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Spherically symmetric Finsler metrics with scalar flag curvature

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Abstract: Finsler geometry is Riemannian geometry without quadratic restriction, and the projectively flat Finsler metrics are very important in Finsler geometry. Here a new example of projectively flat spherically symmetric Finsler metric was given by investigating a PDE equivalent, and using Shen Zhongmin's result, its flag curvature was obtained.

Key words: locally projectively flat; spherically symmetric; error function; scalar flag curvature

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具有标量旗曲率的球对称 Finsler 度量

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摘要:Finsler 几何是没有二次型限制的黎曼几何,射影平坦是 Finsler 几何中非常重要的问题.通过对一个微分方程的研究得到了新的球对称射影平坦的 Finsler 度量并利用沈忠民的结果得到其旗曲率.

关键词:局部射影平坦;球对称;误差函数;标量旗曲率

0 Introduction

It is an important problem in Finsler geometry to study and characterize projectively flat Finsler metrics on an open domain in R^n. The flag curvature is the most important Riemannian quantity in Finsler geometry because it is an analogue of sectional curvature in Riemannian geometry[1].

Huang and Mo[2] has discussed a class of interesting Finsler metrics. They are of the form $F=|y|\phi(|x|,\frac{\langle x,y\rangle}{|y|})$, such metrics are said to be spherically symmetric in Ref. [2]. Obviously, spherically symmetric Finsler metrics are the simplest and most important general (α,β)-metrics[3].

Recently, the study of spherically symmetric Finsler metrics has attracted a lot of attention. Refs. [4-6] classified the projective spherically

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symmetric Finsler metrics with constant flag curvature.

Ref. [2] obtained the second differential equation for F to be projectively flat.

Lemma 0.1[2] Let

$$F(x,y)=|y|\phi(|x|,\frac{\langle x,y\rangle}{|y|})$$

be a spherically symmetric Finsler metric on $B^n(r)\subset R^n$. Then $F=F(x,y)$ is projectively flat if and only if $\phi=\phi(s)$ satisfies

$$s\phi_{bs}+b\phi_{ss}-\phi_b=0 \tag{1}$$

where $b:=\|\beta\|_\alpha, s:=\frac{\beta}{\alpha}$. Note that ϕ_b means derivation of ϕ with respect to the first variable b.

By solving the equation which characterizes projectively flat spherically symmetric Finsler matric, Refs. [7-8] manufacture projectively flat Finsler metrics using the integral form. In this paper, by investigating PDE(1), we manufacture projectively flat spherically symmetric Finsler metrics in terms of error functions. We have the following:

Theorem 0.1 Let

$$F(x,y)=|y|\phi(|x|,\frac{\langle x,y\rangle}{|y|})$$

be a spherically symmetric Finsler metric. $\phi(b,s)$ be a function defined by

$$\phi(b,s)=-\sqrt{\pi}\,e^{-b^2}s\cdot\frac{1}{i}\operatorname{erf}(is)+e^{s^2-b^2}+Cs \tag{2}$$

where C is an arbitrary constant and i is imaginary, then it has the following properties:

① The spherically symmetric Finsler metric given in Eq. (2) is projectively flat on $B^n(r)\subset R^n$, and its projective factor P is given by

$$P=-\langle x,y\rangle+\frac{|y|}{2}H \tag{3}$$

② F is of scalar flag curvature and its flag curvature is given by

$$K=\frac{s^2+1+\frac{3}{4}H^2-sH}{\phi^2}-\frac{s(6C+4Cs^2)-2e^{s^2-b^2}}{2\phi^3} \tag{4}$$

where

$$H=\frac{2Cs^2+C-\frac{1}{i}\sqrt{\pi}\,e^{-b^2}\operatorname{erf}(is)}{\phi} \tag{5}$$

1 Preliminaries

A Finsler metric is a Riemannian metric without quadratic restriction[9]. Precisely, a function $F(x,y)$ on TM is called a Finsler metric on a manifold M if it has the following properties:

① regularity: $F(x,Y)$ is C^∞ on $TM\backslash\{0\}$;

② positive homogeneity: $F(ty)=tF(y),\forall t>0, y\in T_xM$;

③ strong convexity: the $n\times n$ matrix $(\frac{\partial^2F^2}{\partial y^i\partial y^j}(x,y))(y\neq 0)$ is positively definite.

A Finsler metric is said to be locally projectively flat if at any point there is a local coordinate in which the geodesics are straight lines as point sets. It is known that every locally projectively flat Finsler metrics is of scalar curvature[9].

A Finsler metric $F=F(x,y)$ on an open domain $U\subset R^n$ is said to be projectively flat in U if all geodesics are straight lines. Let G^i denote the spray coefficients of F, which are given by

$$G^i=\frac{1}{4}g^{il}\{[F^2]_{x^my^l}y^m-[F^2]_{x^l}\} \tag{6}$$

where $(g^{ij})=(\frac{1}{2}[F^2]_{y^iy^j})$. In this case the flat curvature K is a scalar function on TU given by

$$K=\frac{P^2-P_{x^m}y^m}{F^2} \tag{7}$$

where

$$P=\frac{F_{x^k}y^k}{2F} \tag{8}$$

is said to be projective factor[10].

We need the following lemma for later use.

Lemma 1.1[11] Let M be an n-dimensional mainfold. $F=\alpha\phi(b,\frac{\beta}{\alpha})$ is a Finsler metric on M for any Riemannian metric α and 1-form β with $\|\beta\|_\alpha<b_0$ if and only if $\phi=\phi(b,s)$ is a positive C^∞ function satisfying

$$\phi-s\phi_2>0, \phi-s\phi_2+(b^2-s^2)\phi_{22}>0,$$

when $n \geqslant 3$ or

$$\phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0,$$

when $n = 2$, where s and b are arbitrary numbers with $|s| \leqslant b < b_0$. ϕ_2 means derivation of ϕ with respect to the second variable s.

2 The solution to the PDE

In order to solve the following linear PDE

$$s\phi_{bs} + b\phi_{ss} - \phi_b = 0 \tag{9}$$

Let

$$\psi(b,s) = \phi - s\phi_s \tag{10}$$

then

$$\psi_s = \phi_s - \phi_s - s\phi_{ss} = -s\phi_{ss} \tag{11}$$

$$\psi_b = \phi_b - s\phi_{sb} \tag{12}$$

Substituting Eqs. (11),(12) into Eq. (9) gives

$$s\psi_b + b\psi_s = 0 \tag{13}$$

The special solution to Eq. (13) is given by

$$\psi(b,s) = e^{s^2-b^2} \tag{14}$$

Substituting Eq. (14) into Eq. (10) gives

$$\phi - s\phi_s = e^{s^2-b^2} \tag{15}$$

Eq. (15) is a nonhomogeneous linear differential equation, whose corresponding to the homogeneous linear differential equation is given by

$$\phi - s\phi_s = 0 \tag{16}$$

The general solution to Eq. (16) is given by $\phi = Cs$. Then let

$$\phi = U(s)s \tag{17}$$

Substituting Eq. (17) into Eq. (15) gives

$$U' = -\frac{1}{s^2}e^{s^2-b^2}.$$

Then

$$U(s) = \frac{1}{s}e^{s^2-b^2} - 2e^{-b^2}\int e^{s^2}\,ds.$$

Thus Eq. (15) can be solved as

$$\phi(b,s) = -\sqrt{\pi}\,e^{-b^2}s \cdot \frac{1}{i}\operatorname{erf}(is) + e^{s^2-b^2} + Cs \tag{18}$$

where erf(,) denotes the error function and it is defined by

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt \tag{19}$$

and C is an arbitrary constant. Thus (18) is the solution of (9).

3 Proof

Now we will manufacture a class of projectively flat Finsler metric by Eq. (18), and we will get its Scalar flag curvature.

Let

$$F = \alpha\phi(b,s): T\Omega \to [0, +\infty),$$
$$\alpha = |y|, \beta = \langle x, y\rangle,$$
$$\phi(b,s) = -\sqrt{\pi}\,e^{-b^2}s \cdot \frac{1}{i}\operatorname{erf}(is) + e^{s^2-b^2} + Cs \tag{20}$$

Direct computations yield

$$\phi - s\phi_2 = e^{s^2-b^2} > 0,$$
$$\phi - s\phi_2 + (b^2 - s^2)\phi_{22} = (1 - b^2 + s^2)e^{s^2-b^2} > 0.$$

By Lemmas 0.1 and 1.1, we know

$$F = \alpha\phi(b,s) = \alpha\left(-\sqrt{\pi}\,e^{-b^2}s \cdot \frac{1}{i}\operatorname{erf}(is) + e^{s^2-b^2} + Cs\right) \tag{21}$$

is a projectively flat spherically symmetric Finsler metric.

By a simple calculation, we get

$$\left.\begin{aligned} &\alpha_{x^k}y^k = 0, s_{x^k}y^k = |y|, [s^2]_{x^k}y^k = 2\langle x,y\rangle, \\ &b_{x^k}y^k = \frac{\langle x,y\rangle}{|x|}, [b^2]_{x^k}y^k = 2\langle x,y\rangle \end{aligned}\right\} \tag{22}$$

$$\begin{aligned} F_{x^k}y^k = {} & \alpha\Big[\sqrt{\pi}\,e^{-b^2}[b^2]_{x^k}y^k s \cdot \frac{1}{i}\operatorname{erf}(is) - \\ & \sqrt{\pi}\,e^{-b^2} \cdot \frac{1}{i}\operatorname{erf}(is)s_{x^k}y^k - \\ & 2se^{s^2-b^2}|y| + e^{s^2-b^2}[s^2]_{x^k}y^k - \\ & e^{s^2-b^2}[b^2]_{x^k}y^k + Cs_{x^k}y^k\Big] = \\ & \alpha\Big[\sqrt{\pi}\,e^{-b^2}s \cdot \frac{1}{i}\operatorname{erf}(is)2\langle x,y\rangle - \\ & \sqrt{\pi}\,e^{-b^2}\frac{1}{i}\operatorname{erf}(is)|y| - 2se^{s^2-b^2}|y| + C|y|\Big] = \\ & -2\langle x,y\rangle\alpha\phi(b,s) + \\ & \alpha\left(2Cs^2 + C - \sqrt{\pi}\,e^{-b^2} \cdot \frac{1}{i}\operatorname{erf}(is)\right)|y| \end{aligned} \tag{23}$$

$$\begin{aligned} P = \frac{F_{x^k}y^k}{2F} = {} & -\langle x,y\rangle + \frac{2Cs^2 + C - \sqrt{\pi}\,e^{-b^2} \cdot \frac{1}{i}\operatorname{erf}(is)}{2\phi}|y| = \\ & -\langle x,y\rangle + \frac{|y|}{2}H \end{aligned} \tag{24}$$

where

$$H=\frac{2Cs^2+C-\sqrt{\pi}\,\mathrm{e}^{-b^2}\cdot\frac{1}{i}\mathrm{erf}(is)}{\phi} \tag{25}$$

$$H_{x^k}y^k=\frac{4C\langle x,y\rangle+2\sqrt{\pi}\,\mathrm{e}^{-b^2}\cdot\frac{1}{i}\mathrm{erf}(is)\langle x,y\rangle-2\mathrm{e}^{s^2-b^2}\,|y|}{\phi}+H(2\langle x,y\rangle-H\,|y|)=\frac{(6C+4Cs^2)\langle x,y\rangle-2\mathrm{e}^{s^2-b^2}\,|y|}{\phi}-H^2\,|y| \tag{26}$$

By Eq. (26), we get

$$P_{x^k}y^k=-|y|^2+\frac{|y|}{2}H_{x^k}y^k=-|y|^2+\frac{|y|}{2}\left(\frac{(6C+4Cs^2)\langle x,y\rangle-2\mathrm{e}^{s^2-b^2}\,|y|}{\phi}-H^2\,|y|\right) \tag{27}$$

$$P^2-P_{x^k}y^k=\langle x,y\rangle^2+\left(1+\frac{3}{4}H^2\right)|y|^2-\langle x,y\rangle\,|y|\left(H+\frac{6C+4Cs^2}{2\phi}\right)+\frac{\mathrm{e}^{s^2-b^2}}{\phi}\,|y|^2 \tag{28}$$

By Eqs. (7) and (28), we get

$$K=\frac{P^2-P_{x^k}y^k}{F^2}=\frac{\langle x,y\rangle^2+\left(1+\frac{3}{4}H^2\right)|y|^2-\langle x,y\rangle\,|y|\left(H+\frac{6C+4Cs^2}{2\phi}\right)+\frac{\mathrm{e}^{s^2-b^2}}{\phi}\,|y|^2}{F^2}=\frac{s^2+1+\frac{3}{4}H^2-sH}{\phi^2}-\frac{s(6C+4Cs^2)-2\mathrm{e}^{s^2-b^2}}{2\phi^3}.$$

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