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Anticipated time-dependent backward stochastic evolution equations

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Abstract: A class of anticipated time-dependent backward stochastic evolution equation in a Hilbert space was discussed. The existence and uniqueness of the evolution solution was proved. As an application, the evolution solution for a class of anticipated backward stochastic partial differential equations was derived. Some well-known results were generalized and extended.

Key words: anticipated backward stochastic evolution equation; time-dependent; evolution solution

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时间相依的超前倒向随机发展方程

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摘要:讨论了Hilbert空间中一类时间相依的超前倒向随机发展方程,给出了其解的存在唯一性定理.作为应用,探讨了一类超前倒向随机偏微分方程的演化解.所得结果推广了已有相关结论.

关键词:超前倒向随机发展方程;时间相依;演化解

0 Introduction

Since Pardoux and Peng[1] established the theory of nonlinear backward stochastic differential equations (BSDEs), many interesting generalized forms have been introduced. Among them, Peng and Yang[2] considered a kind of new BSDEs, called anticipated BSDEs, with the following form:

$$\left.\begin{array}{l} -\mathrm{d}Y(t)=f(t,Y(t),Z(t),Y(t+\mu(t)),\\ Z(t+\upsilon(t)))\mathrm{d}t-Z(t)\mathrm{d}W(t),\ t\in[0,T];\\ Y(t)=\xi(t),\ Z(t)=\eta(t),\ t\in[T,T+M]\end{array}\right\} \tag{1}$$

where $\mu(\cdot)$: $[0,T]\rightarrow\mathbb{R}^+\backslash\{0\}$ and $\upsilon(\cdot)$: $[0,T]\rightarrow\mathbb{R}^+\backslash\{0\}$ are continuous functions satisfying that

(a1) there exists a constant $M\geqslant 0$ such that for each $t\in[0,T]$,

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$t+\mu(t) \leqslant T+M$, $t+\upsilon(t) \leqslant T+M$;

(a2) there exists a constant $0 \leqslant L < 1$ such that for each $t \in [0,T]$ and each nonnegative integrable function $g(\cdot)$,

$$\int_t^T g(s+\mu(s))\mathrm{d}s \leqslant L\int_t^{T+M} g(s)\mathrm{d}s,$$
$$\int_t^T g(s+\upsilon(s))\mathrm{d}s \leqslant L\int_t^{T+M} g(s)\mathrm{d}s.$$

In Eq. (1), the generator f contains not only the values of solutions of the present but also the future. Peng and Yang[2] proved the existence and uniqueness of the solutions under Lipschitz conditions. Furthermore, they gave a duality relation between stochastic differential delay equations and anticipated BSDEs, which is a useful tool in the analysis of stochastic optimal control problems, see e. g. Refs. [3-4]. The above works on BSDEs and anticipated BSDEs have been in the framework of finite dimension.

On the other hand, Hu and Peng[5] introduced a kind of backward stochastic evolution equations (BSEEs) in a Hilbert space H with the following form:

$$\left.\begin{array}{l} \mathrm{d}Y(t)=[AY(t)+ \\ f(t, Y(t), Y(t))]\mathrm{d}t-Z(t)\mathrm{d}W(t),\ t\in[0,\ T]; \\ Y(T)=\xi \end{array}\right\} \tag{2}$$

where $A:D(A) \subset H \to H$ is a linear operator which generates a C_0-semigroup $\{S(t)\}_{0\leqslant t\leqslant T}$ on H. They proved the existence and uniqueness of the mild solution for Eq. (2). Since then, Dauer et al.[6] have examined the approximate controllability for the system (2) with a kind of non-Lipschitz coefficients. Mahmudov and McKibben[7] proved the existence and uniqueness of the mild solution to Eq. (2) and obtained a stochastic maximum principle for the optimal control of stochastic systems governed by BSEE (2). In addition, Al-Hussein[8] considered a class of time-dependent BSEE in a Hilbert space H of the following form:

$$\left.\begin{array}{l} -\mathrm{d}Y(t)=[A(t)Y(t)+ \\ f(t, Y(t),Z(t))]\mathrm{d}t-Z(t)\mathrm{d}W(t),t\in[0,\ T]; \\ Y(t)=\xi \end{array}\right\} \tag{3}$$

where $A(t)$, $t \geqslant 0$, are unbounded operators which generate a strong evolution operator $U(t,r)$, $0\leqslant r\leqslant t\leqslant T$. The author proved the existence and uniqueness of the evolution solution of Eq. (3) with non-Lipschitz coefficients.

To our best knowledge, there have been no works reported on anticipated BSEEs. Originally, we wanted to prove the existence and uniqueness of the evolution solution for the following anticipated BSEE:

$$\left.\begin{array}{l} -\mathrm{d}Y(t)=[A(t)Y(t)+ \\ f(t,\ Y(t),\ Z(t),Y(t+\mu(t)),\ Z(t+\upsilon(t)))]\mathrm{d}t- \\ Z(t)\mathrm{d}W(t), \\ Y(t)=\xi(t),\ Z(t)=\eta(t),\ t\in[T,\ T+M] \end{array}\right\} \tag{4}$$

Due to technical difficulty, we can not as yet solve the above general form. The difficulty lies in the existence part. We can not define an efficient Picard iteration. We will try our best to solve this problem in our further study.

Motivated by the aforementioned works, being left with nothing better than the second choice, the present paper deals with a class of anticipated BSEEs in a Hilbert space H with the following form:

$$\left.\begin{array}{l} -\mathrm{d}Y(t)=[A(t)Y(t)+ \\ f(t,\ Y(t),\ Z(t),Y(t+\mu(t)))]\mathrm{d}t- \\ Z(t)\mathrm{d}W(t),\ t\in[0,\ T]; \\ Y(t)=\xi(t),\ Z(t)=\eta(t),\ t\in[T,\ T+M] \end{array}\right\} \tag{5}$$

where $\mu(\cdot)$: $[0,\ T] \to \mathbb{R}^+\backslash\{0\}$ is a continuous function satisfying the assumptions (a1) and (a2).

As the first step, which is also the fundamental step, we prove the existence and uniqueness of the evolution solutions to Eq. (5) with the coefficient satisfying Lipschitz conditions. Of course, our results could be extended to the case of the coefficient satisfying some non-Lipschitz conditions as appeared in Refs. [6-7]. We expect to do more research works on anticipated BSEE (4). Based on our obtained results, the existence and uniqueness of the evolution solution for a class of anticipated backward stochastic partial differential equation is derived.

1 Preliminaries

Let $T > 0$ be fixed throughout this paper. Assume that K, H are two separable Hilbert spaces with inner product $\langle \cdot, \cdot \rangle_K$ and $\langle \cdot, \cdot \rangle_H$, respectively. We denote their norms by $\| \cdot \|_K$ and $\| \cdot \|_H$. In case without confusion, we just use $\langle \cdot, \cdot \rangle$ for the inner product and $\| \cdot \|$ for the norm. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete filtered probability space. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of K, $\{W(t): t \in [0,T]\}$ be a cylindrical K-valued Brownian motion, which can be written formally as the infinite sum $W(t) = \sum_{i=1}^{\infty} w_i(t) e_i$, where $\{w_i(t)\}_{i=1}^{\infty}$ are mutually independent one-dimensional standard Brownian motions. We assume that the filtration is generated by the cylindrical Brownian motion $W(\cdot)$ and augmented, that is

$$\mathscr{F}_t = \sigma\{W(s); s \leqslant t\} \vee \mathscr{N}, 0 \leqslant t \leqslant T,$$

where $\mathscr{N}$ is the class of $\mathbb{P}$-null sets. In the sequel, let $L^2(K, H)$ denote the set of Hilbert-Schmidt operators from K to H. For more details, one can see Ref. [5] and the references therein.

In what follows, we need the following facts on evolution operator defined on a separable Hilbert space H, which appeared in Refs. [9-10].

Lemma 1.1 Let $\{A(t)\}_{0 \leqslant t \leqslant T}$ be a family of closed linear operators satisfying the so-called Acquistapace-Terreni conditions (ATC), that is there exist constants $L_0 > 0$, $\alpha_1, \alpha_2, \cdots, \alpha_k, \gamma_1, \gamma_2, \cdots, \gamma_k$ with $0 \leqslant \gamma_i < \alpha_i \leqslant 2$, $i = 1, 2, \cdots, k$ such that

$$\| A(t)(\lambda - A(t))^{-1}(A(t)^{-1} - A(s)^{-1}) \| \leqslant L_0 \sum_{i=1}^{k} (t-s)^{\alpha_i} | \lambda |^{\gamma_i - 1},$$

for $s, t \in \mathbb{R}$, $\lambda \in S_\theta \backslash \{0\}$, where

$$\rho(A(t)) \supset S_\theta = \{\lambda \in \mathbb{C} : | \arg\lambda | \leqslant \theta\} \cup \{0\}, \theta \in \left(\frac{\pi}{2}, \pi\right)$$

and there exists a constant $L_1 \geqslant 0$ such that

$$\| (\lambda - A(t))^{-1} \| \leqslant \frac{L_1}{1 + |\lambda|}, \lambda \in S_\theta.$$

Then, there exists a unique evolution operator $\{U(t,s), 0 \leqslant s \leqslant t \leqslant T\}$ satisfying that

(i) $U(t,s)U(s,r) = U(t,r)$,

(ii) $U(t,t) = I$ for $r \leqslant s \leqslant t$,

(iii) $(t,s) \mapsto U(t,s)$ is continuous for $t > s$,

(iv) $\frac{\partial U}{\partial t}(t,s) = A(t)U(t,s)$ and

$$\| A(t)^k U(t,s) \| \leqslant L_0 (t-s)^{-k}$$

for $0 < t - s \leqslant 1$, $k = 0, 1$.

Remark 1.1 Generally, $\{U(t,s), 0 \leqslant s \leqslant t \leqslant T\}$ is called an evolution operator or a two-parameters semigroup, and the family $\{A(t)\}_{0 \leqslant t \leqslant T}$ is called the infinitesimal operator of $\{U(t,s), 0 \leqslant s \leqslant t \leqslant T\}$.

Remark 1.2 If $A(t)$, $t \geqslant 0$, is a second order differential operator A, i. e., $A(t) = A$ for $t \geqslant 0$. Then, A generates a C_0-semigroup $\{e^{At}, t \geqslant 0\}$, which is discussed in Refs. [5-7].

In what follows, we need the following space. For $s, S \in [0, T]$, let $L^2_{\mathscr{F}}(s, S; H)$ denote the space of $\mathscr{F}_S$-measurable processes $\psi: \Omega \times [s, S] \to H$ such that $\mathbb{E}\int_s^S | \psi(u) \|^2 du < \infty$.

We give the following assumptions which will be used in the proof of results.

(H1) Assume that $f(s, \omega, y, z, \xi): [0,T] \times \Omega \times H \times L^2(K;H) \times L^2_{\mathscr{F}}(s, T+M; H) \to H$, which is $\mathscr{P} \otimes \mathscr{B}(H) \otimes \mathscr{B}(L^2(K;H)) \otimes \mathscr{B}(H)/\mathscr{B}(H)$ measurable such that $f(\cdot, 0, 0, 0) \in L^2_{\mathscr{F}}(0, T; H)$.

(H2) There exists a constant $C \geqslant 0$, such that for all $s \in [0, T]$, $y(s), y'(s) \in H$, $z(s), z'(s) \in L^2(K;H)$, $\xi(\cdot), \xi'(\cdot) \in L^2_{\mathscr{F}}(s, T+M; H)$, $r \in [s, T+M]$,

$$\| f(s, y(s), z(s), \xi(r), \eta(\bar{r})) - f(s, y'(s), z'(s), \xi'(r), \eta'(\bar{r})) \| \leqslant C(\| y(s) - y'(s) \| + \| z(s) - z'(s) \| + \mathbb{E}^{\mathscr{F}_s} \| \xi(r) - \xi'(r) \|).$$

Definition 1.1 A pair of $\mathscr{F}_t$-adapted stochastic processes $(Y, Z) = (Y(t), Z(t))_{0 \leqslant t \leqslant T+M}$ is called the evolution solution (or simply a solution) to the anticipated BSEE (5) if $(Y, Z) \in L^2_{\mathscr{F}}(0, T+M; H) \times L^2_{\mathscr{F}}(0, T+M; L^2(K;H))$ such that

$$\left.\begin{aligned}&Y(t)=U(T,t)\xi(T)+\\&\int_t^T U(s,t)f(s,Y(s),Z(s),Y(s+\mu(s)))\mathrm{d}s-\\&\int_t^T U(s,t)Z(s)\mathrm{d}W(s),\ 0\leqslant t\leqslant T;\\&Y(t)=\xi(t),\ Z(t)=\eta(t),\ t\in[T,T+M]\end{aligned}\right\}\tag{6}$$

2 Existence and uniqueness of the evolution solution

In this section, we aim to derive the existence and uniqueness result for the solution of anticipated BSEE (5). Before stating our main theorem, we recall an existence, a uniqueness and an estimate of the solution for the following BSEE appeared in Ref. [8].

$$\left.\begin{aligned}&-\mathrm{d}Y(t)=[A(t)Y(t)+f(t)]\mathrm{d}t-\\&\qquad Z(t)\mathrm{d}W(t),\ t\in[0,T];\\&Y(t)=\xi\end{aligned}\right\}\tag{7}$$

In the sequel, we let $k:=\sup_{0\leqslant t\leqslant s\leqslant T}\|U(s,t)\|$.

Lemma 2.1[8, Lemma 3.1] If $\xi\in L^2(\Omega,\mathscr{F}_T,\mathbb{P};H)$ and $f\in L^2_{\mathscr{F}}(0,T;H)$, there exists a unique pair $(Y,Z)\in L^2_{\mathscr{F}}(0,T;H)\times L^2_{\mathscr{F}}(0,T;L^2(K;H))$ for $0\leqslant t\leqslant T$ such that

$$Y(t)=U(T,t)\xi+\int_t^T U(s,t)f(s)\mathrm{d}s-\int_t^T U(s,t)Z(s)\mathrm{d}W(s)\tag{8}$$

Moreover, for $0\leqslant t\leqslant T$, it holds that

$$\mathbb{E}\|Y(t)\|^2\leqslant 2k^2\mathbb{E}\|\xi\|^2+2k^2(T-t)\mathbb{E}\int_t^T\|f(s)\|^2\mathrm{d}s\tag{9}$$

and

$$\mathbb{E}\int_t^T\|Z(s)\|^2\mathrm{d}s\leqslant 8k^2\mathbb{E}\|\xi\|^2+8k^2(T-t)\mathbb{E}\int_t^T\|f(s)\|^2\mathrm{d}s\tag{10}$$

The following existence and uniqueness theorem is the main result of this section.

Theorem 2.1 Assume the assumptions (H1)～(H2) hold, and μ satisfies (a1) and (a2). Then, for any given terminal conditions $\xi(\cdot)\in L^2_{\mathscr{F}}(T,T+M;H)$ and $\eta(\cdot)\in L^2_{\mathscr{F}}(T,T+M;L^2(K;H))$, the anticipated BSEE (5) has a unique solution.

Proof Uniqueness. Let $(Y_i,Z_i)\in L^2_{\mathscr{F}}(0,T+M;H)\times L^2_{\mathscr{F}}(0,T+M;L^2(K;H))$, $i=1,2$, be two solutions of anticipated BSEE (5). Define $\hat{Y}=Y_1-Y_2$, $\hat{Z}=Z_1-Z_2$ and

$$\hat{f}(s)=f(s,Y_1(s),Z_1(s),Y_1(s+\mu(s)))-f(s,Y_2(s),Z_2(s),Y_2(s+\mu(s))),$$

we then have

$$\left.\begin{aligned}&\hat{Y}(t)=\int_t^T U(s,t)\hat{f}(s)\mathrm{d}s-\int_t^T U(s,t)\hat{Z}(s)\mathrm{d}W(s),\\&t\in[0,T];\\&\hat{Y}(t)=0,\ \hat{Z}(t)=0,\ t\in[T,T+M]\end{aligned}\right\}\tag{11}$$

From Lemma 2.1 and (H2), we have

$$\begin{aligned}&\mathbb{E}\|\hat{Y}(t)\|^2\leqslant 2k^2(T-t)\mathbb{E}\int_t^T\|\hat{f}(s)\|^2\mathrm{d}s\leqslant\\&6k^2C^2(T-t)\mathbb{E}\int_t^T(\|\hat{Y}(s)\|^2+\|\hat{Z}(s)\|^2)\mathrm{d}s+\\&6k^2C^2(T-t)\mathbb{E}\int_t^T\mathbb{E}^{\mathscr{F}_s}\|\hat{Y}(s+\mu(s))\|^2\mathrm{d}s\leqslant\\&6k^2C^2(T-t)\mathbb{E}\int_t^T(\|\hat{Y}(s)\|^2+\|\hat{Z}(s)\|^2)\mathrm{d}s+\\&6k^2C^2(T-t)L\mathbb{E}\int_t^T\|\hat{Y}(s)\|^2\mathrm{d}s=\\&6k^2C^2(T-t)(1+L)\mathbb{E}\int_t^T\|\hat{Y}(s)\|^2\mathrm{d}s+\\&6k^2C^2(T-t)\mathbb{E}\int_t^T\|\hat{Z}(s)\|^2\mathrm{d}s\end{aligned}\tag{12}$$

and

$$\begin{aligned}&\mathbb{E}\int_t^T\|\hat{Z}(s)\|^2\mathrm{d}s\leqslant\\&8k^2(T-t)\mathbb{E}\int_t^T\|\hat{f}(s)\|^2\mathrm{d}s\leqslant\\&24k^2C^2(T-t)(1+L)\int_t^T\mathbb{E}\|\hat{Y}(s)\|^2\mathrm{d}s+\\&24k^2C^2(T-t)\int_t^T\mathbb{E}\|\hat{Z}(s)\|^2\mathrm{d}s\end{aligned}\tag{13}$$

Now, letting $\eta=\dfrac{1}{48k^2C^2}$, for $t\in[T-\eta,T]$, we get

$$\mathbb{E}\int_t^T\|\hat{Z}(s)\|^2\mathrm{d}s\leqslant(1+L)\int_t^T\mathbb{E}\|\hat{Y}(s)\|^2\mathrm{d}s$$

and

$$\mathbb{E}\|\hat{Y}(t)\|^2\leqslant\frac{1+L}{4}\mathbb{E}\int_t^T\|\hat{Y}(s)\|^2\mathrm{d}s.$$

The Gronwall inequality implies that

$$\mathbb{E}\ \|\widehat{Y}(t)\|^2=0,\ \mathbb{E}\ \|\widehat{Z}(t)\|^2=0,$$

i. e. , $Y_1(t)=Y_2(t)$ and $Z_1(t)=Z_2(t)$ $\mathbb{P}$ -a. s. for $t\in[T-\eta,T+M]$. With the same procedure, we can show the uniqueness for $t\in[T-2\eta, T-\eta]$. Thus, the uniqueness is proved.

Existence. Letting $Y^0(t)=0$ for $t\in[0,T+M]$, in virtue of Ref. [8, Theorem 3. 1], we define recursively $(Y^{n+1}, Z^{n+1})\in L^2_{\mathscr{F}}(0, T+M; H)\times L^2_{\mathscr{F}}(0, T+M; L^2(K;H))$ as the unique solution to the following BSEE:

$$\left.\begin{aligned}&Y^{n+1}(t)=U(T,t)\xi(T)+\int_t^T U(s,t)f(s, Y^n(s), Z^{n+1}(s),Y^n(s+\mu(s)))\mathrm{d}s-\\&\qquad\int_t^T U(s,t)Z^{n+1}(s)\mathrm{d}W(s),\ 0\leqslant t\leqslant T;\\&Y^{n+1}(t)=\xi(t),\ Z^{n+1}(t)=\eta(t),\ t\in[T, T+M]\end{aligned}\right\}\tag{14}$$

For $0\leqslant t\leqslant T$, we have

$$\begin{cases}Y^{n+1}(t)-Y^n(t)=\displaystyle\int_t^T U(s,t)[f(s, Y^n(s), Z^{n+1}(s),Y^n(s+\mu(s)))-\\\qquad f(s, Y^{n-1}(s), Z^n(s),Y^{n-1}(s+\mu(s)))]\mathrm{d}s-\displaystyle\int_t^T U(s,t)(Z^{n+1}(s)-Z^n(s))\mathrm{d}W(s);\\Y^{n+1}(t)-Y^n(t)=0,\ Z^{n+1}(t)-Z^n(t)=0,\ t\in[T, T+M].\end{cases}$$

From Lemma 2. 1 and (H2), we have

$$\begin{aligned}&\mathbb{E}\ \|Y^{n+1}(t)-Y^n(t)\|^2\leqslant 2k^2(T-t)\,\mathbb{E}\int_t^T\|[f(s, Y^n(s), Z^{n+1}(s),Y^n(s+\mu(s)))-\\&\qquad f(s, Y^{n-1}(s), Z^n(s),Y^{n-1}(s+\mu(s)))]\mathrm{d}s\|^2\mathrm{d}s\leqslant\\&6k^2C^2(T-t)\,\mathbb{E}\int_t^T(\|Y^n(s)-Y^{n-1}(s)\|^2+\|Z^{n+1}(s)-Z^n(s)\|^2)\mathrm{d}s+\\&6k^2C^2(T-t)\,\mathbb{E}\int_t^T\mathbb{E}^{\mathscr{F}_s}[\|Y^n(s+\mu(s))-Y^{n-1}(s+\mu(s))\|^2]\mathrm{d}s\leqslant\\&6k^2C^2(T-t)\,\mathbb{E}\int_t^T(\|Y^n(s)-Y^{n-1}(s)\|^2+\|Z^{n+1}(s)-Z^n(s)\|^2)\mathrm{d}s+\\&6k^2C^2L(T-t)\,\mathbb{E}\int_t^T\|Y^n(s)-Y^{n-1}(s)\|^2\mathrm{d}s=\\&6k^2C^2(T-t)(1+L)\,\mathbb{E}\int_t^T\|Y^n(s)-Y^{n-1}(s)\|^2\mathrm{d}s+6k^2C^2(T-t)\,\mathbb{E}\int_t^T\|Z^{n+1}(s)-Z^n(s)\|^2\mathrm{d}s\end{aligned}\tag{15}$$

and

$$\begin{aligned}&\mathbb{E}\int_t^T\|Z^{n+1}(s)-Z^n(s)\|^2\mathrm{d}s\leqslant\\&8k^2(T-t)\,\mathbb{E}\int_t^T\|[f(s, Y^n(s), Z^{n+1}(s),Y^n(s+\mu(s)))-f(s, Y^{n-1}(s), Z^n(s),Y^{n-1}(s+\mu(s)))]\mathrm{d}s\|^2\mathrm{d}s\leqslant\\&24k^2C^2(T-t)(1+L)\,\mathbb{E}\int_t^T\|Y^n(s)-Y^{n-1}(s)\|^2\mathrm{d}s+24k^2C^2(T-t)\,\mathbb{E}\int_t^T\|Z^{n+1}(s)-Z^n(s)\|^2\mathrm{d}s\end{aligned}\tag{16}$$

Let $\eta=\dfrac{1}{48k^2C^2}$. For $t\in[T-\eta,T]$, we get

$$\mathbb{E}\int_t^T\|Z^{n+1}(s)-Z^n(s)\|^2\mathrm{d}s\leqslant\frac{1+L}{2}\,\mathbb{E}\int_t^T\|Y^n(s)-Y^{n-1}(s)\|^2\mathrm{d}s\tag{17}$$

$$\mathbb{E}\ \|Y^{n+1}(t)-Y^n(t)\|^2\leqslant\frac{3(1+L)}{16}\,\mathbb{E}\int_t^T\|Y^n(s)-Y^{n-1}(s)\|^2\mathrm{d}s\tag{18}$$

Let $u^n(t)=\int_t^T \mathbb{E}\ \| Y^n(s)-Y^{n-1}(s)\|^2 \mathrm{d}s$, it follows from (18) that

$$-\frac{\mathrm{d}u^{n+1}(t)}{\mathrm{d}t}\leqslant \frac{3(1+L)}{16}u^n(t),\ u^{n+1}(T)=0.$$

Let $C_1=\frac{3(1+L)}{16}$. Integration shows that

$$u^{n+1}(t)\leqslant C_1\int_t^T u^n(s)\mathrm{d}s.$$

Iterating the above inequality, we have

$$u^{n+1}(0)\leqslant \frac{C_1^n}{n!}\ u^1(0).$$

This implies that $\{Y^n\}$ is a Cauchy sequence in $L^2_{\mathscr{F}}(T-\eta,\ T+M;H)$. Then by (17), $\{Z^n\}$ is a Cauchy sequence in $L^2_{\mathscr{F}}(T-\eta,\ T+M;\ L^2(K;H))$. Let

$$Y:=\lim_{n\to\infty}Y^n,\ Z:=\lim_{n\to\infty}Z^n.$$

By virtue of Lipschitz assumption of f , one can easily check that for any $t\in[T-\eta,T+M]$, $\mathbb{P}$ -a. s. ,

$$\mathbb{E}\int_t^T \| U(s,t)f(s,\ Y^n(s),\ Z^{n+1}(s),Y^n(s+\mu(s)))-f(s,\ Y(s),\ Z(s),Y(s+\mu(s)))\|^2\mathrm{d}s\to 0,\quad n\to\infty.$$

Now, letting $n\to\infty$ in (14) yields that

$$\begin{cases} Y(t)=U(T,t)\xi(T)+\int_t^T U(s,t)\cdot \\ \qquad f(s,\ Y(s),\ Z(s),Y(s+\mu(s)))\mathrm{d}s- \\ \qquad \int_t^T U(s,t)Z(s)\mathrm{d}W(s),\ 0\leqslant t\leqslant T; \\ Y(t)=\xi(t),Z(t)=\eta(t),\ t\in[T,\ T+M], \end{cases}$$

as desired, which means that (Y,Z) solves the anticipated BSEE (5) for $t\in[T-\eta,T+M]$. For $t\in[T-2\eta,T-\eta]$, we can show the existence with the above procedure. Because η is fixed, we can establish the existence of solutions for BSEE (5) as above for $t\in[0,T+M]$. The proof is complete.

2 An example

In this section, an example is provided to illustrate the theory obtained.

Example 2. 1 Letting $D=[0,\pi]$, we consider the following anticipated backward stochastic partial differential equations of the form:

$$\left.\begin{aligned} &-\mathrm{d}v(t,\xi)=\left[\frac{\partial^2}{\partial x^2}v(t,\xi)+b(t,\xi)v(t,\xi)\right]\mathrm{d}t+ \\ &F(t,\xi,v(t,\xi),z(t,\xi),v(t+M,\xi))\mathrm{d}t- \\ &z(t,\xi)\mathrm{d}W(t),\ 0\leqslant\xi\leqslant\pi,t\in[0,T]; \\ &v(t,0)=v(t,\pi)=0,\ t\in[0,T]; \\ &v(t,\xi)=\phi(t,\xi),\ z(t)=\eta(t,\xi), \\ &t\in[T,\ T+M],\ 0\leqslant\xi\leqslant\pi \end{aligned}\right\}\tag{19}$$

where $W(t)$ denotes a standard cylindrical Wiener process defined on a complete probability space $(\Omega,\mathscr{F},P)$, $\phi\in L^2_{\mathscr{F}}(T,\ T+M;\mathbb{R})$, $\eta\in L^2_{\mathscr{F}}(T,\ T+M;\ L^2(\mathbb{R};L^2(D)))$, $v:[0,T+M]\times D\to\mathbb{R}$, $z:\ [0,T+M]\times D\to L^2(\mathbb{R};L^2(D))$, $F(s,\xi,\ y,\ z,\eta):[0,T]\times D\times\mathbb{R}\times L^2(\mathbb{R};L^2(D))\times L^2_{\mathscr{F}}(s,\ T+M;\ L^2(\mathbb{R};L^2(D)))\to\mathbb{R}$.

To rewrite (19) into the abstract form of (5), we consider the space $U=L^2(D)$ and define the operator $A:D(A)\subset U\to U$ by $Az=z''$ with domain $D(A)=\{z\in U,\ z,\ z'$ being absolutely continuous $z''\in U,\ z(0)=z(\pi)=0\}$.

Then, A is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geqslant 0}$ on U. A has a discrete spectrum with eigenvalues $-n^2$, $n\in\mathbb{N}$, and the corresponding normalized eigenfunctions is given by $z_n(\xi)=\sum_{n=1}^{\infty}\exp(-n^2t)\sin(n\xi)$. Moreover, $\{z_n,\ n\in\mathbb{N}\}$ is an orthonormal basis of U.

Now, we define an operator $A(t):D(A)\subset H\to H$ by

$$A(t)x(\xi)=Ax(\xi)+b(t,\xi)x(\xi).$$

Let $b(\cdot)$ be continuous and $b(t,\xi)\leqslant-\gamma(\gamma>0)$ for every $t\in\mathbb{R}$. Then, the system

$$\left.\begin{aligned} &v'(t)=A(t)v(t),\ t\geqslant s; \\ &v(s)=x\in U \end{aligned}\right\}$$

has an associated evolution family given by

$$U(t,s)x(\xi)=[S(t-s)\exp\Big(\int_s^t b(\tau,\xi)\mathrm{d}\tau\Big)x](\xi).$$

From the above expression, it follows that $U(t,s)$ is exponentially stable, and for every $t,s\in J$ with $t>s$

$$\|U(t,x)\| \leqslant \exp(-(1+\gamma)(t-s)).$$

Letting $H = L^2(D), K = \mathbb{R}$, $Y(t)(\cdot) = v(t,\cdot), Z(t)(\cdot) = z(t,\cdot)$, and defining a map $f(s,\omega,y,z,\xi):[0,T]\times\Omega\times H\times L^2(K;H)\times L^2_{\mathcal{F}}(s,T+M;H)\rightarrow H$ by

$$f(t,Y(t),Z(t),Y(t+M))(\xi) = F(t,\xi,v(t,\xi),z(t,\xi),v(t+M,\xi)).$$

Then, we can rewrite (19) as the form of (5). Furthermore, if we impose the conditions on F, ϕ and η are similar to (H1) and (H2). Then, by Theorem 2.1, we can conclude that the system (19) has a unique evolution solution $(Y,Z) \in L^2_{\mathcal{F}}(0,T+M;L^2(\Omega,L^2(D)))\times L^2_{\mathcal{F}}(0,T+M;L^2(\mathbb{R},L^2(\Omega,L^2(D))))$.

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