第45卷第11期 中国科学技术大学学报 Vol. 45, No. 11
2015年11月 JOURNAL OF UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA Nov. 2015

Article ID: 0253-2778(2015)11-0899-07

Impulsive stochastic differential equations of Sobolev-type

YUE Chaohui, WANG Hongxi

(*School of Sciences, Anhui Agricultural University, Hefei* 230036, *China*)

Abstract: A class of impulsive stochastic differential equations of Sobolev-type was studied. The existence and uniqueness of the mild solution with the coefficients satisfying some generalized Lipschitz conditions was proved by means of the successive approximation. Moreover, the continuous dependence of the solutions on the initial values was given.

Key words: stochastic differential equation; Sobolev-type; impulsive effect

CLC number: O211 **Document code**: A doi: 10.3969/j.issn.0253-2778.2015.11.003

2010 Mathematics Subject Classification: Primary 60H10; Secondary 60H15

Citation: Yue Chaohui, Wang Hongxi. Impulsive stochastic differential equations of Sobolev-type[J]. Journal of University of Science and Technology of China, 2015,45(11):899-905.

具有脉冲的 Sobolev 型随机微分方程

岳超慧,汪宏喜

(安徽农业大学理学院,安徽合肥 230036)

摘要:研究了一类具有脉冲的 Sobolev 型随机微分方程,利用迭代序列证明了在方程系数满足一类推广的 Lipschitz 条件时,其适度解的存在唯一性.进一步的,给出了方程的解对于初值的连续依赖性.

关键词:随机微分方程;Sobolev 型;脉冲作用

0 Introduction

For the applications in modeling various physical phenomena such as the fluid flow through fissured rocks, thermodynamics and shear in second order fluids, Sobolev-type differential equations attracted researchers' great interest. One can see Refs. [1-4] and the references therein for more details. Especially, the existence, uniqueness, continuation and other properties of solutions of various special forms of the deterministic Sobolev-type differential equations have been established by using different techniques. One can see Refs. [5-8] and the references therein. The deterministic models often fluctuate due to noise, which is random or at least appears to be so. Therefore, we must move from deterministic problems to stochastic problems. As

Received: 2015-03-11; **Revised**: 2015-11-16

Foundation item: Supported by the Natural Science Foundation of Education Department of Anhui Province (KJ2015A318) and the Youth Fund of Anhui Agricultural University.

Biography: YUE Chaohui, female, born in 1979, master/associate Prof. Research field: stochastic differential equations. E-mail: huihui.eyre5166@163.com

Corresponding author: WANG Hongxi, Prof. E-mail: wanghongxi@ahau.edu.cn

the generalization of deterministic Sobolev-type integro-differential equations, Ref. [1] proved the existence and uniqueness of the mild solution for a class of first-order abstract stochastic Sobolev-type integro-differential equations in a real separable Hilbert space. In addition, the theory of impulsive differential equations has become an active area of investigation due to their applications in fields such as mechanics, electrical engineering, medicine biology, ecology and so on (see Ref. [9] and the references therein).

However, it should be emphasized that to the best of our knowledge there is no result on impulsive stochastic differential equations of Sobolev-type and the aim of this paper is to fill this gap. In this paper, we study a class of impulsive stochastic differential equations of Sobolev-type. We prove the existence and uniqueness of the mild solution with the coefficients satisfying some generalized Lipschitz conditions by means of the successive approximation. Moreover, we show the continuous dependence of the solutions on the initial values.

1 Preliminaries

In this section, we propose some preliminaries for our analysis. Throughout this paper, let $(H, \|\cdot\|)$ and $(K, \|\cdot\|)$ be two real separable Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$. Let $\mathscr{L}(K,H)$ be the space of bounded linear operators from K into H. In the sequel, without confusion, we also employ the inner product and the norm denoted, respectively, by $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ for $\mathscr{L}(K,H)$. Let $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geqslant 0},P)$ be a complete filtered probability space satisfying that $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets of $\mathscr{F}$. Let $W=(W_t)_{t\geqslant 0}$ represent a Q-Wiener process defined on $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geqslant 0},\mathbb{P})$ with the covariance operator Q such that $TrQ<\infty$. Further, we assume that there exists a complete orthonormal system $\{e_k\}_{k\geqslant 1}$ in K, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k=\lambda_k e_k, k=1, 2,\cdots$, and a sequence of independent Wiener processes $\{\beta_k\}_{k\geqslant 1}$ such that

$$\langle w(t),e\rangle_K = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\langle e_k,e\rangle_K\beta_k(t),\ e\in K, t\geqslant 0.$$

Let $\mathscr{L}_2^0=\mathscr{L}_2(Q^{\frac{1}{2}}K,H)$ be the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}K$ to H with the inner product $\langle\varphi,\psi\rangle_{\mathscr{L}_2^0}=Tr[\varphi Q\psi *]$.

The purpose of this paper is to study the following impulsive stochastic differential equations of Sobolev-type with the form

$$\left.\begin{aligned}
&\mathrm{d}(Bx(t)) = (Ax(t)+F(x)(t))\mathrm{d}t + \\
&\qquad \sigma(t,x(t))\mathrm{d}W(t),\ t\in J:=[0,T], t\neq t_k,\\
&x(0)=x_0,\\
&\Delta x(t_k) = x(t_k^+)-x(t_k^-) = I_k(x(t_k)),\\
&\qquad k=1,2,\cdots,m
\end{aligned}\right\} \tag{1}$$

in a real separable Hilbert space H, where the state $x(\cdot)$ takes values in the separable real Hilbert space H; A and B are linear operators on H, $F:([0,T];\mathscr{L}^2(\Omega;H))\to\mathscr{L}^2((0,T);\mathscr{L}^2(\Omega;H))$ is a given mapping, $\sigma: J\times\mathscr{L}^2(\Omega;H)\to BL(K,H)$ is an appropriate mapping. Furthermore, the fixed times t_k satisfies $0=t_0<t_1<t_2<\cdots<t_m<T$, $x(t_k^+)$ and $x(t_k^-)$ denote the right and left limits of $x(t)$ at $t=t_k$. $I_k(x(t_k))=x(t_k^+)-x(t_k^-)$ represents the jump in the state x at time t_k, where I_k determines the size of the jump. The operators $A: D(A)\subset H\to H$ and $B: D(B)\subset H\to H$ satisfy the following conditions:

(M_1) A and B are closed linear operators,

(M_2) $D(B)\subset D(A)$ and B is bijective,

(M_3) $B^{-1}: H\to D(B)$ is continuous.

Here, (M_1) ~ (M_3) and the closed graph theorem imply the boundedness of the linear operator AB^{-1}: $H\to H$ and AB^{-1} generates a strongly continuous semigroup $\{T(t)\}_{t\geqslant 0}$ in H. In what follows, $\max_{t\in J}\|T(t)\|^2=M$, $B^{-1}=M_B$, $\|B\|^2=\widetilde{M}_B$.

Definition 1.1 A càdlàg H-valued process $x(t)_{0\leqslant t\leqslant T}$ is said to be a mild solution of (1) if

(i) $x(t)$ is $\mathscr{F}_t$-adapted,

(ii) $x(0)=x_0\in\mathscr{L}^2(\Omega,H)$,

(iii) for each $t\in J$, $x(t)$ satisfies the following integral equation:

$$x(t)=B^{-1}T(t)Bx_0+\int_0^t B^{-1}T(t-s)F(x)(s)\mathrm{d}s+\int_0^t B^{-1}T(t-s)\sigma(s,x(s))\mathrm{d}W(s)+\sum_{0<t_k<t}B^{-1}T(t-t_k)I_k(x(t_k)).$$

Now, we assume the following assumptions:

(H1)

$F:([0,T];\mathscr{L}^2(\Omega;H))\to\mathscr{L}^2((0,T);\mathscr{L}^2(\Omega;H))$ and $\sigma:J\times\mathscr{L}^2(\Omega;H)\to BL(K,H)$ satisfy for all $t\in J$, $x,y\in H$ such that

$$\|F(x)(t)-F(y)(t)\|^2\vee\|\sigma(t,x)-\sigma(t,y)\|^2\leqslant\kappa(\|x-y\|^2).$$

(H2) for all $t\in J$, it follows that $F(0)(t)$, $\sigma(t,0)\in L^2$ such that

$$\|F(0)(t)\|^2\vee\|\sigma(t,0)\|^2\leqslant\widetilde{K},$$

where $\widetilde{K}$ is a constant.

(H3) The function I_k: $H\to H$ are continuous and there is a positive constant q_k, $k=1,2,\cdots,m$, such that

$$\|I_k(x)-I_k(y)\|^2\leqslant q_k\|x-y\|^2,\quad q_k>0,\ k=1,2,\cdots,m$$

for each $x,y\in H$, and $I_k(0)=0$, $k=1,2,\cdots,m$.

Remark 1.1 The function $\kappa(\cdot)$ is a generality of classic Lipschitz condition. Let us illustrate it using concrete functions. Let $K_1>0$ and let $\delta\in(0,1)$ be sufficiently small. Define

$$\kappa_1(u)=K_1u,\ u\geqslant0,$$

$$\kappa_2(u)=\begin{cases}u\ln(u^{-1}),\ 0\leqslant u\leqslant\delta,\\ \delta\ln(\delta^{-1})+\kappa_2'(\delta-)(u-\delta),\ u>\delta,\end{cases}$$

$$\kappa_3(u)=\begin{cases}u\ln(u^{-1})\ln\ln(u^{-1}),\ 0\leqslant u\leqslant\delta,\\ \delta\ln(\delta^{-1})\ln\ln(\delta^{-1})+\kappa_3'(\delta-)(u-\delta),\\ \quad u>\delta,\end{cases}$$

where κ' denotes the derivative of function κ. They are all concave nondecreasing functions satisfying $\int_{0^+}\frac{\mathrm{d}u}{\kappa_i(u)}=+\infty\ (i=1,2,3)$. In particular, we see that the Lipschitz condition is a special case of the proposed conditions.

Lemma 1.2[10] (Bihari inequality) Let $T>0$ and $u_0\geqslant0$, $u(t)$ and $v(t)$ be two continuous functions on $[0,\mathrm{T}]$. Let $\kappa:\mathbb{R}_+\to\mathbb{R}_+$ be a concave continuous and nondecreasing function such that $\kappa(r)>0$ for $r>0$. If

$$u(t)\leqslant u_0+\int_0^t v(s)\kappa(u(s))\mathrm{d}s \text{ for all } 0\leqslant t\leqslant T,$$

then

$$u(t)\leqslant G^{-1}(G(u_0)+\int_0^t v(s)\mathrm{d}s)$$

for all such $t\in[0,T]$ that

$$G(u_0)+\int_0^t v(s)\mathrm{d}s\in\mathrm{Dom}(G^{-1})$$

where $G(r)=\int_1^r\frac{\mathrm{d}s}{\kappa(s)}$, $r\geqslant0$ and G^{-1} is the inverse function of G. In particular, if, moreover, $u_0=0$ and $\int_{0^+}\frac{\mathrm{d}s}{\kappa(s)}=\infty$, then $u(t)=0$ for all $0\leqslant t\leqslant T$.

In order to obtain the stability of solutions, we give the extended Bihari inequality which appeared in Ref. [11, Lemma 3.2] and its corollary.

Lemma 1.3 Let the assumptions of Lemma 1.3 hold. If

$$u(t)\leqslant u_0+\int_t^T v(s)\kappa(u(s))\mathrm{d}s,\ 0\leqslant t\leqslant T,$$

then

$$u(t)\leqslant G^{-1}(G(u_0)+\int_t^T v(s)\mathrm{d}s),$$

for all $t\in[0,T]$ such that

$$G(u_0)+\int_t^T v(s)\mathrm{d}s\in\mathrm{Dom}(G^{-1}),$$

where $G(r)=\int_1^r\frac{\mathrm{d}s}{\kappa(s)}$, $r\geqslant0$ and G^{-1} is the inverse function of G.

Corollary 1.4 Let the assumptions of Lemma 1.3 hold and $v(t)\geqslant0$ for $t\in[0,T]$. If for all $\varepsilon>0$, there exists $t_1\geqslant0$ such that for $0\leqslant u_0<\varepsilon$, $\int_{t_1}^T v(s)\mathrm{d}s\leqslant\int_{u_0}^{\varepsilon}\frac{\mathrm{d}s}{\kappa(s)}$ holds. Then for every $t\in[t_1,T]$, the estimate $u(t)\leqslant\varepsilon$ holds.

2 Main results

We construct the sequence of successive approximations defined as follows:

$$\left.\begin{aligned}&x^0(t)=B^{-1}T(t)Bx_0,\ t\in J,\\&x^n(t)=B^{-1}T(t)Bx_0+\int_0^t B^{-1}T(t-s)\mathscr{F}(x^{n-1})(s)\mathrm{d}s+\\&\int_0^t B^{-1}T(t-s)\sigma(s,x^{n-1}(s))\mathrm{d}W(s)+\\&\sum_{0<t_k<t}B^{-1}T(t-t_k)I_k(x^{n-1}(t_k)),\ t\in J,\ n\geqslant1\end{aligned}\right\}\tag{2}$$

Lemma 2.1 Assume the assumptions (H1)～(H3) are satisfied and

$$1-4M_BMm\sum_{k=1}^{m}q_k>0.$$

Then, for all $n\geqslant 1$, it holds that

$$E\|x^n(t)\|^2\leqslant M_1,$$

where M_1 is a positive constant.

Proof From (2), we have

$$\begin{aligned}
E\|x^n(t)\|^2 &\leqslant 4E\|B^{-1}T(t)Bx_0\|^2+4E\|\int_0^t B^{-1}T(t-s)\mathscr{F}(x^{n-1})(s)\mathrm{d}s\|^2+\\
&4E\|\int_0^t B^{-1}T(t-s)\sigma(s,x^{n-1}(s))\mathrm{d}W(s)\|^2+4E\|\sum_{0<t_k<t}B^{-1}T(t-t_k)I_k(x^{n-1}(t_k))\|^2\leqslant\\
&4M_BM\widetilde{M}_B^2E\|x_0\|^2+4M_BMTE\int_0^t\|\mathscr{F}(x^{n-1})(s)-\mathscr{F}(0)+\mathscr{F}(0)\|^2\mathrm{d}s+\\
&4M_BME\int_0^t\|\sigma(s,x^{n-1}(s))-\sigma(s,0)+\sigma(s,0)\|^2\mathrm{d}s+4M_BMmE\sum_{0<t_k<t}\|I_k(x^{n-1}(t_k))\|^2\leqslant\\
&4M_BM\widetilde{M}_B^2E\|x_0\|^2+8M_BMTE\int_0^t(\|\mathscr{F}(x^{n-1})(s)-\mathscr{F}(0)\|^2+\|\mathscr{F}(0)\|^2)\mathrm{d}s+\\
&8M_BME\int_0^t(\|\sigma(s,x^{n-1}(s))-\sigma(s,0)\|^2+\|\sigma(s,0)\|^2)\mathrm{d}s+4M_BMm\sum_{k=1}^{k=m}q_kE\|x^{n-1}\|^2\leqslant\\
&4M_BM\widetilde{M}_B^2E\|x_0\|^2+8M_BM(T+1)(E\int_0^t\kappa(\|x^{n-1}\|^2)\mathrm{d}s+\widetilde{K}T)+4M_BMm\sum_{k=1}^{k=m}q_kE\|x^{n-1}\|^2\leqslant\\
&C_1+8M_BM(T+1)(E\int_0^t\kappa(\|x^{n-1}\|^2)\mathrm{d}s)+4M_BMm\sum_{k=1}^{k=m}q_kE\|x^{n-1}\|^2,
\end{aligned}$$

where

$$C_1=4M_BM\widetilde{M}_B^2E\|x_0\|^2+8M_BM\widetilde{K}T^2+8M_BM\widetilde{K}T.$$

Given that $\kappa(\cdot)$ is concave and $\kappa(0)=0$, there exists a pair of positive constants a and b such that

$$\kappa(u)\leqslant a+bu,\ \text{for all } u\geqslant 0.$$

So, we obtain

$$\begin{aligned}
E\|x^n(t)\|^2&\leqslant\\
&C_1+8M_BM(T+1)(E\int_0^t(a+b\|x^{n-1}\|^2)\mathrm{d}s)+\\
&4M_BMm\sum_{k=1}^{m}q_kE\|x^{n-1}\|^2\leqslant\\
&C_2+8M_BM(T+1)b(E\int_0^t\|x^{n-1}\|^2\mathrm{d}s)+\\
&4M_BMm\sum_{k=1}^{m}q_kE\|x^{n-1}\|^2,
\end{aligned}$$

where $C_2=C_1+8M_BM(T+1)aT$. For

$$\begin{aligned}
\max_{1\leqslant n\leqslant k}E\|x^n(t)\|^2&=\\
&\max\{E\|B^{-1}T(t)Bx_0\|^2,E\|x^1(s)\|^2,\cdots,\\
&E\|x^{k-1}(s)\|^2\}\leqslant\\
&\max\{M_BM\widetilde{M}_BE\|x_0\|^2,E\|x^1(s)\|^2,\cdots,\\
&E\|x^{k-1}(s)\|^2,E\|x^k(s)\|^2\}=\\
&\{M_BM\widetilde{M}_BE\|x_0\|^2,\max_{1\leqslant n\leqslant k}E\|x^n(s)\|^2\}\leqslant\\
&C_3+\max_{1\leqslant n\leqslant k}E\|x^n(s)\|^2,
\end{aligned}$$

where $C_3=M_BM\widetilde{M}_BE\|x_0\|^2$. Thus, we have

$$\begin{aligned}
\max_{1\leqslant n\leqslant k}E\|x^n(t)\|^2&\leqslant\\
&C_2+8M_BM(T+1)b\int_0^T\max_{1\leqslant n\leqslant k}E\|x^n(s)\|^2\mathrm{d}s+\\
&C_3 8M_BM(T+1)b\cdot T+\\
&4M_BMm\sum_{k=1}^{k=m}q_k(C_3+\max_{1\leqslant n\leqslant k}E\|x^n(s)\|^2).
\end{aligned}$$

Moreover, we get

$$\max_{1\leqslant n\leqslant k}E\|x^n(t)\|^2\leqslant C_4+C_5\int_0^T\max_{1\leqslant n\leqslant k}E\|x^n(s)\|^2\mathrm{d}s,$$

where

$$C_4=\frac{C_2+8C_3M_BM(T+1)bT+4M_BMmC_3\sum_{k=1}^{k=m}q_k}{1-4M_BMm\sum_{k=1}^{k=m}q_k}$$

and

$$C_5 = \frac{8M_BM(T+1)b}{1-4M_BMm\sum_{k=1}^{k=m}q_k}.$$

From the Gronwall inequality, we have

$$\max_{1\leqslant n\leqslant k} E\parallel x^n(t)\parallel^2 \leqslant C_4 e^{C_5T}.$$

For k is arbitrary, we have

$$\sup_{0\leqslant t\leqslant T} E\parallel x^n(t)\parallel^2 \leqslant C_4 e^{C_5T},\ n\geqslant 1.$$

Thus, we get the desired result with $M_1 = \max\{C_4 e^{C_5T}, E\parallel x_0\parallel^2\}$. □

Theorem 2.2 Under the assumptions of Lemma 2.1, there exists a unique mild solution of (1).

Proof Existence For $n\geqslant 1$, $t\in J$, form (2), we have

$$x^{n+1}(t)-x^n(t) = \int_0^t B^{-1}T(t-s)[\mathscr{F}(x^n)-\mathscr{F}(x^{n-1})]ds + \int_0^t B^{-1}T(t-s)[\sigma(s,x^n)-\sigma(s,x^{n-1})]ds + \sum_{0<t_k<t} B^{-1}T(t-t_k)[I_k(x^n(t_k))-I_k(x^{n-1}(t_k))].$$

Furthermore,

$$E\parallel x^{n+1}(t)-x^n(t)\parallel^2 \leqslant 3M_BMTE\int_0^t \parallel\mathscr{F}(x^n)-\mathscr{F}(x^{n-1})\parallel^2 ds + 3M_BME\int_0^t\parallel\sigma(s,x^n)-\sigma(s,x^{n-1})\parallel^2 ds + 3M_BMm\sum_{k=1}^{k=m}q_kE\parallel x^n(t)-x^{n-1}(t)\parallel^2 \quad (3)$$

Choosing $T_1\in[0,T)$ such that

$$\kappa\Big(\frac{(t-s)^n}{n!}C_7\Big)\leqslant \frac{(t-s)^n}{n!}C_7, 0\leqslant t\leqslant T_1,$$

where C_7 is a positive constant. Moreover, for $0\leqslant t\leqslant T$, (H2), (H3), (H4) and the Jensen inequality show that

$$E\parallel x^1(t)-x^0(t)\parallel^2 \leqslant 3M_BMTE\int_0^t\parallel\mathscr{F}(x^0)\parallel^2ds + 3M_BME\int_0^t\parallel\sigma(s,x^0)\parallel^2ds + 3M_BMm\sum_{k=1}^{k=m}q_kE\parallel x^0(t)\parallel^2 \leqslant$$
$$6M_BMTE(\int_0^T\kappa(\parallel x^0\parallel^2)ds+\widetilde{K}T) + 6M_BME(\int_0^T\kappa(\parallel x^0\parallel^2)ds+\widetilde{K}T) + 3M_BMm\sum_{k=1}^{k=m}q_kE\parallel x^0\parallel^2 \leqslant$$
$$6M_BM\widetilde{K}T(T+1) + 6M_BM(T+1)E\int_0^t\kappa(\parallel x^0\parallel^2)ds + 3M_BMm\sum_{k=1}^{k=m}q_kE\parallel x^0\parallel^2.$$

From Lemma 2.1, we have

$$E\parallel x^1(t)-x^0(t)\parallel^2 \leqslant 6M_BM\widetilde{K}T(T+1) + 6M_BM(T+1)E\int_0^T\kappa(M_1)ds + 3M_BMmM_1\sum_{k=1}^{k=m}q_k := C_7 \quad (4)$$

(3) and (4) show that

$$\sup_{0\leqslant t\leqslant T_1} E\parallel x^{n+1}(t)-x^n(t)\parallel^2 \leqslant$$
$$6M_BMT_1E\int_0^t\kappa\Big(\frac{(t-s)^n}{n!}\sup_{0\leqslant s\leqslant T_1}E\parallel x^1(s)-x^0(s)\parallel^2\Big)ds + 3M_BMm\sum_{k=1}^{k=m}q_k\frac{T_1^n}{n!}(\sup_{0\leqslant s\leqslant T_1}E\parallel x^1(s)-x^0(s)\parallel^2) \leqslant$$
$$6M_BMT_1E\int_0^t\kappa\Big(\frac{(t-s)^n}{n!}C_7\Big)ds + 3M_BMm\sum_{k=1}^{k=m}q_k\frac{T_1^n}{n!}C_7 \leqslant$$
$$6M_BMT_1E\int_0^t\frac{(t-s)^n}{n!}C_7ds + 3M_BMm\sum_{k=1}^{k=m}q_k\frac{T_1^n}{n!}C_7 \leqslant C_8\frac{T_1^n}{n!},\ n\geqslant 0,$$

where $C_8>0$ is a constant. Therefore, for any $1\leqslant n<k$, we obtain

$$\sup_{0\leqslant t\leqslant T_1}E\parallel x^k(t)-x^n(t)\parallel^2 \leqslant \sum_{r=n}^{r=k-1}\sup_{0\leqslant t\leqslant T_1}E\parallel x^{r+1}(t)-x^r(t)\parallel^2 \leqslant \sum_{r=n}^{r=k-1}C_8\frac{T_1^n}{n!}\to 0,\ \text{as } n\to\infty.$$

Thus, we can see that $\{x_n\}$ is a Cauchy sequence. The Borel-Cantelli Lemma shows that $x^n(t)$ uniformly converges to $x(t)$ for $t\in[0,T_1]$. From this, $x(t)$ is a mild solution of (1).

Uniqueness Let $x(t)$ and $y(t)$ be two solutions of (1). We have

$$E\|x(t)-y(t)\|^2 \leqslant 3E\left\|\int_0^t B^{-1}T(t-s)[\mathscr{F}(x)-\mathscr{F}(y)]\mathrm{d}s\right\|^2+$$
$$3E\left\|\int_0^t B^{-1}T(t-s)[\sigma(s,x(s))-\sigma(s,y(s))]\mathrm{d}W(s)\right\|^2+$$
$$3E\left\|\sum_{0<t_k<t} B^{-1}T(t-t_k)[I_k(x(t_k))-I_k(y(t_k))]\right\|^2 \leqslant$$
$$3M_BMTE\int_0^t \kappa(\|x-y\|^2)\mathrm{d}s+3M_BME\int_0^t \kappa(\|x-y\|^2)\mathrm{d}s+3M_BME\sum_{k=1}^{k=m}q_k\|x-y\|^2 \leqslant$$
$$3M_BM(T+1)E\int_0^t \kappa(\|x-y\|^2)\mathrm{d}s+3M_BMm\sum_{k=1}^{k=m}q_kE\|x-y\|^2.$$

From the Jensen inequality, we get

$$E\|x(t)-y(t)\|^2 \leqslant \frac{3M_BM(T+1)}{1-3M_BMm\sum_{k=1}^{k=m}q_k}\int_0^t \kappa(E\|x(s)-y(s)\|^2)\mathrm{d}s.$$

Moreover, the Bihari shows that

$$\sup_{0\leqslant t\leqslant T} E\|x(t)-y(t)\|^2=0.$$

Thus, $x(t)=y(t)$, for all $0\leqslant t\leqslant T$. □

3 Stability of the solutions

In what follows, we aim to derive the continuous dependence of the solution on the initial value. To do so, we propose the definition of the stability in mean square.

Definition 3.1 A mild solution $X^{\xi,x}(t)$ of (1) with initial value (ξ,x) is said to be stable in mean square if for all $\varepsilon>0$ there exists $\delta>0$ such that

$$E\|X^{\xi}(t)-Y^{\eta}(t)\|^2 \leqslant \varepsilon \tag{5}$$

when $E\|\xi-\eta\|^2<\delta$, where $Y^{\eta,y}(t)$ is another solution of (1) with initial value (η,y).

Theorem 3.2 Let $X^{\xi}(t)$ and $Y^{\eta}(t)$ be solutions of (1) with initial values ξ and η, respectively. Assume the assumptions of Theorem 2.2 are satisfied. Then, the solution of (1) is stable in mean square.

Proof From (1), we have

$$x(t)=B^{-1}T(t)B\xi+\int_0^t B^{-1}T(t-s)\mathscr{F}(x)(s)\mathrm{d}s+$$
$$\int_0^t B^{-1}T(t-s)\sigma(s,x(s))\mathrm{d}W(s)+$$
$$\sum_{0<t_k<t} B^{-1}T(t-t_k)I_k(x(t_k))$$

and

$$y(t)=B^{-1}T(t)B\eta+\int_0^t B^{-1}T(t-s)\mathscr{F}(y)(s)\mathrm{d}s+$$
$$\int_0^t B^{-1}T(t-s)\sigma(s,y(s))\mathrm{d}W(s)+$$
$$\sum_{0<t_k<t} B^{-1}T(t-t_k)I_k(y(t_k)).$$

Thus, we have

$$E\|X(t)-Y(t)\|^2 \leqslant 4M_BM\widetilde{M}_BE\|\xi-\eta\|^2+$$
$$4M_BM(T+1)\int_0^t \kappa(E\|X-Y\|^2)\mathrm{d}s+$$
$$4M_BMm\sum_{k=1}^{k=m}q_kE\|X-Y\|^2.$$

Moreover,

$$E\|X(t)-Y(t)\|^2 \leqslant \frac{4M_BM\widetilde{M}_B}{1-4M_BMm\sum_{k=1}^{k=m}q_k}E\|\xi-\eta\|^2+$$
$$\frac{4M_BM(T+1)}{1-4M_BMm\sum_{k=1}^{k=m}q_k}\int_0^t \kappa(E\|X-Y\|^2)\mathrm{d}s.$$

Let

$$\kappa_1(u)=\frac{4M_BM(T+1)}{1-4M_BMm\sum_{k=1}^{k=m}q_k}\kappa(u),$$

for κ is a concave increasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\kappa(0)=0, \kappa(u)>0$ for $u>0$ and $\int_{0^+}\frac{\mathrm{d}u}{\kappa(u)}=+\infty$. So, $\kappa_1(u)$ is obvious a concave function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\kappa_1(0)=0, \kappa(u)\geqslant \kappa(1)u$, for any $0\leqslant u\leqslant 1$ and $\int_{0^+}\frac{\mathrm{d}u}{\kappa_1(u)}=\infty$. So, for any $\varepsilon>0, \varepsilon_1 \triangleq \frac{1}{2}\varepsilon$, we have $\lim_{s\to 0}\int_s^{\varepsilon_1}\frac{\mathrm{d}u}{\kappa_1(u)}=$

$+\infty$. So, there is a positive constant $\delta<\varepsilon_1$ such that $\int_\delta^{\varepsilon_1}\frac{\mathrm{d}u}{\kappa_1(u)}\geqslant T$. From Corollary 1.4, let

$$u_0=\frac{4M_BM\widetilde{M}_B}{1-4M_BMm\sum_{k=1}^{k=m}q_k}E\|\xi-\eta\|^2,$$

$$u(t)=E\|X(s)-Y(s)\|^2,\ v(t)=1.$$

When $u_0\leqslant\delta\leqslant\varepsilon_1$, we have

$$\int_{u_0}^{\varepsilon_1}\frac{\mathrm{d}u}{\kappa_1(u)}\geqslant\int_\delta^{\varepsilon_1}\frac{\mathrm{d}u}{\kappa_1(u)}\geqslant T=\int_0^T v(s)\mathrm{d}s.$$

So, for any $t\in[0,T]$, the estimate $u(t)\leqslant\varepsilon_1$ holds. This completes the proof of the theorem. □

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