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The expected discounted penalty function under the compound Poisson risk model with tax payments and a threshold dividend strategy

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Abstract: The compound Poisson risk model was considered in which taxes were paid according to a loss-carry forward system and dividends were paid under a threshold strategy. For this model, the ruin quantities were discussed by defining an expected discounted penalty function at ruin and the analytical integro-differential equation satisfied by the expected discounted penalty function was derived. Finally, in the case where the individual claims follow an exponential distribution, explicit expressions for the ruin probability were given.

Key words: compound Poisson risk process; expected discounted penalty function; threshold dividend strategy

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带赋税与门槛分红的复合泊松风险模型的Gerber-Shiu函数

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摘要:研究了一类复合泊松风险模型,其在安全负载体系下进行赋税,且按门槛策略进行分红.讨论了此模型破产时的变量期望折现罚金函数且得到了此函数满足的积分-微分方程和相关的表达式.最后,在单独索赔量为指数分布的特例下,给出了破产概率的一般表达式.

关键词:复合 Poisson 风险过程;期望折现罚金函数;门槛分红策略

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0 Introduction

The compound Poisson risk model (or the classical Cramér-Lundberg risk model) describes the surplus process of an insurance portfolio by a stochastic process $U=\{U(t), t\geqslant 0\}$ with

$$R(t)=u+ct-S(t) \tag{1}$$

where $u\geqslant 0$ is the initial capital, the constant $c>0$ is the premium rate, and $S(t):=\sum_{n=1}^{N(t)}X_n$ is aggregate amounts of claims with the innovation number process $\{N(t), t\geqslant 0\}$ being a Poisson process (with jump intensity $\lambda>0$) denoting the number of claims up to time t, and $\{X_n, n\geqslant 1\}$ (representing the amounts of claims and independent of $\{N(t), t\geqslant 0\}$) being a sequence of independent and identically distributed nonnegative random variables with a common distribution function F.

Due to its practical importance, the surplus process with certain dividend strategies has been receiving remarkable attention. In his original paper, De Finetti[1] laid the foundations of what would become an increasingly popular branch of risk theory: dividend strategies. Gerber et al.[2] considered the compound Poisson risk model with a constant dividend barrier and obtained the optimal constant dividend barrier level. In the well cited paper of Gerber et al.[3], the so-called expected discounted penalty function (also called Gerber-Shiu function) is introduced and in which an obvious direction is to turn one's attention to the the joint distribution of the time of ruin, the deficit at ruin and the wealth prior to ruin. Lin et al.[4] also considered the compound Poisson risk model with constant dividend barrier, and analyzed the Gerber-Shiu function. Lin et al.[5] considered the compound Poisson risk model with the threshold strategy and discussed the Gerber-Shiu function as well. More results on dividend strategies can be found in Refs. [6-12].

In recent papers, the loss-carry forward tax system (the amount of tax payments should not lead to bunkruptsy) has been investigated extensively. Albrecher et al.[13] discussed how tax payments affect the behavior of a compound Poisson surplus process, a remarkably simple relationship between the ruin probabilities of the surplus process with and without tax was established. More recent papers[14-20] on this topic can be found.

In our model, we consider a compound Poisson risk model with taxes paid according to the loss-carry forward tax system and dividends paid under a threshold strategy. The loss-carry forward tax system is as follows: taxes are paid at a fixed rate $\gamma\in[0,1)$ of the insurer's income, whenever the surplus is at a running maximum (or, the portfolio is in a profitable situation): $R_{\gamma,\alpha,b}(t)=\max\{R_{\gamma,\alpha,b}(s): s\leqslant t\}$. Meanwhile, when the surplus reaches a barrier of constant level b, dividends are distributed at a constant rate $\alpha<c(1-\gamma)$, where $c>0$ is the premium rate in the classical compound Poisson risk model. The dynamics of the surplus process $\{R_{\gamma,\alpha,b}(t), t\geqslant 0\}$ thus are determined by

$$\left.\begin{aligned}
&dR_{\gamma,\alpha,b}(t)= \\
&\quad (c-\alpha-c\gamma\mathbf{1}_{\{R_{\gamma,\alpha,b}(t)=\max_{0\leqslant s\leqslant t}R_{\gamma,\alpha,b}(s)\}})\mathbf{1}_{\{R_{\gamma,\alpha,b}(t)\geqslant b\}}dt+ \\
&\quad (c-c\gamma\mathbf{1}_{\{R_{\gamma,\alpha,b}(t)=\max_{0\leqslant s\leqslant t}R_{\gamma,\alpha,b}(s)\}})\mathbf{1}_{\{R_{\gamma,\alpha,b}(t)<b\}}dt- \\
&\quad d(\sum_{n=1}^{N(t)}X_n), \\
&R_{\gamma,\alpha,b}(0)=u
\end{aligned}\right\} \tag{2}$$

where $u\geqslant 0$ is the initial capital, $c>0$ the constant premium rate, α the threshold dividend rate, $\mathbf{1}_A$ the indicator function of a set A, $\{N(t), t\geqslant 0\}$ a Poisson process with intensity $\lambda>0$ denoting the number of claims up to time t, and $\{X_n, n\geqslant 1\}$, representing the amounts of claims and being independent of $\{N(t), t\geqslant 0\}$, a sequence of independent and identically distributed nonnegative random variables with a common distribution function $F(x)$ which has a positive mean

$$\mu=\int_0^\infty \overline{F}(x)dx<\infty.$$

Here, $\overline{F}(x)=1-F(x)$ is the tail function of $F(x)$.

We denote the time of ruin by $T_{\gamma,\alpha,b}$, that is

$$T_{\gamma,\alpha,b}=\inf\{t: R_{\gamma,\alpha,b}(t)<0\} \tag{3}$$

and $T_{\gamma,\alpha,b}=\infty$ if $R_{\gamma,\alpha,b}(t)\geqslant 0$ for all $t\geqslant 0$. Clearly, $R_{\gamma,\alpha,b}(T_{\gamma,\alpha,b}-)$ and $|R_{\gamma,\alpha,b}(T_{\gamma,\alpha,b})|$ are the surplus immediately prior to ruin and the deficit at ruin. We define the Gerber-Shiu function by

$$\Phi_{\gamma,\alpha,b}(u)=E[e^{-\delta T_{\gamma,\alpha,b}}w(R_{\gamma,\alpha,b}(T_{\gamma,\alpha,b}-), |R_{\gamma,\alpha,b}(T_{\gamma,\alpha,b})|)\mathbf{1}_{\{T_{\gamma,\alpha,b}<\infty\}} \mid R_{\gamma,\alpha,b}(0)=u] \tag{4}$$

where $w(x_1,x_2)$, $x_1\geqslant 0$, $x_2>0$, is a nonnegative function which denotes the penalty due at ruin, $\delta\geqslant 0$ can be viewed as the argument for the Laplace transform of $T_{\gamma,\alpha,b}$ or an interest force for the calculation of the present value of the penalty. It is clear that the ruin probability for the process $\{R_{\gamma,\alpha,b}(t), t\geqslant 0\}$, denoted by

$$\Psi_{\gamma,\alpha,b}(u)=P(T_{\gamma,\alpha,b}<\infty \mid R_{\gamma,\alpha,b}(0)=u):=P_u(T_{\gamma,\alpha,b}<\infty),$$

is obtained from $\Phi_{\gamma,\alpha,b}(u)$ by letting $\delta\downarrow 0$ and $w\equiv 1$.

We write

$$\Phi_{\gamma,\alpha,b}(u)=(\Phi_{\gamma,\alpha,b})_1(u),$$
$$\Psi_{\gamma,\alpha,b}(u)=(\Psi_{\gamma,\alpha,b})_1(u)$$

for $0\leqslant u<b$ and

$$\Phi_{\gamma,\alpha,b}(u)=(\Phi_{\gamma,\alpha,b})_2(u),$$
$$\Psi_{\gamma,\alpha,b}(u)=(\Psi_{\gamma,\alpha,b})_2(u)$$

for $u\geqslant b$. We shall drop the subscripts γ and α whenever γ and α are zero, respectively, and drop the subscript b whenever b tends to infinity.

Throughout this paper, we assume that the safety loading factor defined by

$$\theta=\frac{c-\alpha-\lambda\mu}{\lambda\mu} \tag{5}$$

is positive. We also assume that $\lim_{u\to\infty}\Phi_{\gamma,\alpha,b}(u)=0$, which holds naturally when $w(x_1,x_2)$ is a bounded function.

The rest of the paper is organized as follows. In Section 1, analytic expressions for the expected discounted penalty function at ruin, that is, $(\Phi_{\gamma,\alpha,b})_1(u)$ and $(\Phi_{\gamma,\alpha,b})_2(u)$ are derived. In Section 2, for the case that individual claims follow an exponential distribution, explicit expressions for the ruin probability are given.

1　Analytical expressions for the expected discounted penalty function

In this section, we derive the analytical expressions for $(\Phi_{\gamma,\alpha,b})_1(u)$ and $(\Phi_{\gamma,\alpha,b})_2(u)$. It will turn out that $\Phi_{\gamma,\alpha,b}(u)$ is related to $\Phi_{\alpha,b}(u)$ which is extensively investigated in Ref. [5].

Let

$$B_\gamma(u,u_0):=E[e^{-\delta\tau_\gamma(u,u_0)}] \tag{6}$$

denote the Laplace-stieltjes transform of the upper exit time $\tau_\gamma(u,u_0)$ which is the time until the surplus process $\{R_\gamma(t), t\geqslant 0\}$ (with premium rate c and tax rate γ) starting with initial surplus $u<u_0$ reaches u_0 without leading to ruin before that event.

In the following Proposition 1.1 we derive the analytical expression of $B_\gamma(u,u_0)$, which plays an instrumental role in analyzing the expected discounted penalty function.

Proposition 1.1　The resulting Laplace-Stieltjes transform of the upper exit time $\tau_\gamma(u,u_0)$ is a power of that of the upper exit time $\tau(u,u_0)$, that is

$$B_\gamma(u,u_0)=(B(u,u_0))^{\frac{1}{1-\gamma}}=\left(\frac{h(u)}{h(u_0)}\right)^{\frac{1}{1-\gamma}} \tag{7}$$

where $h(u)$ is the solution of the integro-differential equation

$$ch'(x)-(\lambda+\delta)h(x)+\lambda\int_0^x h(x-y)\mathrm{d}F(y)=0 \tag{8}$$

Proof　It follows for instance from Gerber et al.[3] that the second equality in (7) holds. Now, if we condition on the time and the amount of the first claim when $0\leqslant u<u_0$, contingent on this time, there are two options: the first claim occurs before the surplus has attained the level u_0 or it occurs after the surplus attained the level u_0. For the amount of the first claim, there are two possibilities as well: after the claim the process $\{R_\gamma(t), t\geqslant 0\}$ still operates or the first claim leads to ruin. Implementing these considerations and

using the conditioning technique we obtain

$$E[e^{-\delta\tau_\gamma(u,u_0)}]=$$

$$e^{-\lambda\frac{u_0-u}{c(1-\gamma)}}E\left[e^{-\delta\tau_\gamma(u,u_0)}\,\middle|\,T_1>\frac{u_0-u}{c(1-\gamma)}\right]+$$

$$\int_0^{\frac{u_0-u}{c(1-\gamma)}}\lambda e^{-(\lambda+\delta)t}dt\cdot$$

$$\int_0^{u+c(1-\gamma)t}E[E[e^{-\delta\tau_\gamma(u,u_0)}\mathbf{1}_{\{R_\gamma \text{will attain the level } u_0 \text{ before ruin}\}}\mid$$

$$t+\tau(u+c(1-\gamma)t-x,u+c(1-\gamma)t)]\mid$$

$$X_1=x,T_1=t]dF(x)=$$

$$e^{-\lambda\frac{u_0-u}{c(1-\gamma)}}e^{-\delta\frac{u_0-u}{c(1-\gamma)}}+\int_0^{\frac{u_0-u}{c(1-\gamma)}}\lambda e^{-(\lambda+\delta)t}dt\cdot$$

$$\int_0^{u+c(1-\gamma)t}B(u+c(1-\gamma)t-x,u+c(1-\gamma)t)\cdot$$

$$B_\gamma(u+c(1-\gamma)t,u_0)dF(x)$$

Changing variables $s=u+c(1-\gamma)t$ and applying the well known identity $B(u_1,u_2)=\frac{h(u_1)}{h(u_2)}$, $0\leqslant u_1\leqslant u_2$, we have

$$B_\gamma(u,u_0)=e^{-(\lambda+\delta)\frac{u_0-u}{c(1-\gamma)}}+e^{\frac{\lambda+\delta}{c(1-\gamma)}u}\frac{\lambda}{c(1-\gamma)}\cdot$$

$$\int_u^{u_0}e^{-\frac{\lambda+\delta}{c(1-\gamma)}s}B_\gamma(s,u_0)ds\int_0^s\frac{h(s-x)}{h(s)}dF(x)\quad(9)$$

Multiplying Eq. (9) with $e^{-\frac{\lambda+\delta}{c(1-\gamma)}u}$ then differentiating it with respect to u leads to

$$\frac{\frac{\partial}{\partial u}B_\gamma(u,u_0)}{B_\gamma(u,u_0)}=\frac{1}{c(1-\gamma)}\frac{1}{h(u)}\cdot$$

$$((\lambda+\delta)h(u)-\lambda\int_0^u h(u-x)dF(x)).$$

Applying (8) and using the boundary condition $B_\gamma(u_0,u_0)=1$ leads to (7). Proposition 1.1 is proved. □

Let

$$B^{\alpha,b}(u,u_0):=E[e^{-\delta\tau_{\alpha,b}(u,u_0)}]\quad(10)$$

denote the Laplace-stieltjes transform of the upper exit time $\tau_{\alpha,b}(u,u_0)$ which is the time until the surplus process $\{R_{\alpha,b}(t),t\geqslant0\}$ (with premium rate c, dividend rate α and threshold b) starting with initial surplus $u<u_0$ reaches $u_0\geqslant b$ without leading to ruin before that event. Clearly, if we let $\delta\downarrow0$ in $B^{\alpha,b}(u,u_0)$, it is reduced to the probability that the surplus process $\{R_{\alpha,b}(t),t\geqslant0\}$ starting from initial surplus $u<u_0$ reaches $u_0\geqslant b$ before ruin, which is denoted by $(B^{\alpha,b})_0(u,u_0)$. We write

$$B^{\alpha,b}(u,u_0)=B_1^{\alpha,b}(u,u_0),$$

$$(B^{\alpha,b})_0(u,u_0)=(B_1^{\alpha,b})_0(u,u_0)$$

for $0\leqslant u<b$ and

$$B^{\alpha,b}(u,u_0)=B_2^{\alpha,b}(u,u_0),$$

$$(B^{\alpha,b})_0(u,u_0)=(B_2^{\alpha,b})_0(u,u_0)$$

for $u\geqslant b$.

Now we provide integro-differential equations for the function $B^{\alpha,b}(u,u_0)$ in the following Proposition 1.2 (proved by Ref. [19]), which will help us to prove Theorem 1.1.

Proposition 1.2 The function $B^{\alpha,b}(u,u_0)$ satisfies the following integro-differential equations.

When $0\leqslant u<b$,

$$\frac{\partial}{\partial u}B_1^{\alpha,b}(u,u_0)=\frac{\lambda+\delta}{c}B_1^{\alpha,b}(u,u_0)-\frac{\lambda}{c}\int_0^u B_1^{\alpha,b}(u-x,u_0)dF(x)\quad(11)$$

and, when $u\geqslant b$,

$$\frac{\partial}{\partial u}B_2^{\alpha,b}(u,u_0)=\frac{\lambda+\delta}{c-\alpha}B_2^{\alpha,b}(u,u_0)-\frac{\lambda}{c-\alpha}\Big(\int_0^{u-b}B_2^{\alpha,b}(u-x,u_0)dF(x)+\int_{u-b}^u B_1^{\alpha,b}(u-x,u_0)dF(x)\Big)\quad(12)$$

Now, we derive the expressions of $(\Phi_{\gamma,\alpha,b})_1(u)$ and $(\Phi_{\gamma,\alpha,b})_2(u)$ as follows.

Theorem 1.1 When $0\leqslant u<b$,

$$(\Phi_{\gamma,\alpha,b})_1(u)=\Phi_\gamma(u)-\left(\frac{h(u)}{h(b)}\right)^{\frac{1}{1-\gamma}}\Phi_\gamma(b)+\left(\frac{h(u)}{h(b)}\right)^{\frac{1}{1-\gamma}}(\Phi_{\gamma,\alpha,b})_2(b)\quad(13)$$

and, when $u\geqslant b$,

$$(\Phi_{\gamma,\alpha,b})_2(u)=\frac{c-\alpha}{c(1-\gamma)-\alpha}(\Phi_{\alpha,b})_2(u)-\frac{c\gamma}{c(1-\gamma)-\alpha}\cdot\exp\{\int_0^u M(t)dt\}\int_u^\infty(\Phi_{\alpha,b})_2(s)M(s)\cdot\exp\{-\int_0^s M(t)dt\}ds\quad(14)$$

where

$$M(t)=\frac{1}{c(1-\gamma)-\alpha}\{\lambda+\delta-\lambda(\int_0^{t-b}B_2^{\alpha,b}(t-x,t)dF(x)+\int_{t-b}^t B_1^{\alpha,b}(t-x,t)dF(x))\}\quad(15)$$

$\Phi_\gamma(u)$ is given by Ref. [18, Eq. (10)], that is

$$\Phi_\gamma(u) = \frac{1}{c(1-\gamma)}\exp\left\{-\frac{1}{c(1-\gamma)}\int_0^u V_1(s)\mathrm{d}s\right\}\cdot \int_u^\infty V_2(t)\exp\left\{\frac{1}{c(1-\gamma)}\int_0^t V_1(s)\mathrm{d}s\right\}\mathrm{d}t,$$

with

$$V_1(t) = \lambda\int_0^t B(t-x,t)\mathrm{d}F(x) - (\lambda+\delta),$$

$$V_2(t) = (\lambda+\delta-\lambda\int_0^t B(t-x,t)\mathrm{d}F(x))\Phi(t) - c\Phi'(t).$$

Proof　When $0\leqslant u<b$, let $A:=\{$the process $\{R_{\gamma,\alpha,b}(s), s\geqslant 0\}$ will attain level b before ruin$\}$ and A^c the complement of A. Clearly, if the process $\{R_{\gamma,\alpha,b}(t), t\geqslant 0\}$ does not attain the level b until ruin, then the trajectories (until ruin) of the process $\{R_{\gamma,\alpha,b}(t), t\geqslant 0\}$ is identical to those of the process $\{R_\gamma(t), t\geqslant 0\}$. Implementing the considerations above we have

$$\begin{aligned}(\Phi_{\gamma,\alpha,b})_1(u) = {} & E_u[E_u[\mathrm{e}^{-\delta T_{\gamma,\alpha,b}} w(R_{\gamma,\alpha,b}(T_{\gamma,\alpha,b}-), |R_{\gamma,\alpha,b}(T_{\gamma,\alpha,b})|)\mathbf{1}_{\{T_{\gamma,\alpha,b}<\infty\}}\mathbf{1}_{\{A\}} \mid \tau_\gamma(u,b)]] + \\ & E_u[\mathrm{e}^{-\delta T_\gamma} w(R_\gamma(T_\gamma-), |R_\gamma(T_\gamma)|)\mathbf{1}_{\{T_\gamma<\infty\}}\mathbf{1}_{\{R_\gamma \text{never attains level } b \text{ before ruin}\}}] = \\ & B_\gamma(u,b)(\Phi_{\gamma,\alpha,b})_2(b) + E_u[\mathrm{e}^{-\delta T_\gamma} w(R_\gamma(T_\gamma-), |R_\gamma(T_\gamma)|)\mathbf{1}_{\{T_\gamma<\infty\}}] - \\ & E_u[\mathrm{e}^{-\delta T_\gamma} w(R_\gamma(T_\gamma-), |R_\gamma(T_\gamma)|)\mathbf{1}_{\{T_\gamma<\infty\}}\mathbf{1}_{\{R_\gamma \text{will attain } b \text{ before ruin}\}}] = \\ & B_\gamma(u,b)(\Phi_{\gamma,\alpha,b})_2(b) + \Phi_\gamma(u) - B_\gamma(u,b)\Phi_\gamma(b).\end{aligned}$$

Applying Proposition 1.1, one can arrive at (13).

When $u\geqslant b$, by considering whether or not there is a claim during the infinitesimal time interval from 0 to $\mathrm{d}t$ and using the conditioning idea of Ref.[17], we have

$$\begin{aligned}(\Phi_{\gamma,\alpha,b})_2(u) = {} & (1-\lambda\mathrm{d}t)\mathrm{e}^{-\delta\mathrm{d}t}(\Phi_{\gamma,\alpha,b})_2(u+(c(1-\gamma)-\alpha)\mathrm{d}t) + \\ & \lambda\mathrm{d}t\mathrm{e}^{-\delta\mathrm{d}t}\left\{\int_{u+(c(1-\gamma)-\alpha)\mathrm{d}t}^\infty w(u+(c(1-\gamma)-\alpha)\mathrm{d}t, x-u-(c(1-\gamma)-\alpha)\mathrm{d}t)\mathrm{d}F(x) + \right. \\ & \int_{u+(c(1-\gamma)-\alpha)\mathrm{d}t-b}^{u+(c(1-\gamma)-\alpha)\mathrm{d}t}(B_1^{\alpha,b}(u+(c(1-\gamma)-\alpha)\mathrm{d}t-x, u+(c(1-\gamma)-\alpha)\mathrm{d}t)(\Phi_{\gamma,\alpha,b})_2(u+(c(1-\gamma)-\alpha)\mathrm{d}t) + \\ & (\Phi_{\alpha,b})_1(u+(c(1-\gamma)-\alpha)\mathrm{d}t-x) - B_1^{\alpha,b}(u+(c(1-\gamma)-\alpha)\mathrm{d}t-x, u+(c(1-\gamma)-\alpha)\mathrm{d}t)\cdot \\ & (\Phi_{\alpha,b})_2(u+(c(1-\gamma)-\alpha)\mathrm{d}t))\mathrm{d}F(x) + \int_0^{u+(c(1-\gamma)-\alpha)\mathrm{d}t-b}((\Phi_{\alpha,b})_2(u+(c(1-\gamma)-\alpha)\mathrm{d}t-x) - \\ & B_2^{\alpha,b}(u+(c(1-\gamma)-\alpha)\mathrm{d}t-x, u+(c(1-\gamma)-\alpha)\mathrm{d}t)(\Phi_{\alpha,b})_2(u+(c(1-\gamma)-\alpha)\mathrm{d}t) + \\ & \left. B_2^{\alpha,b}(u+(c(1-\gamma)-\alpha)\mathrm{d}t-x, u+(c(1-\gamma)-\alpha)\mathrm{d}t)(\Phi_{\gamma,\alpha,b})_2(u+(c(1-\gamma)-\alpha)\mathrm{d}t))\mathrm{d}F(x)\right\} + o(\mathrm{d}t).\end{aligned}$$

Taylor expansion and collection of terms of order $\mathrm{d}t$ yield

$$\begin{aligned}(\Phi_{\gamma,\alpha,b})_2'(u) = {} & \frac{\lambda+\delta}{c(1-\gamma)-\alpha}(\Phi_{\gamma,\alpha,b})_2(u) - \frac{\lambda}{c(1-\gamma)-\alpha}\left\{\int_u^\infty w(u,x-u)\mathrm{d}F(x) + \right. \\ & \int_{u-b}^u (B_1^{\alpha,b}(u-x,u)(\Phi_{\gamma,\alpha,b})_2(u) + (\Phi_{\alpha,b})_1(u-x) - B_1^{\alpha,b}(u-x,u)(\Phi_{\alpha,b})_2(u))\mathrm{d}F(x) + \\ & \left. \int_0^{u-b}((\Phi_{\alpha,b})_2(u-x) - B_2^{\alpha,b}(u-x,u)(\Phi_{\alpha,b})_2(u) + B_2^{\alpha,b}(u-x,u)(\Phi_{\gamma,\alpha,b})_2(u))\mathrm{d}F(x)\right\}\end{aligned} \tag{16}$$

In addition, from Ref.[5] we know that

$$(\Phi_{\alpha,b})_2'(u) = \frac{\lambda+\delta}{c-\alpha}(\Phi_{\alpha,b})_2(u) - \frac{\lambda}{c-\alpha}\left\{\int_0^{u-b}(\Phi_{\alpha,b})_2(u-x)\mathrm{d}F(x) + \int_{u-b}^u (\Phi_{\alpha,b})_1(u-x)\mathrm{d}F(x) + A(u)\right\} \tag{17}$$

where

$$A(u) = \int_u^\infty w(u,x-u)\mathrm{d}F(x).$$

Plugging (17) into (16), we have

$$(\Phi_{\gamma,\alpha,b})'_2(u) = \frac{1}{c(1-\gamma)-\alpha}\cdot$$
$$\Big\{\lambda+\delta-\lambda(\int_0^{u-b} B_2^{\alpha,b}(u-x,u)\mathrm{d}F(x)+$$
$$\int_{u-b}^{u} B_1^{\alpha,b}(u-x,u)\mathrm{d}F(x))\Big\}(\Phi_{\gamma,\alpha,b})_2(u)-$$
$$\frac{1}{c(1-\gamma)-\alpha}\Big\{\lambda+\delta-\lambda(\int_0^{u-b} B_2^{\alpha,b}(u-x,u)\mathrm{d}F(x)+$$
$$\int_{u-b}^{u} B_1^{\alpha,b}(u-x,u)\mathrm{d}F(x))\Big\}(\Phi_{\alpha,b})_2(u)+$$
$$\frac{c-\alpha}{c(1-\gamma)-\alpha}(\Phi_{\alpha,b})'_2(u) =$$
$$M(u)(\Phi_{\gamma,\alpha,b})_2(u) - M(u)(\Phi_{\alpha,b})_2(u)+$$
$$\frac{c-\alpha}{c(1-\gamma)-\alpha}(\Phi_{\alpha,b})'_2(u).$$

Thus,

$$(\Phi_{\gamma,\alpha,b})_2(u) = \exp\{\int_b^u M(t)\mathrm{d}t\}\cdot$$
$$\Big(C-\int_b^u\Big(M(s)(\Phi_{\alpha,b})_2(s)-$$
$$\frac{c-\alpha}{c(1-\gamma)-\alpha}(\Phi_{\alpha,b})'_2(s)\Big)\cdot$$
$$\exp\{-\int_b^s M(t)\mathrm{d}t\}\mathrm{d}s\Big),$$

where C is some constant.

Noting that $M(t)\geqslant\frac{\delta}{c(1-\gamma)-\alpha}$ (>0) and $\lim_{u\to\infty}(\Phi_{\gamma,\alpha,b})_2(u)=0$, we have

$$C=\int_b^\infty\Big[M(s)(\Phi_{\alpha,b})_2(s)-\frac{c-\alpha}{c(1-\gamma)-\alpha}(\Phi_{\alpha,b})'_2(s)\Big]\cdot$$
$$\exp\{-\int_b^s M(t)\mathrm{d}t\}\mathrm{d}s.$$

Hence

$$(\Phi_{\gamma,\alpha,b})_2(u) = \exp\{\int_b^u M(t)\mathrm{d}t\}\cdot$$
$$\int_u^\infty\Big(M(s)(\Phi_{\alpha,b})_2(s)-\frac{c-\alpha}{c(1-\gamma)-\alpha}(\Phi_{\alpha,b})'_2(s)\Big)\cdot$$
$$\exp\{-\int_b^s M(t)\mathrm{d}t\}\mathrm{d}s \tag{18}$$

Furthermore,

$$\frac{c-\alpha}{c(1-\gamma)-\alpha}(\Phi_{\alpha,b})_2(u)-$$
$$\frac{c-\alpha}{c(1-\gamma)-\alpha}\exp\{\int_b^u M(t)\mathrm{d}t\}\cdot$$
$$\int_u^\infty M(s)(\Phi_{\alpha,b})_2(s)\exp\{-\int_b^s M(t)\mathrm{d}t\}\mathrm{d}s=$$
$$-\frac{c-\alpha}{c(1-\gamma)-\alpha}\exp\{\int_b^u M(t)\mathrm{d}t\}\cdot$$
$$\int_u^\infty(\Phi_{\alpha,b})'_2(s)\exp\{-\int_b^s M(t)\mathrm{d}t\}\mathrm{d}s \tag{19}$$

Plugging (19) into (18), we arrive at (14). The proof of Theorem 1.1 is completed. □

We remark that the Eq. (14) for $(\Phi_{\gamma,\alpha,b})_2(u)$ is independent of $(\Phi_{\gamma,\alpha,b})_1(u)$. However, $(\Phi_{\gamma,\alpha,b})_1(u)$ is involved with $(\Phi_{\gamma,\alpha,b})_2(u)$ by the boundary condition $(\Phi_{\gamma,\alpha,b})_1(b-)=(\Phi_{\gamma,\alpha,b})_2(b)$, which can be obtained by letting $u\uparrow b$ in Eq. (13).

Remark 1.1 Letting $\delta\downarrow 0$ and $w\equiv 1$, (14) can be rewritten as

$$(\Psi_{\gamma,\alpha,b})_2(u) =$$
$$\frac{c-\alpha}{c(1-\gamma)-\alpha}(\Psi_{\alpha,b})_2(u)-\frac{c\gamma}{c(1-\gamma)-\alpha}\cdot$$
$$\exp\{\int_0^u M_0(t)\mathrm{d}t\}\int_u^\infty(\Psi_{\alpha,b})_2(s)M_0(s)\cdot$$
$$\exp\{-\int_0^s M_0(t)\mathrm{d}t\}\mathrm{d}s \tag{20}$$

where

$$M_0(t) = \frac{1}{c(1-\gamma)-\alpha}\cdot$$
$$\Big\{\lambda-\lambda\Big(\int_0^{u-b}(B_2^{\alpha,b})_0(u-x,u)\mathrm{d}F(x)+$$
$$\int_{u-b}^{u}(B_1^{\alpha,b})_0(u-x,u)\mathrm{d}F(x)\Big)\Big\} \tag{21}$$

Note that

$$\lim_{\alpha\downarrow 0}(B_i^{\alpha,b})_0(u-x,u) =$$
$$P_{u-x}(R(t)\text{ reaches the level }u$$
$$\text{without leading to ruin}) =$$
$$\frac{1-\Psi(u-x)}{1-\Psi(u)}, 0\leqslant x< u, i=1,2,$$

and $\lim_{\alpha\downarrow 0}(\Psi_{\gamma,\alpha,b})_i(u)=\Psi_\gamma(u), 0\leqslant u<b, i=1,2$, we have

$$\Psi_\gamma(u) =$$
$$\Psi(u)-\frac{\gamma}{1-\gamma}\int_u^\infty\Psi'(s)\exp\Big\{-\frac{1}{c(1-\gamma)}\cdot$$
$$\int_0^s\Big[\lambda-\lambda\int_0^t\frac{1-\Psi(t-x)}{1-\Psi(t)}\mathrm{d}F(x)\Big]\mathrm{d}t\Big\}\mathrm{d}s\cdot$$
$$\exp\Big\{\frac{1}{c(1-\gamma)}\int_0^u\Big[\lambda-\lambda\int_0^t\frac{1-\Psi(t-x)}{1-\Psi(t)}\mathrm{d}F(x)\Big]\mathrm{d}t\Big\}=$$
$$\Psi(u)-\exp\Big\{\frac{1}{(1-\gamma)}\int_0^u\frac{(1-\Psi(t))'}{1-\Psi(t)}\mathrm{d}t\Big\}\cdot$$
$$\int_u^\infty\frac{\gamma}{1-\gamma}\Psi'(s)\exp\Big\{-\frac{1}{(1-\gamma)}\int_0^s\frac{(1-\Psi(t))'}{1-\Psi(t)}\mathrm{d}t\Big\}\mathrm{d}s=$$

$$\Psi(u)-(1-\Psi(u))^{\frac{1}{1-\gamma}}\cdot$$
$$\int_u^\infty \frac{\gamma}{1-\gamma}\Psi'(s)(1-\Psi(s))^{-\frac{1}{1-\gamma}}\mathrm{d}s=$$
$$1-(1-\Psi(u))^{\frac{1}{1-\gamma}} \tag{22}$$

where the second equation follows from the integro-differential equation for $\Psi(u)$:

$$-\lambda\Psi(u)+c\Psi'(u)+\lambda\int_0^u \Psi(u-x)\mathrm{d}F(x)+\lambda\overline{F}(u)=0.$$

Obviously, Eq. (22) coincides with Ref. [13, Eq. (1)].

2 Probability of ruin under exponential distribution

In this section, we assume that the individual claim amount is exponentially distributed with parameter $\beta>0$. Our objective is to calculate the closed-form expressions of probability of ruin, that is, the expressions of $(\Psi_{\gamma,\alpha,b})_1(u)$ and $(\Psi_{\gamma,\alpha,b})_2(u)$. Following from (12), (21), and the results (16), (17) in Ref. [19], one can get that

$$M_0(t)=\frac{\frac{c-\alpha}{c(1-\gamma)-\alpha}\frac{\lambda}{c}(\lambda-c\beta)\left(\frac{\lambda}{c-\alpha}-\beta\right)e^{\left(\frac{\lambda}{c-\alpha}-\beta\right)t}}{(\lambda-c\beta)\frac{\lambda}{c}\left(e^{\left(\frac{\lambda}{c-\alpha}-\beta\right)t}-e^{\left(\frac{\lambda}{c-\alpha}-\beta\right)b}\right)+((c-\alpha)\beta-\lambda)\left(\beta e^{\left(\frac{\lambda}{c-\alpha}-\frac{\lambda}{c}\right)b}-\frac{\lambda}{c}e^{\left(\frac{\lambda}{c-\alpha}-\beta\right)b}\right)} \tag{23}$$

which implies that

$$\exp\left\{\int_b^u M_0(t)\mathrm{d}t\right\}:=\left(\frac{(\lambda-c\beta)\frac{\lambda}{c}e^{\left(\frac{\lambda}{c-\alpha}-\beta\right)u}+q_3(b)}{(\lambda-c\beta)\frac{\lambda}{c}e^{\left(\frac{\lambda}{c-\alpha}-\beta\right)b}+q_3(b)}\right)^{\frac{c-\alpha}{c(1-\gamma)-\alpha}} \tag{24}$$

where

$$q_3(b)=-(\lambda-c\beta)\frac{\lambda}{c}e^{\left(\frac{\lambda}{c-\alpha}-\beta\right)b}+((c-\alpha)\beta-\lambda)\left(\beta e^{\left(\frac{\lambda}{c-\alpha}-\frac{\lambda}{c}\right)b}-\frac{\lambda}{c}e^{\left(\frac{\lambda}{c-\alpha}-\beta\right)b}\right).$$

In addition, from in Ref. [5, Example 6.1], we have

$$(\Psi_{\alpha,b})_2(u)=\frac{\lambda}{(c-\alpha)\beta}\left(1-Q(b)+Q(b)e^{\left(\frac{\lambda}{c}-\beta\right)b}\right)e^{\left(\frac{\lambda}{c-\alpha}-\beta\right)(u-b)},\ u\geqslant b \tag{25}$$

where

$$Q(b)=\frac{c\beta((c-\alpha)\beta-\lambda)}{\alpha\beta\lambda e^{\left(\frac{\lambda}{c}-\beta\right)b}+c\beta((c-\alpha)\beta-\lambda)} \tag{26}$$

Therefore, by Eqs. (14) and (19) we have

$$\begin{aligned}(\Psi_{\gamma,\alpha,b})_2(u)=&\frac{\lambda\left(1-Q(b)+Q(b)e^{\left(\frac{\lambda}{c}-\beta\right)b}\right)}{(c-\alpha)\beta}e^{\left(\frac{\lambda}{c-\alpha}-\beta\right)(u-b)}-\\&\frac{c\left(1-Q(b)+Q(b)e^{\left(\frac{\lambda}{c}-\beta\right)b}\right)}{(c-\alpha)(\lambda-c\beta)\beta}e^{-\left(\frac{\lambda}{c-\alpha}-\beta\right)b}\left((\lambda-c\beta)\frac{\lambda}{c}e^{\left(\frac{\lambda}{c-\alpha}-\beta\right)u}+q_3(b)\right)^{\frac{c-\alpha}{c(1-\gamma)-\alpha}}\cdot\\&\left(\left((\lambda-c\beta)\frac{\lambda}{c}e^{\left(\frac{\lambda}{c-\alpha}-\beta\right)u}+q_3(b)\right)^{-\frac{c-\alpha}{c(1-\gamma)-\alpha}+1}-(q_3(b))^{-\frac{c-\alpha}{c(1-\gamma)-\alpha}+1}\right)\end{aligned} \tag{27}$$

By (13) and the classical ruin probability identity

$$\Psi_\gamma(u)=1-\left(1-\frac{\lambda}{c\beta}e^{\left(\frac{\lambda}{c}-\beta\right)u}\right)^{\frac{1}{1-\gamma}},$$

when $0\leqslant u<b$, we have

$$(\Psi_{\gamma,\alpha,b})_1(u)=1-\left(1-\frac{\lambda}{c\beta}e^{\left(\frac{\lambda}{c}-\beta\right)u}\right)^{\frac{1}{1-\gamma}}-\left(\frac{\beta-\frac{\lambda}{c}e^{\left(\frac{\lambda}{c}-\beta\right)u}}{\beta-\frac{\lambda}{c}e^{\left(\frac{\lambda}{c}-\beta\right)b}}\right)^{\frac{1}{1-\gamma}}\left(1-\left(1-\frac{\lambda}{c\beta}e^{\left(\frac{\lambda}{c}-\beta\right)b}\right)^{\frac{1}{1-\gamma}}\right)+$$

$$\left(\frac{\beta-\frac{\lambda}{c}\mathrm{e}^{\left(\frac{\lambda}{c}-\beta\right)u}}{\beta-\frac{\lambda}{c}\mathrm{e}^{\left(\frac{\lambda}{c}-\beta\right)b}}\right)^{\frac{1}{1-\gamma}}\Bigg[\frac{\lambda(1-Q(b)+Q(b)\mathrm{e}^{\left(\frac{\lambda}{c}-\beta\right)b})}{(c-\alpha)\beta}-$$

$$\frac{c(1-Q(b)+Q(b)\mathrm{e}^{\left(\frac{\lambda}{c}-\beta\right)b})}{(c-\alpha)(\lambda-c\beta)\beta}\mathrm{e}^{-\left(\frac{\lambda}{c-\alpha}-\beta\right)b}\left[(\lambda-c\beta)\frac{\lambda}{c}\mathrm{e}^{\left(\frac{\lambda}{c-\alpha}-\beta\right)b}+q_3(b)\right]^{\frac{c-\alpha}{c(1-\gamma)-\alpha}}\cdot$$

$$\left(\left[(\lambda-c\beta)\frac{\lambda}{c}\mathrm{e}^{\left(\frac{\lambda}{c-\alpha}-\beta\right)b}+q_3(b)\right]^{-\frac{c-\alpha}{c(1-\gamma)-\alpha}+1}-(q_3(b))^{-\frac{c-\alpha}{c(1-\gamma)-\alpha}+1}\right)\Bigg] \tag{28}$$

We point out that when $0\leqslant u<b$, the probability of a drop below the initial level is $(\Psi_{\gamma,\alpha,b-u})_1(0)$ under the present model. When $u\geqslant b$, the probability of a drop below the initial level is reduced to the same probability under the compound Poisson model with tax rate γ and premium rate $c-\alpha$ considered by Ref. [13], which equals

$$1-\left(1-\frac{\lambda}{(c-\alpha)\beta}\mathrm{e}^{\left(\frac{\lambda}{c-\alpha}-\beta\right)u}\right)^{\frac{1}{1-\gamma}}.$$

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