag{21}$$

$$\bar{\beta}(t,\varepsilon)=\sum_{i=0}^{m}[U_i(t)+V_i(\frac{|t-t_0|}{\varepsilon})+ W_{1i}(\frac{t}{\varepsilon})]\varepsilon^{\alpha i}+\delta\varepsilon^{\alpha(m+1)} \tag{22}$$

where δ is a positive constant large enough to be chosen below, and m is an arbitrary fixed positive integer.

Obviously, we have

$$\bar{\alpha}(t,\varepsilon)\leqslant\bar{\beta}(t,\varepsilon) \tag{23}$$

$$\left.\begin{aligned} &(D_t^\alpha)^j\bar{\alpha}(0,\varepsilon)\leqslant A_j\leqslant(D_t^\alpha)^j\bar{\beta}(0,\varepsilon),\\ &j=0,1,\cdots,n-1\end{aligned}\right\} \tag{24}$$

Now we prove that

$$\left.\begin{aligned} &\sum_{r=1}^{n}\varepsilon^{\alpha r}a_r(D_t^\alpha)^r\bar{\alpha}+\bar{\alpha}-f(t,T\bar{\alpha},\varepsilon)\geqslant 0,\\ &0<t<\infty\end{aligned}\right\} \tag{25}$$

$$\left.\begin{aligned} &\sum_{r=1}^{n}\varepsilon^{\alpha r}a_r(D_t^\alpha)^r\bar{\beta}+\bar{\beta}-f(t,T\bar{\beta},\varepsilon)\leqslant 0,\\ &0<t<\infty\end{aligned}\right\} \tag{26}$$

In fact, from the hypotheses $[H_1]$～$[H_3]$ and Eqs. (4),(7),(9),(14) and (16) for ε small enough, there is a positive constant d, such that

$$\begin{aligned} &\sum_{r=1}^{n}\varepsilon^{\alpha r}a_r(D_t^\alpha)^r\bar{\beta}+\bar{\beta}-f(t,T\bar{\beta},\varepsilon)=\\ &\sum_{r=1}^{n}\varepsilon^{\alpha r}a_r(D_t^\alpha)^r\sum_{i=0}^{m}[U_i(t)+V_i(\frac{|t-t_0|}{\varepsilon})+W_i(\frac{t}{\varepsilon})]\varepsilon^{i}+\\ &\sum_{i=0}^{m}[U_i(t)+V_i(\frac{|t-t_0|}{\varepsilon})+W_i(\frac{t}{\varepsilon})]\varepsilon^{\alpha i}+\\ &\delta\varepsilon^{\alpha(m+1)}-f(t,T[\sum_{i=0}^{m}[U_i(t)+V_i(\frac{|t-t_0|}{\varepsilon})+\\ &W_i(\frac{t}{\varepsilon})]\varepsilon^{\alpha i}]+\delta\varepsilon^{\alpha(m+1)},\varepsilon)+\\ &f(t,T[\sum_{i=0}^{m}[U_i(t)+V_i(\frac{|t-t_0|}{\varepsilon})+W_i(\frac{t}{\varepsilon})]\varepsilon^{\alpha i}],\varepsilon)\leqslant\\ &U_0-f(t,TU_0,0)+\\ &\sum_{i=1}^{m}[U_i(t)-F_i(t)+G_i(t)]\varepsilon^{\alpha i}+\\ &\sum_{r=1}^{n}\varepsilon^{\alpha r}a_r(D_\sigma^\alpha)^rV_0+V_0-f(t_0,T(U_0+V_0),0)+\\ &f(t_0,TU_0,0)+\sum_{i=1}^{m}[\sum_{r=1}^{n}\varepsilon^{\alpha r}a_r(D_\sigma^\alpha)^rV_i+V_i-\\ &f_{Tu}(t_0,T(U_0+V_0,0)T(U_i+V_i)-\bar{F}_{ii}]\varepsilon^{\alpha i}+\\ &\sum_{r=1}^{n}\varepsilon^{\alpha r}a_r(D_t^\alpha)^rW_0+W_0-f(0,T(U_0+W_0,0))+\\ &f(t,TU_0,0)+\sum_{i=1}^{m}[\sum_{r=1}^{n}\varepsilon^{\alpha r}a_r(D_\tau^\alpha)^rW_i+W_i-\\ &f_{Tu}(0,T(U_0+W_0,0)T(U_i+W_i))-\bar{G}_i]\varepsilon^{\alpha i}-\\ &f_{Tu}(t,T[\sum_{i=0}^{m}[U_i(t)+V_i(\frac{|t-t_0|}{\varepsilon})+\end{aligned}$$

$$W_{1i}(\frac{t}{\varepsilon})]\varepsilon^{\alpha i}+\delta\varepsilon^{\alpha(m+1)}],\varepsilon)+$$

$$dt^{-\alpha}\varepsilon^{\alpha(m+1)}\leqslant(-c\delta+d)t^{-\alpha}\varepsilon^{\alpha(m+1)}.$$

Selecting $\delta\geqslant d/c$, then we have Eq. (26). Analogously, we can prove Eq. (25). From Eqs. (23)～(26), $\bar{\alpha}$ and $\bar{\beta}$ are upper and lower solutions to the initial value problem (1)～(2) respectively. From the theory of differential inequality[10-11], there is a solution $u(t,\varepsilon)$ to the nonlinear nonlocal singular perturbation initial value problem (1)～(2) such that $\bar{\alpha}(t,\varepsilon)\leqslant u(t,\varepsilon)\leqslant\bar{\beta}(t,\varepsilon)$. And from Eqs. (21) and (22), we have the relation (20). The proof of Theorem 4.1 is completed.

5 Conclusion

This paper dealt with a class of singular perturbation Cauchy problem of the fractional order differential equation. We obtained the outer solution to original problem and constructed the interior shock and initial corrective terms of the asymptotic solutions to the original problem by using the stretched variables and using the method of singular perturbation. By means of the theory of differential inequality, the uniform validity of the asymptotic solution was proved. Then the shock wave asymptotic solution to a class of singular perturbation problem possessed simple, valid and higher accuracy peculiarity.

The fractional order differential equation can be applied to the following physical models:

The time delay atmospheric mid-latitude western wind field anomaly will cause mid-latitude upper layer ocean flow anomaly, which can be solved using the fractional order differential equation.

Virus transmission is a complicated phenomenon. We can study the quality and quantitative behaviors of the human group for the infected and the susceptible populations in the infested area by using the fractional order differential equation.

The nonlinear evolution fractional order differential equations of BKK mechanism is a disturbed physical model. It can have the solution of the nonlinear BKK mechanism disturbed physical model using the approximation theory of the fractional order differential equation.

The method of the singular perturbation is an approximate analytic method, which differs from the general numerical method. From the approximate analytic expansions of the nonlinear nonlocal singularly perturbed fractional order equation Cauchy problem, we can also execute the continuous analytic operations, such as differential and integral operations, etc, and thus further study the qualitative and quantitative behaviors of the fractional order equation Cauchy problem.

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