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The computing formula for generalized Euler functions

LIAO Qunying

(*School of Mathematical Sciences, Sichuan Normal University, Chengdu* 610068, *China*)

Abstract: Let n and e be positive integers. Based on elementary methods and techniques, the explicit formula for $\varphi_e(n)$ $(e = p^r, \prod_{i=1}^{t} q_i)$ was given for the case $q_1 \equiv \cdots \equiv q_t \equiv 1 \pmod p$ or $q_1 \equiv \cdots \equiv q_t \equiv -1 \pmod p$, where $p, q_1, \cdots, q_t$ are distinct primes, t and r are both positive integers, thus generalizing the previous results.

Key words: Euler function; generalized Euler function; Möbius function

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广义欧拉函数的计算公式

廖群英

(四川师范大学数学科学学院,四川成都 610068)

摘要:设 n 和 e 均为正整数. 利用初等的方法和技巧,给出了广义欧拉函数 $\varphi_e(n)$ $(e=p^r, \prod_{i=1}^{t} q_i)$ 在所有的 q_i 同余于 p 均模 1 或者均模 -1 时的准确计算公式,其中,$p, q_1, \cdots, q_t$ 为不同的素数,t 和 r 为正整数. 这推广了前人的结果.

关键词:欧拉函数;广义欧拉函数;莫比乌斯函数

0 Introduction

In the 18th century, Euler first defined the Euler function $\varphi(n)$ of a positive integer n to be the number of positive integers not greater than n but prime to n [1]. It's well known that as one of the important number theory functions, Euler function has been applied widely[1-4]. In 2002 and 2007, Cai[5] and Cai et al.[6] generalized the definition of Euler function to be the generalized Euler function.

Definition 0.1 For two positive integer n and e, the generalized Euler function of n related to e is

$$\varphi_e(n) = \sum_{i=1, \gcd(i,n)=1}^{\left[\frac{n}{e}\right]} 1,$$

i.e., $\varphi_e(n)$ is the number of positive integers not

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Biography: LIAO Qunying, female, born in 1974, PhD/Prof. Research field: Coding and cryptology theory.
E-mail: qunyingliao@sicnu.edu.cn

greater than $[\frac{n}{e}]$ but prime to n, where e is a positive integer and $[x]$ is the greatest integer not greater than x. It's easy to verify that

$$\varphi_e(n) = \sum_{d|n} \mu(\frac{n}{d})[\frac{d}{e}] \qquad (1)$$

where $\mu(n)$ is the Möbius function, i. e.,

$$\mu(n) = \begin{cases} 1, & n = 1; \\ (-1)^k, & n \geqslant 2 \text{ and } \alpha_1 = \cdots = \alpha_k = 1; \\ 0, & n \geqslant 2 \text{ and there is some } \alpha_i > 1 (1 \leqslant i \leqslant k) \end{cases} \qquad (2)$$

when $n = \prod_{i=1}^{k} p_i^{\alpha_i} (\alpha_i \geqslant 0)$ is a positive integer and $p_i (i = 1, \cdots, k)$ are distinct primes. It is easy to see that for any positive integer $n \geqslant 2$,

$$\sum_{d|n} \mu(\frac{n}{d}) = 0 \qquad (3)$$

It's well known that for the positive integer $n = \prod_{i=1}^{k} p_i^{\alpha_i} (\alpha_i \geqslant 1)$, the Euler function

$$\varphi(n) = \prod_{i=1}^{k} (p_i^{\alpha_i} - p_i^{\alpha_i - 1}).$$

And from the definition one can get $\varphi(n) = \varphi_1(n)$. Naturally we ask the following:

Question 0.1 For any fixed positive integer e, determine the explicit algorithm formula for the generalized Euler function $\varphi_e(n)$.

Fixed a positive integer $n = \prod_{i=1}^{k} p_i^{\alpha_i} \geqslant 2$, where $p_1, \cdots, p_k$ are distinct primes and $\alpha_1, \cdots, \alpha_k$ are positive integers, denote $\Omega(n) = \sum_{i=1}^{k} \alpha_i$ and $\omega(n) = k$. Especially, $\Omega(1) = \omega(1) = 0$.

In recent years, Cai et al.[7-8] obtained the accurate calculation formula for $\varphi_e(n) (e = 2, 3, 4, 6)$, and then, by using properties for Legendre or Jacobi symbols, they also got some necessary and sufficient conditions for that $\varphi_e(n)$ and $\varphi_e(n+1)$ $(e = 2, 3, 4)$ are both odd or even numbers.

Proposition 0.1[7-8] Let $p_1, \cdots, p_k$ be distinct primes, $\alpha_1, \cdots, \alpha_k$ positive integers, α, β nonnegative integers, and $n_1 = \prod_{i=1}^{k} p_i^{\alpha_i}$.

① If $\gcd(p_i, 3) = 1 (i = 1, \cdots, k)$ and $n = 3^{\alpha} n_1 > 3$, then

$$\varphi_3(n) = \begin{cases} \frac{\varphi(n)}{3} + \frac{(-1)^{\Omega(n)} 2^{\omega(n) - \alpha - 1}}{3}, & \alpha \in \{0,1\} \text{ and } p_i \equiv 2 (\bmod 3) (i = 1, \cdots, k); \\ \frac{\varphi(n)}{3}, & \text{otherwise.} \end{cases}$$

② If $n = 2^{\alpha} n_1 > 4$, then

$$\varphi_4(n) = \begin{cases} \frac{\varphi(n)}{4} + \frac{(-1)^{\Omega(n)} 2^{\omega(n) - \alpha}}{4}, & \alpha \in \{0,1\} \text{ and } p_i \equiv 3 (\bmod 4) (i = 1, \cdots, k); \\ \frac{\varphi(n)}{4}, & \text{otherwise.} \end{cases}$$

③ If $\gcd(p_i, 6) = 1 (i = 1, \cdots, k)$ and $n = 2^{\alpha} 3^{\beta} n_1 > 6$, then

$$\varphi_6(n) = \begin{cases} \frac{1}{6}\varphi(n) + \frac{(-1)^{\Omega(n)} 2^{\omega(n) + 1 - \beta}}{6}, & \alpha = 0, \beta \in \{0,1\} \text{ and } p_i \equiv 5 (\bmod 6) (i = 1, \cdots, k); \\ \frac{1}{6}\varphi(n) + \frac{(-1)^{\Omega(n)} 2^{\omega(n) - 1 - \beta}}{6}, & \alpha = 1, \beta \in \{0,1\} \text{ and } p_i \equiv 5 (\bmod 6) (i = 1, \cdots, k); \\ \frac{1}{6}\varphi(n) - \frac{(-1)^{\Omega(n)} 2^{\omega(n) - \beta}}{6}, & \alpha \geqslant 2, \beta \in \{0,1\} \text{ and } p_i \equiv 5 (\bmod 6) (i = 1, \cdots, k); \\ \frac{1}{6}\varphi(n), & \text{otherwise.} \end{cases}$$

Recently, Wang and Liao[9] obtained the formula for $\varphi_5(n)$ and some sufficient conditions for $2 \mid \varphi_5(n)$. The present paper continues the study, based on elementary methods and techniques, generalizes the main results in Refs. [7-9], and gives the explicit formula for $\varphi_e(n)$ $(e = p^r, \prod_{i=1}^{t} q_i)$, where $p, q_1, \cdots, q_t$ are distinct primes, t and r are both positive integers. In fact we prove the following main results.

Theorem 0.1 Let $p_1, \cdots, p_k$ be distinct primes, and $\alpha_1, \cdots, \alpha_k$ be positive integers. Suppose that e is a positive integer, and $n=\prod_{i=1}^{k} p_i^{\alpha_i}$, then

$$\varphi_e(n)=\begin{cases}\dfrac{1}{e}\varphi(n),\ p_i\equiv 1(\mathrm{mod}\ e)(i=1,\cdots,k);\\ \dfrac{1}{e}\varphi(n)+\dfrac{(e-2)(-1)^{\Omega(n)}2^{\omega(n)-1}}{e},\\ \qquad p_i\equiv -1(\mathrm{mod}\ e)(i=1,\cdots,k).\end{cases}$$

Theorem 0.2 Let $p, p_1, \cdots, p_k$ be distinct primes, $t, \alpha, \alpha_1, \cdots, \alpha_k$ be positive integers. If $e=p^t$, $n_1=\prod_{i=1}^{k} p_i^{\alpha_i}$ and $n=e^{\alpha}n_1$, then

$$\varphi_e(n)=\begin{cases}\dfrac{\varphi(pn_1)}{p},\\ \alpha=1,\ p_i\equiv 1(\mathrm{mod}\ p)(i=1,\cdots,k);\\ \dfrac{\varphi(pn_1)}{p}-\dfrac{(p-2)(-1)^{\Omega(n)-t}2^{\omega(n)-2}}{p},\\ \alpha=1,\ p_i\equiv -1(\mathrm{mod}\ p)(i=1,\cdots,k);\\ \dfrac{1}{e}\varphi(n),\ \alpha\geqslant 2.\end{cases}$$

Theorem 0.3 Let $q_1, \cdots, q_t, p_1, \cdots, p_k$ be distinct primes, and $t, \alpha, \alpha_1, \cdots, \alpha_k$ be positive integers. If $e=\prod_{i=1}^{t} q_i$, $n_1=\prod_{i=1}^{k} p_i^{\alpha_i}$ and $n=e^{\alpha}n_1$, then

$$\varphi_e(n)=\begin{cases}\dfrac{1}{e}\varphi(n),\ \alpha\geqslant 2,\ \text{or}\ \alpha=1,\\ \qquad p_i\equiv 1(\mathrm{mod}\ e)(i=1,\cdots,k);\\ \dfrac{1}{e}\varphi(n)+M,\ \alpha=1,\\ \qquad p_i\equiv -1(\mathrm{mod}\ e)(i=1,\cdots,k).\end{cases}$$

where

$$M=(-1)^{\Omega(n)-t}2^{\omega(n)-t-1}\cdot \sum_{r=1}^{t}(-1)^r\sum_{1\leqslant i_1<\cdots<i_r\leqslant t}\frac{(\prod_{j=1}^{r}q_{i_j}-2)}{\prod_{j=1}^{r}q_{i_j}}.$$

Remark 0.1 Firstly, by taking $e=3$ in Theorem 0.1 and $p=3, t=1$ in Theorems 0.2, one can get (1) of Proposition 0.1. Secondly, by taking $\alpha=2r$ in (2) of Proposition 0.1, one can get Theorem 0.1 for the case $e=4$ and Theorem 0.2 for the case $p=2$, where r is a nonnegative integer. Lastly, by taking $e=6$ in Theorems 0.1 and 0.3 one can get (3) of Proposition 0.1 for the case $\alpha=\beta$. We leave the detailed proofs to interested readers.

1 Proofs for main results

1.1 Proof for Theorem 0.1

① If $p_i\equiv 1(\mathrm{mod}\ e)(i=1,\cdots,k)$, i.e., for any $d\mid n$, $d\equiv 1(\mathrm{mod}\ e)$. Then by (1)～(3) we have

$$\varphi_e(n)=\sum_{d\mid n}\mu\left(\frac{n}{d}\right)\left[\frac{d}{e}\right]= \sum_{d\mid n}\mu\left(\frac{n}{d}\right)\cdot\left(\frac{d-1}{e}\right)= \frac{1}{e}\sum_{d\mid n}\mu\left(\frac{n}{d}\right)d-\frac{1}{e}\sum_{d\mid n}\mu\left(\frac{n}{d}\right)=\frac{1}{e}\varphi(n) \tag{4}$$

② If $p_i\equiv -1(\mathrm{mod}\ e)(i=1,\cdots,k)$, then for any $d\mid n$, $d\equiv \pm 1(\mathrm{mod}\ e)$. And so by Eqs. (2)～(4), we have

$$\begin{aligned}\varphi_e(n)&=\sum_{d\mid n}\mu\left(\frac{n}{d}\right)\left[\frac{d}{e}\right]=\\ &\sum_{d\mid n, d\equiv 1(\mathrm{mod}\ e)}\mu\left(\frac{n}{d}\right)\left[\frac{d}{e}\right]+\sum_{d\mid n, d\equiv -1(\mathrm{mod}\ e)}\mu\left(\frac{n}{d}\right)\left[\frac{d}{e}\right]=\\ &\sum_{d\mid n, d\equiv 1(\mathrm{mod}\ e)}\mu\left(\frac{n}{d}\right)\left(\frac{d-1}{e}\right)+\\ &\sum_{d\mid n, d\equiv -1(\mathrm{mod}\ e)}\mu\left(\frac{n}{d}\right)\left(\frac{d-e+1}{e}\right)=\\ &\frac{\varphi(n)}{e}-\frac{1}{e}\sum_{d\mid n, d\equiv 1(\mathrm{mod}\ e)}\mu\left(\frac{n}{d}\right)-\\ &\left(\frac{e-1}{e}\right)\sum_{d\mid n, d\equiv -1(\mathrm{mod}\ e)}\mu\left(\frac{n}{d}\right)=\\ &\frac{\varphi(n)}{e}+\left(\frac{e-2}{e}\right)\sum_{d\mid n, d\equiv 1(\mathrm{mod}\ e)}\mu\left(\frac{n}{d}\right)=\\ &\frac{\varphi(n)}{e}+\left(\frac{e-2}{e}\right)\sum_{\substack{(\prod_{i=1}^{k}p_i^{\alpha_i-1})t\mid n\\ (\prod_{i=1}^{k}p_i^{\alpha_i-1})t\equiv 1(\mathrm{mod}\ e)}}\mu\left(\frac{n}{(\prod_{i=1}^{k}p_i^{\alpha_i-1})t}\right)=\\ &\frac{\varphi(n)}{e}+\left(\frac{e-2}{e}\right)\sum_{\substack{t\mid \prod_{i=1}^{k}p_i\\ (-1)^{\Omega(n)-k}t\equiv 1(\mathrm{mod}\ e)}}\mu\left(\frac{\prod_{i=1}^{k}p_i}{t}\right)=\\ &\frac{\varphi(n)}{e}+\left(\frac{e-2}{e}\right)A\end{aligned} \tag{5}$$

(a) If $2\mid \Omega(n)$ and $2\mid k$, then

$$A=\sum_{\substack{t\mid \prod_{i=1}^{k}p_i\\ t\equiv 1(\mathrm{mod}\ e)}}\mu\left(\frac{\prod_{i=1}^{k}p_i}{t}\right)=$$

$$\sum_{j=0}^{\frac{k}{2}}\binom{k}{2j}(-1)^{k-2j}=2^{k-1}=(-1)^{\Omega(n)}2^{\omega(n)-1}.$$

(b) If $2\mid\Omega(n)$ and $2\nmid k$, then

$$A=\sum_{t\mid\prod_{i=1}^{k}p_i,\,t\equiv-1(\mathrm{mod}\,e)}\mu\left(\frac{\prod_{i=1}^{k}p_i}{t}\right)=$$

$$\sum_{j=1}^{\frac{k+1}{2}}\binom{k}{2j-1}(-1)^{k-2j+1}=$$

$$2^{k-1}=(-1)^{\Omega(n)}2^{\omega(n)-1}.$$

For the case $2\nmid\Omega(n)$ and $2\nmid k$ or $2\nmid\Omega(n)$ and $2\mid k$, similarly we can get

$$A=(-1)^{\Omega(n)}2^{\omega(n)-1}\tag{6}$$

Hence by Eqs. (5) and (6) we immediately have

$$\varphi_e(n)=\frac{1}{e}\varphi(n)+\frac{(e-2)(-1)^{\Omega(n)}2^{\omega(n)-1}}{e}\tag{7}$$

This completes the proof for Theorem 0.1.

1.2 Proof for Theorem 0.2

① For $\alpha=1$ and $t=1$, i.e., $e=p$, $n=pn_1$, then by $\gcd(p,n_1)=1$ we have

$$\left.\begin{aligned}&\varphi(n)=\varphi(pn_1)=(p-1)\varphi(n_1),\\&\Omega(n)=\Omega(n_1)+1,\omega(n)=\omega(n_1)+1\end{aligned}\right\}\tag{8}$$

And so by Eqs. (1)~(3), we can get

$$\varphi_e(n)=\varphi_p(pn_1)=\sum_{d\mid pn_1}\mu\left(\frac{pn_1}{d}\right)\left[\frac{d}{p}\right]=$$

$$\sum_{d\mid n_1}\mu\left(\frac{n_1}{d}\right)d+\sum_{d\mid n_1}\mu\left(\frac{pn_1}{d}\right)\left[\frac{d}{p}\right]=\varphi(n_1)-\varphi_p(n_1)\tag{9}$$

Now by $\gcd(p,n_1)=1$, Eqs. (8)~(9) and Theorem 0.1 we can get

$$\varphi_e(n)=\varphi(n_1)-\varphi_p(n_1)=$$

$$\varphi(n_1)-\begin{cases}\dfrac{\varphi(n_1)}{p},\ p_i\equiv1(\mathrm{mod}\ p)(i=1,\cdots,k);\\ \left(\dfrac{\varphi(n_1)}{p}+\dfrac{(p-2)(-1)^{\Omega(n_1)}2^{\omega(n_1)-1}}{p}\right),\\ \qquad p_i\equiv-1(\mathrm{mod}\ p)(i=1,\cdots,k).\end{cases}=$$

$$\begin{cases}\dfrac{\varphi(pn_1)}{p},\ p_i\equiv1(\mathrm{mod}\ p)(i=1,\cdots,k);\\ \dfrac{\varphi(pn_1)}{p}-\dfrac{(p-2)(-1)^{\Omega(n)-t}2^{\omega(n)-2}}{p},\\ \qquad p_i\equiv-1(\mathrm{mod}\ p)(i=1,\cdots,k)\end{cases}\tag{10}$$

② For $\alpha=1$ and $t\geqslant2$, i.e., $e=p^t$, $n=p^tn_1$. Then by $\gcd(p,n_1)=1$ we know that

$$\left.\begin{aligned}&\varphi(n)=(p^{t\alpha}-p^{t\alpha-1})\varphi(n_1),\\&\Omega(n)=\Omega(n_1)+t,\omega(n)=\omega(n_1)+1\end{aligned}\right\}\tag{11}$$

and so by $\gcd(p,n_1)=1$ and Eqs. (1)~(3) we have

$$\varphi_e(n)=\varphi_{p^t}(p^tn_1)=\sum_{d\mid p^tn_1}\mu\left(\frac{p^tn_1}{d}\right)\left[\frac{d}{p^t}\right]=$$

$$\sum_{d\mid n_1}\mu\left(\frac{n_1}{d}\right)\left[\frac{p^td}{p^t}\right]+\sum_{d\mid n_1}\mu\left(\frac{p^tn_1}{d}\right)\left[\frac{d}{p^t}\right]+$$

$$\sum_{d\mid n_1,1\leqslant\beta\leqslant t-1}\mu\left(\frac{p^{t-\beta}n_1}{d}\right)\left[\frac{p^\beta d}{p^t}\right]=$$

$$\varphi(n_1)+\sum_{d\mid n_1}\mu\left(\frac{pn_1}{d}\right)\left[\frac{p^{t-1}d}{p^t}\right]=$$

$$\varphi(n_1)-\sum_{d\mid n_1}\mu\left(\frac{n_1}{d}\right)\left[\frac{d}{p}\right]=$$

$$\varphi(n_1)-\varphi_p(n_1).$$

Thus by $\gcd(p,n_1)=1$, (11) and Theorem 0.1 we know that

$$\varphi_e(n)=\varphi(n_1)-\varphi_p(n_1)=$$

$$\varphi(n_1)-\begin{cases}\dfrac{\varphi(n_1)}{p},\\ \qquad p_i\equiv1(\mathrm{mod}\ p)(i=1,\cdots,k);\\ \dfrac{\varphi(n_1)}{p}-\dfrac{(p-2)(-1)^{\Omega(n_1)}2^{\omega(n_1)-1}}{p},\\ \qquad p_i\equiv-1(\mathrm{mod}\ p)(i=1,\cdots,k).\end{cases}=$$

$$\begin{cases}\dfrac{(p-1)\varphi(n_1)}{p},\\ \qquad p_i\equiv1(\mathrm{mod}\ p)(i=1,\cdots,k);\\ \dfrac{(p-1)\varphi(n_1)}{p}-\dfrac{(p-2)(-1)^{\Omega(n)-t}2^{\omega(n)-2}}{p},\\ \qquad p_i\equiv-1(\mathrm{mod}\ p)(i=1,\cdots,k)\end{cases}\tag{12}$$

③ For $\alpha\geqslant2$, i.e., $e=p^t$, $n=p^{t\alpha}n_1$. Then by $\gcd(p,n_1)=1$ we know that

$$\varphi(n)=(p^{t\alpha}-p^{t\alpha-1})\varphi(n_1),$$

and so by Eqs. (1)~(3) we have

$$\varphi_e(n)=\varphi_{p^t}(p^{t\alpha}n_1)=\sum_{d\mid p^{t\alpha}n_1}\mu\left(\frac{p^{t\alpha}n_1}{d}\right)\left[\frac{d}{p^t}\right]=$$

$$\sum_{d\mid n_1}\mu\left(\frac{n_1}{d}\right)\left[\frac{p^{t\alpha}d}{p^t}\right]+\sum_{d\mid n_1}\mu\left(\frac{p^{t\alpha}n_1}{d}\right)\left[\frac{d}{p^t}\right]+$$

$$\sum_{d\mid n_1,1\leqslant\beta\leqslant t\alpha-1}\mu\left(\frac{p^{t\alpha-\beta}n_1}{d}\right)\left[\frac{p^\beta d}{p^t}\right]=$$

$$p^{t\alpha-t}\varphi(n_1)+\sum_{d\mid n_1}\mu\left(\frac{pn_1}{d}\right)\left[\frac{p^{t\alpha-1}d}{p^t}\right]=$$

$$p^{t\alpha-t}\varphi(n_1)-p^{t\alpha-t-1}\varphi(n_1)=\frac{\varphi(n)}{e}\tag{13}$$

Now by Eqs. (10) and (12)～(13), Theorem 0.2 is proved.

1.3 Proof for Theorem 0.3

① For $\alpha=1$, i.e., $n=en_1$, and then

$$\Omega(n)=\Omega(n_1)+t,\omega(n)=\omega(n_1)+t\tag{14}$$

Thus by Eqs. (1)～(3) we have

$$\varphi_e(n)=\varphi_e(en_1)=\sum_{d\mid en_1}\mu\left(\frac{en_1}{d}\right)\left[\frac{d}{e}\right]=\sum_{r=1}^{t}A_r+\varphi(n_1)\tag{15}$$

where for any $r=1,\cdots,t$,

$$A_r=\sum_{1\leqslant i_1<\cdots<i_r\leqslant t}\sum_{d\mid n_1}\mu\left(\frac{\prod_{j=1}^{r}q_{i_j}n_1}{d}\right)\left[\frac{d}{\prod_{j=1}^{r}q_{i_j}}\right]=\sum_{1\leqslant i_1<\cdots<i_r\leqslant t}(-1)^r\varphi_{\prod_{j=1}^{r}q_{i_j}}(n_1)\tag{16}$$

If $p_i\equiv 1(\bmod e)(i=1,\cdots,k)$, then by $\gcd(n_1,e=\prod_{i=1}^{t}p_i)=1$, (16) and Theorem 0.1, we have

$$A_r=\sum_{1\leqslant i_1<\cdots<i_r\leqslant t}(-1)^r\varphi_{\prod_{j=1}^{r}q_{i_j}}(n_1)=\sum_{1\leqslant i_1<\cdots<i_r\leqslant t}(-1)^r\frac{\varphi(n_1)}{\prod_{j=1}^{r}q_{i_j}}\tag{17}$$

Now by (15) and (17) we know that

$$\varphi_e(n)=\varphi(n_1)(1+(-1)\sum_{i=1}^{t}\frac{1}{q_i}+(-1)^2\sum_{1\leqslant i<j\leqslant t}\frac{1}{q_iq_j}+\cdots+(-1)^{t-1}\sum_{i_1<\cdots<i_{t-1}}\frac{1}{q_{i_1}\cdots q_{i_{t-1}}}+(-1)^t\frac{1}{e})=\varphi(n_1)\cdot\frac{\prod_{i=1}^{t}(q_i-1)}{e}=\frac{\varphi(n)}{e}\tag{18}$$

If $p_i\equiv -1(\bmod e)(i=1,\cdots,k)$, then by $\gcd(n_1,e=\prod_{i=1}^{t}p_i)=1$, Theorem 0.1 and (14) we have

$$A_r=\sum_{1\leqslant i_1<\cdots<i_r\leqslant t}(-1)^r\cdot\left(\frac{\varphi(n_1)}{\prod_{j=1}^{r}q_{i_j}}+\frac{(\prod_{j=1}^{r}q_{i_j}-2)(-1)^{\Omega(n_1)}2^{\omega(n_1)-1}}{\prod_{j=1}^{r}q_{i_j}}\right)=\varphi(n_1)\sum_{1\leqslant i_1<\cdots<i_r\leqslant t}(-1)^r\frac{1}{\prod_{j=1}^{r}q_{i_j}}+(-1)^r\sum_{1\leqslant i_1<\cdots<i_r\leqslant t}\frac{(\prod_{j=1}^{r}q_{i_j}-2)(-1)^{\Omega(n)-t}2^{\omega(n)-t-1}}{\prod_{j=1}^{r}q_{i_j}}\tag{19}$$

Now by (15) and (18)～(19), we can get

$$\varphi_e(n)=\varphi(n_1)\Big(1+(-1)\sum_{i=1}^{t}\frac{1}{q_i}+(-1)^2\sum_{1\leqslant i<j\leqslant t}\frac{1}{q_iq_j}+\cdots+(-1)^{t-1}\sum_{i_i<\cdots<i_{t-1}}\frac{1}{q_{i_1}\cdots q_{i_j}}+(-1)^t\frac{1}{e}\Big)+(-1)^{\Omega(n)-t}2^{\omega(n)-t-1}\sum_{r=1}^{t}(-1)^r\cdot\sum_{1\leqslant i_1<\cdots<i_r\leqslant t}\frac{(\prod_{j=1}^{r}q_{i_j}-2)}{\prod_{j=1}^{r}q_{i_j}}=\frac{\varphi(n)}{e}+(-1)^{\Omega(n)-t}2^{\omega(n)-t-1}\sum_{r=1}^{t}(-1)^r\cdot\sum_{1\leqslant i_1<\cdots<i_r\leqslant t}\frac{(\prod_{j=1}^{r}q_{i_j}-2)}{\prod_{j=1}^{r}q_{i_j}}\tag{20}$$

② For $\alpha\geqslant 2$, i.e., $n=e^{\alpha}n_1$, then

$$\varphi(n)=e^{\alpha-1}\prod_{i=1}^{t}(q_i-1)\varphi(n_1)\tag{21}$$

and by Eqs. (1)～(3) we have

$$\varphi_e(n)=\sum_{d\mid e^{\alpha}n_1}\mu\left(\frac{e^{\alpha}n_1}{d}\right)\left[\frac{d}{e}\right]=\sum_{d\mid n_1}\mu\left(\frac{e^{\alpha}n_1}{d}\right)\left[\frac{d}{e}\right]+\sum_{d\mid n_1}\mu\left(\frac{n_1}{d}\right)\left[\frac{e^{\alpha}d}{e}\right]+\sum_{r=1}^{t}B_r=e^{\alpha-1}\varphi(n_1)+\sum_{r=1}^{t}B_r\tag{22}$$

where for any $r=1,\cdots,t$,

$$B_r=\sum_{1\leqslant i_1<\cdots<i_r\leqslant t}\sum_{d\mid n_1}\mu\left(\frac{\prod_{j=1}^{r}q_{i_j}n_1}{d}\right)\left[\frac{e^{\alpha-1}d}{\prod_{j=1}^{r}q_{i_j}}\right]=(-1)^re^{\alpha-1}\varphi(n_1)\sum_{1\leqslant i_1<\cdots<i_r\leqslant t}\frac{1}{\prod_{j=1}^{r}q_{i_j}}\tag{23}$$

Now from Eqs. (18) and (21)～(23) we can get

$$\varphi_e(n)=e^{\alpha-1}\varphi(n_1)+ e^{\alpha-1}\varphi(n_1)\sum_{r=1}^{t}(-1)^r\sum_{1\leqslant i_1,\cdots,i_r\leqslant t}\frac{1}{\prod_{j=1}^{r}q_{i_j}}= e^{\alpha-1}\varphi(n_1)+e^{\alpha-1}\varphi(n_1)\left(\frac{\prod_{i=1}^{t}(q_i-1)}{e}-1\right)= \frac{\varphi(n)}{e} \tag{24}$$

Thus from Eqs. (18), (20) and (24), we complete the proof for Theorem 0.3.

2 Conclusion

Let n and e be positive integers. Based on elementary methods and techniques, the present paper gave the explicit formula for $\varphi_e(n)(e=p^r, \prod_{i=1}^{t}q_i)$, where $p, q_1, \cdots, q_t$ are distinct primes, t and r are both positive integers. Our results are generalizations of the main results in Refs. [7-8] to some degree.

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