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Large induced subgraph with restricted degrees in trees

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Abstract: A problem was proposed to determine for a tree T the size of the largest $S \subseteq V(T)$ such that all vertices in $T[S]$ have either degree 1 or degree 0 (mod k). Here it was proved that, for integer $k \geqslant 2$, every tree T contains an induced subgraph of order at least $c_k |V(T)|$ with all degrees either equal to 1 or 0 (mod k), where $c_k = 3/4$ when $k=2$, and $c_k = 2/3$ when $k \geqslant 3$. Moreover, the bounds are best possible. This gives a good answer to the problem.

Key words: tree; induced subgraph; degree

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树图中度数受限的大导出子图

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摘要：有文献提出公开问题:对树 T,求最大的集合 $S \in V(T)$使得导出子图 $T[S]$每个点的度为 1 或 0 (mod k). 证明了,对给定的整数 $k \geqslant 2$, 每一棵树 T 都包含一个阶数至少为 $c_k |V(T)|$ 的导出子图使得所有的度为 1 或 0(mod k), 这里当 $k=2$ 时, $c_k=3/4$;当 $k \geqslant 3$ 时 $c_k=2/3$, 且下界是最好的.这个结果解决了上述问题.

关键词：树;导出子图;度

0 Introduction

A classical result of Gallai[1] asserts that for any graph G, the vertex set $V(G)$ can be partitioned into two sets, each of which induces a subgraph with all degrees even. From this we can conclude that every graph of order n contains an induced subgraph of order at least $\lceil \frac{n}{2} \rceil$ with all degrees even, and this is best possible by considering a path.

A natural question is to ask for the largest size

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of an induced subgraph with all degrees odd in a graph of given order. There are many results related to the problem (see for example in Refs. [2-6]. In particular, for trees, Radcliffe and Scott[7] proved that every tree of order n contains an induced subgraph of order at least $2\lceil\frac{n+1}{3}\rceil$ with all degrees odd. Berman et al.[8] further extended this result to an induced subgraph having all degrees congruent to 1 modulo k. In the same paper, they proposed the following interesting problem. Write $G[S]$ for the subgraph of graph G induced by $S\subseteq V(G)$.

Problem 0.1[8] For any tree T, determine the size of the largest $S\subseteq V(T)$ such that all vertices in $T[S]$ have either degree 1 or degree 0 (mod k).

In this paper, we give an answer to Problem 0.1 in the following theorem.

Theorem 0.1 For every tree T and every integer $k\geqslant 2$, there is a set $S\subseteq V(T)$ such that $|S|\geqslant c_k|V(T)|$ and $T[S]$ has all degrees either 1 or 0(mod k), where $c_k=\frac{3}{4}$ for $k=2$ and $c_k=\frac{2}{3}$ for $k\geqslant 3$. Moreover, the bound of $|S|$ is best possible.

The tightness of c_k can be shown by considering a path P_{3n} on $3n$ vertices for $k\geqslant 3$ and the following tree T_{4n} on $4n$ vertices as shown in Fig. 1 for $k=2$.

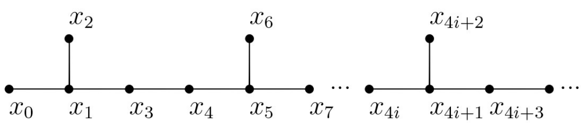

Fig. 1 Tree T_{4n}

The rest of the paper is arranged as follows. We give the proof of Theorem 0.1 in Section 1. In Section 2, we give some remarks and discussions.

1 Proof of Theorem 0.1

We call an $S\subset V(T)$ a good subset of T if $|S|\geqslant c_k|V(T)|$ and $T[S]$ has all degrees 1 or 0 (mod k). Our proof is by contradiction. Suppose to the contrary that there is a tree T such that T contains no good set $S\subseteq V(T)$. Without loss of generality, we may assume T is a smallest counterexample. Clearly, $|V(T)|>2$. If $2\leqslant \text{diam}(T)\leqslant 3$ then T is a star or a double-star. It is an easy task to check that T has a good set, a contradiction. So we may assume that $\text{diam}(T)\geqslant 4$, where $\text{diam}(T)$ is the diameter of T.

Let L_0 be the set of leaves of T and L_1 be the set of leaves of $T-L_0$ and L_2 be the set of leaves of $T-(L_0\cup L_1)$. For a vertex v of T, write $N_i(v)$ for $N(v)\cap L_i$ and $d_i(v)$ for $|N_i(v)|$, $i=0, 1, 2$. Since $\text{diam}(T)\geqslant 4$, L_2 is non-empty. By the definition, we have $d_0(v)>0$ for $v\in L_1$ and $d_1(v)>0$ for $v\in L_2$.

Claim 1.1 Let $x\in L_2$. Then for each $w\in N_1(x)$, we have $d_0(w)=1$ and $d_T(w)=2$.

Proof Let w be any vertex in $N_1(x)$. Note that $w\in L_1$. Then $d_0(w)>0$.

Case 1 $k\geqslant 3$.

If $d_0(w)\geqslant 2$, let $T_0=T[N_0(w)]$, then T_0 is an empty graph on $N_0(w)$. Now let $T'=T-(N_0(w)\cup\{w\})$. Then T' is a tree smaller than T. Hence T' has a good set S'. Therefore, $S'\cup N_0(w)$ is a good set of T since

$$|N_0(w)|=|V(T_0)|\geqslant \frac{2}{3}(|V(T_0)|+1),$$

a contradiction.

Case 2 $k=2$.

If $d_0(w)\geqslant 3$, with a same argument with $d_0(w)\geqslant 2$ for $k\geqslant 3$, we can find a good set $S'\cup N_0(w)$ with order at least $\frac{3}{4}|V(T)|$ of T, a contradiction.

If $d_0(w)=2$, let $S_0=\{w\}\cup N_0(w)$ and $T'=T-(S_0\cup\{x\})$. Then T' has a good set S'. Note that $T[S_0]$ is a path of length 2. Then S_0 is a good set of $T[S_0]$. Since $|S_0|=|V(T_0)|=3$, we have $|S_0|\geqslant\frac{3}{4}(|V(T_0)|+1)$. Therefore, $S'\cup S_0$ is a good set of T, a contradiction.

Note that $N_T(w)=N_0(w)\cup\{x\}$, we have $d_T(w)=d_0(w)+1=2$.

Claim 1.2 If $k=2$, then for each $x\in L_2$, we have $d_1(x)=1$, $d_0(x)=0$ and $d_T(x)=2$.

Proof Since $x\in L_2$, we have $d_1(x)\geqslant 1$. Let $S_0=N_0(x)\cup N_1(x)\cup(\bigcup_{w\in N_1(x)}N_0(w))$ and $T_0=T[S_0]$. Denote $d_1(x)=a$ and $d_0(x)=b$. By Claim 1.1, T_0 consists of a independent edges and b independent vertices. So S_0 is a good set of T_0. Let $T'=T-(S_0\cup\{x\})$. Then T' is a tree smaller than T. By the minimality of T, T' has a good set S'. If $a\geqslant 2$ or $b\geqslant 1$ then $|S_0|=2a+b\geqslant 3$. So $|S_0|=|V(T_0)|\geqslant\frac{3}{4}(|V(T_0)|+1)$. Therefore, $S'\cup S_0$ is a good set of T, a contradiction.

Note that for each $x\in L_2$, we have $d_T(x)=d_0(x)+d_1(x)+1=2$.

Proof of Theorem 0.1 Choose a vertex $x\in L_2$.

Case 1 $k\geqslant 3$.

By Claim 1.1, we can find a vertex $w\in N_1(x)$ with $d_0(w)=1$. Denote $N_0(w)=\{v\}$. Let $T'=T-\{w,v,x\}$ and $T_0=T[\{w,v\}]$. By the minimality of T, T' has a good set S'. Let $S_0=\{w,v\}$. Note that S_0 is a good set of T_0 and $|S_0|=2\geqslant\frac{2}{3}(|V(T_0)|+1)$. $S'\cup S_0$ is a good set of T, a contradiction.

Case 2 $k=2$.

By Claims 1.1 and 1.2, we may assume $N(x)=\{w,y\}$ with $w\in N_1(x)$ and $N_0(w)=\{v\}$. Let $S_0=\{x,w,v\}$, $T_0=T[S_0]$ and $T'=T-(S_0\cup\{y\})$. Note that each component of T' has a good set by the minimality of T. So T' has a good set S'. Since $|S_0|=3\geqslant\frac{3}{4}(|V(T_0)|+1)$, $S'\cup S_0$ is a good set of T, a contradiction.

2 Conclusion

In this paper, we proved that, for integer $k\geqslant 2$, every tree T contains an induced subgraph of order at least $c_k|V(T)|$ with all degrees either equal to 1 or 0 (mod k), where $c_k=\frac{3}{4}$ when $k=2$, and $c_k=\frac{2}{3}$ when $k\geqslant 3$. Moreover, the bounds are best possible. This solved Problem 0.1 proposed by Berman et al. As a further step, for given integer $k\geqslant 2$ and general graph G, it is an interesting challenge to determine the size of the largest $S\subseteq V(G)$ such that all vertices in $G[S]$ have either degree 1 or degree 0 (mod k).

References

[1] LOVÁSZ L. Combinatorial Problems and Exercises [M]. Amsterdam: North-Holland, 1979.

[2] CARO Y. On induced subgraphs with odd degrees[J]. Discrete Math, 1994, 132:23-28.

[3] CARO Y, KRASIKOV I, RODITTY Y. On induced subgraphs of trees with restricted degrees[J]. Discrete Math, 1994,125: 101-106.

[4] HOU X, YU L, LI J, et al. Odd induced subgraphs in graphs with treewidth at most two[J]. Graphs and Combin, 2018, 34 (4): 535-544.

[5] TAO X, LIU B, HOU X. Weak internal partition of regular graphs[J]. Commun Math Stat, 2017, 5(3): 335-338.

[6] SCOTT A D. Large induced subgraphs with all degrees odd[J]. Comb Probab Comput, 1992, 1 (4): 335-349.

[7] RADCLIFFE A J, SCOTT A D. Every tree contains a large induced subgraph with all degrees odd [J]. Discrete Math, 1995, 140: 275-279.

[8] BERMAN D M, RADCLIFFE A J, SCOTT A D. All trees contain a large induced subgraph having all degrees 1 (mod k)[J]. Discrete Math, 1997, 175: 35-40.