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The asymptotic solution to singularly perturbed boundary value problems for nonlinear nonlocal elliptic equations of higher order with two parameters

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Abstract: The singularly perturbed boundary value problem for a class of nonlinear nonlocal elliptic equation of higher order was considered. Under suitable conditions, the outer solution of the original problem was obtained. Then, applying the multiple scales variable and the method of component expansion, the first and second boundary layer corrective terms were constructed and the formal asymptotic expansion was obtained. Finally, applying the theory of differential inequalities the asymptotic expansion of a solution for the boundary value problem with two parameters was studied. Some relational inequalities were educed. And the existence of the solution for the original problem and the uniformly valid asymptotic estimation were discussed.

Key words: nonlocal; elliptic equation; singular perturbation

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两参数奇摄动非线性非局部高阶椭圆型方程边值问题的渐近解

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摘要:研究了一类非线性非局部高阶椭圆型方程奇摄动边值问题.在适当的条件下,首先求出了原问题的外部解.然后利用多重尺度变量、合成展开法构造出解的第一、第二边界层项,并得到解的形式渐近展开式.最后,利用微分不等式理论,研究了两参数边值问题解的渐近展开式.导出了几个有关的不等式.讨论了原问题

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存在一个解和解的一致有效渐近估计式.
关键词:非局部;椭圆型方程;奇异摄动

0 Introduction

The nonlinear singularly perturbed problem is a very attractive object of study in the mathematical circles[1-2]. Many approximation methods have been refined, including the method of averaging, the boundary layer method, the methods of matched asymptotic expansion and multiple scales. Recently, many scholars[3-7] have done a great deal of work. Using the method of differential inequality and others the authors also considered a class of singularly perturbed nonlinear problems[8-20]. This paper studies a class of nonlinear nonlocal singularly perturbed problems for the elliptic equations of higher order with two parameters by means of the boundary layer method. We construct the asymptotic expansion of solution and discuss its asymptotic behavior.

Now we consider the following nonlinear nonlocal boundary value problem with two parameters

$$\left.\begin{aligned}&\varepsilon^{2(m-1)}L^m u+\mu^{2(k-1)}L^k u+Lu=f(x,u,Tu,\varepsilon,\mu),\\&\qquad x\in\Omega\end{aligned}\right\}\tag{1}$$

$$\frac{\partial^l u}{\partial n^l}(x)=g_l(x),\ l=0,1,\cdots,m-1,x\in\partial\Omega\tag{2}$$

where ε and μ are small positive parameters, Ω signifies a bounded convex domain in R^n with boundary $\partial\Omega$ of class C^∞ and

$$\left\{\begin{aligned}&L=\sum_{i,j=1}^n\alpha_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^n\beta_i(x)\frac{\partial}{\partial x_i},\\&\qquad x=(x_1,x_2,\cdots,x_n)\in\Omega,\end{aligned}\right.$$

$$\left\{\begin{aligned}&Tu=\int_\Omega K(x)u(x,\varepsilon,\mu)\mathrm{d}x,\\&\qquad x=(x_1,x_2,\cdots,x_n)\in\Omega\end{aligned}\right.$$

with $1<k<m$, and L signifies a uniformly elliptic type on $\bar{\Omega}$, $\frac{\partial}{\partial n}$ denotes differentiation on the direction of the outward normal at $\partial\Omega$.

We need the following hypotheses:

[H_1] $\sigma=\varepsilon^m/\mu^k\to 0$, as $\mu\to 0$.

[H_2] The α_{ij},β_i,K and g_l are sufficiently smooth functions with respect to variables.

[H_3]

$$f_u(x,u,Tu,\varepsilon,\mu)\geqslant N,$$
$$f_{T_1u}(x,u,Tu,\varepsilon,\mu)\geqslant N,$$

where N is a positive constant.

We now construct the formal asymptotic expansion for the solution to the boundary value problem (1), (2).

1 The outer solution

The reduced problem of problem (1), (2) is

$$Lu=f(x,u,Tu,0,0),\ x\in\Omega\tag{3}$$

$$u=g_0(x),\ x\in\partial\Omega\tag{4}$$

We need also a hypothesis that

[H_4] There exists a unique solution U_{00} for problem (3), (4) in Ω.

Let formal expansion of the outer solution U for the boundary value problem (1), (2):

$$U\sim\sum_{i,j=0}^\infty U_{ij}\varepsilon^i\mu^j\tag{5}$$

Substituting (5) into Eqs. (1) and (2), developing f and g in ε,μ, we equate the coefficients of same powers for $\varepsilon^i\mu^j$ $(i,j=0,1,\cdots,i+j\neq 0)$ respectively. And we have

$$\begin{aligned}LU_{ij}=&f_u(x,U_{00},TU_{00},0,0)U_{ij}+\\&f_{Tu}(x,U_{00},TU_{00},0,0)TU_{ij}-L^mU_{(i-2m+2)j}-\\&L^kU_{i(j-2k+2)}+F_{ij},\ i,j=0,1,\cdots,i+j\neq 0\end{aligned}\tag{6}$$

$$U_{ij}(x)=0,\ i,j=0,1,\cdots,i+j\neq 0,x\in\partial\Omega\tag{7}$$

where

$$F_{ij}=\frac{1}{i!j!}\Big[\frac{\partial^{i+j}}{\partial\varepsilon^i\partial\mu^j}f(x,\sum_{i,j=0}^\infty U_{ij}\varepsilon^i\mu^j,T(\sum_{i,j=0}^\infty U_{ij}\varepsilon^i\mu^j),\varepsilon,\mu)\Big]_{\varepsilon=\mu=0}.$$

In the above text and below, the values of terms for the negative subscript are zero. From the linear boundary value problems (6), (7), we can solve U_{ij} $(i,j=0,1,2,\cdots;\ i+j\neq 0)$ successively. Substituting U_{ij} $(i,j=0,1,2,\cdots)$ into Eq. (5), we obtain the outer solution U for the original boundary value problem (1), (2). But it may not satisfy the boundary condition (2) for $l=1,2,\cdots,m-1$, so we need to construct boundary layer corrective terms.

2 The first boundary layer corrective term

Set up a local coordinate system (ρ,ϕ) near $\partial\Omega$. Define the coordinate of every point Q in the neighborhood of $\partial\Omega$ in the following way: The coordinate $\rho(\leqslant\rho_0)$ is the distance from point Q to boundary $\partial\Omega$, where ρ_0 is small enough such that the inner normals on every point of $\partial\Omega$ do not intersect each other in this neighborhood of $\partial\Omega$. The $\phi=(\phi_1,\phi_2,\cdots,\phi_{n-1})$ is a nonsingular coordinate system in the $(n-1)$-dimensional manifold $\partial\Omega$. The coordinate ϕ of point Q is defined to be equal to the coordinate ϕ of point $P\in\partial\Omega$ at whose inner normal through point Q.

In the neighborhood of $\partial\Omega$: $0\leqslant\rho\leqslant\rho_0$,

$$L=a_{nn}\frac{\partial^2}{\partial\rho^2}+\sum_{i=1}^{n-1}a_{ni}\frac{\partial^2}{\partial\rho\partial\phi_i}+\sum_{i,j=1}^{n-1}a_{ij}\frac{\partial^2}{\partial\phi_i\partial\phi_j}+b_n\frac{\partial}{\partial\rho}+\sum_{i=1}^{n-1}b_i\frac{\partial}{\partial\phi_i} \tag{8}$$

where

$$a_{nn}=\sum_{i,j=1}^{n}\alpha_{ij}\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j},\quad a_{ni}=2\sum_{j,k=1}^{n}\alpha_{jk}\frac{\partial\rho}{\partial x_j}\frac{\partial\phi_i}{\partial x_k},$$

$$a_{ij}=\sum_{k,l=1}^{n}\alpha_{kl}\frac{\partial\phi_i}{\partial x_k}\frac{\partial\phi_j}{\partial x_l},\quad b_n=\sum_{i,j=1}^{n}\alpha_{ij}\frac{\partial^2\rho}{\partial x_i\partial x_j},$$

$$b_i=\sum_{j,k=1}^{n}\alpha_{jk}\frac{\partial^2\phi_i}{\partial x_j\partial x_k}.$$

We introduce the variables of multiple scales[1] on $0\leqslant\rho\leqslant\rho_0$:

$$\tau=\frac{h(\rho,\phi)}{\mu},\quad \bar\rho=\rho,\quad \bar\phi=\phi,$$

where $h(\rho,\phi)$ is a function to be determined. For convenience, we still substitute ρ,ϕ for $\bar\rho,\bar\phi$ below, respectively. From (8), we have

$$L=\frac{1}{\mu^2}K_0+\frac{1}{\mu}K_1+K_0 \tag{9}$$

while $K_0=a_{nn}h_\rho^2\frac{\partial^2}{\partial\tau^2}$ and K_1,K_2 are determined operators and their constructions are omitted.

Let the solution u of the original boundary value problem (1), (2) be

$$u=U+V \tag{10}$$

where V is a first boundary layer corrective function. Substituting Eq. (10) into problem (1), (2), we have

$$\mu^{2(k-1)}L^kV+LV=f(x,U+V,T(U+V),\varepsilon,\mu)-f(x,U,TU,\varepsilon,\mu)-\varepsilon^{2(m-1)}L^mV \tag{11}$$

$$\frac{\partial^lV}{\partial n^l}=g_l(x)-\frac{\partial^lU}{\partial n^l},\ l=0,1,\cdots,k-1,x\in\partial\Omega \tag{12}$$

Set $h(\rho,\phi)=\int_0^\rho\frac{1}{\sqrt{a_{nn}}}\mathrm{d}\rho$. Thus we have $K_0=\frac{\partial^2}{\partial\tau^2}$. And let

$$V\sim\sum_{i,j=0}^{\infty}v_{ij}(\tau,\phi)\varepsilon^i\sigma^j \tag{13}$$

Substituting Eq. (13) into the boundary value problem (11), (12), expanding nonlinear terms in ε,σ, we equate the coefficients of same powers for $\varepsilon^i\sigma^j$ $(i,j=0,1,\cdots)$. And we obtain

$$\frac{\partial^{2k}v_{00}}{\partial\tau^{2k}}+\frac{\partial^2v_{00}}{\partial\tau^2}=0 \tag{14}$$

$$\frac{\partial^lv_{00}}{\partial n^l}=g_l-U_{00},\ l=0,1,\cdots,k-1,x\in\partial\Omega \tag{15}$$

$$\left.\begin{aligned}&\frac{\partial^{2k}v_{ij}}{\partial\tau^{2k}}+\frac{\partial^2v_{ij}}{\partial\tau^2}=\\&f_u(0,\phi,U_{00}+v_{00},T(U_{00}+v_{00}),0,0)v_{i(j-2)}+\\&f_{Tu}(0,\phi,U_{00}+v_{00},T(U_{00}+v_{00}),0,0)Tv_{i(j-2)}+G_{ij},\\&\qquad i,j=0,1,\cdots,k-1,i+j\neq0\end{aligned}\right\} \tag{16}$$

$$\frac{\partial^lv_{ij}}{\partial n^l}=-U_{ij},\ l=0,1,\cdots,k-1,x\in\partial\Omega \tag{17}$$

where $G_{ij}(i,j=0,1,\cdots,k-1,i+j\neq0)$ are determined functions successively, and their constructions are omitted.

From Eqs. (14), (15), we can have solution v_{00}. And form Eqs. (16), (17), we can also obtain v_{ij} $(i,j=0,1,2,\cdots;\ i+j\neq0)$ successively. Substituting $v_{ij}(i,j=0,1,2,\cdots)$ into Eq. (13), we obtain the first boundary layer function V for the original boundary value problem (1), (2).

3 The second boundary layer corrective term

We introduce the variables of multiple

scales[1] on $0 \leqslant \rho \leqslant \rho_0$:

$$\eta = \frac{h(\rho, \phi)}{\nu\mu}, \tilde{\rho} = \rho, \bar{\phi} = \phi,$$

where $0 < \nu = \varepsilon^{m/(m-k)} \ll 1$. For convenience, we still substitute ρ, ϕ for $\bar{\rho}, \bar{\phi}$ below, respectively. From Eq. (8), we have

$$L = \frac{1}{\nu^2\mu^2}\bar{K}_0 + \frac{1}{\nu\mu}\bar{K}_1 + \bar{K}_0 \tag{18}$$

while $\bar{K}_0 = \frac{\partial^2}{\partial \eta^2}$ and $\bar{K}_1, \bar{K}_2$ are determined operators and their constructions are omitted, too.

Let the solution u of original boundary value problem (1), (2) be

$$u = U + W \tag{19}$$

where W is a second boundary layer corrective function. Substituting Eq. (19) into problem (1), (2), we have

$$\varepsilon^{2(m-1)} L^m W + \varepsilon^{2(k-1)} L^k V = f(x, U+W, T(U+W), \varepsilon, \mu) - f(x, U, TU, \varepsilon, \mu) - LW \tag{20}$$

$$\frac{\partial^l W}{\partial n^l} = g_l(x) - \frac{\partial^l U}{\partial n^l},\ l = 0,1,\cdots m-1, x \in \partial\Omega \tag{21}$$

And let

$$W \sim \sum_{i,j=0}^{\infty} w_{ij}(\eta, \phi)\nu^i\mu^j \tag{22}$$

Substituting Eq. (13) into problem (11), (12), expanding nonlinear terms in μ, σ, we equate the coefficients of same powers for $\mu^i\sigma^j (i, j = 0,1,\cdots)$. And we obtain

$$\frac{\partial^{2m} w_{00}}{\partial \eta^{2m}} + \frac{\partial^{2k} w_{00}}{\partial \eta^{2k}} = 0 \tag{23}$$

$$\frac{\partial^l w_{00}}{\partial n^l} = g_l - U_{00},\ l = 0,1,\cdots,m-1, x \in \partial\Omega \tag{24}$$

$$\left.\begin{aligned}&\frac{\partial^{2m} w_{ij}}{\partial \eta^{2m}} + \frac{\partial^{2k} w_{ij}}{\partial \eta^{2k}} = \\ &f(0, U_{00} + w_{00}, T(U_{00} + w_{00}), 0, 0) w_{(i-2m)(j-2m)} + \\ &f(0, U_{00} + w_{00}, T(U_{00} + w_{00}), 0, 0) T w_{(i-2m)(j-2m)} + \bar{G}_{ij}, \\ &\qquad i, j = 0,1,\cdots,m-1, i+j \neq 0\end{aligned}\right\} \tag{25}$$

$$\frac{\partial^l w_{ij}}{\partial n^l} = -U_{ij},\ l = 0,1,\cdots,m-1, x \in \partial\Omega \tag{26}$$

where $\bar{G}_{ij} (i, j = 0,1,\cdots,k-1; i+j \neq 0)$ are successively determined functions, and their constructions are omitted too.

From Eqs. (23)~(26) we can obtain solution $w_{ij} (i, j = 0,1,\cdots)$ successively. Substituting $w_{ij} (i, j = 0,1,2,\cdots)$ into Eq. (22), we obtain the second boundary layer function W for the original boundary value problem (1), (2).

Note From $(\nu\mu)/\mu = \nu \to 0$, we know that the thickness of the second boundary layer for W is less than of the first boundary layer for V near $\partial\Omega$.

4 The main theorem

Let

$$\bar{v}_{ij} = \psi(\rho) v_{ij}, \bar{w}_{ij} = \psi(\rho) w_{ij},$$

where $\psi(\rho)$ is a sufficiently smooth function on $\bar{\Omega}$ and satisfies

$$\psi(\rho) = \begin{cases} 1, & 0 \leqslant \rho \leqslant \frac{1}{3}\rho_0; \\ 0, & \rho \geqslant \frac{2}{3}\rho_0. \end{cases}$$

For convenience, we still substitute v_{ij}, w_{ij} for $\bar{v}_{ij}, \bar{w}_{ij}$ below, respectively. It is easy to see that $v_{ij}, w_{ij} (i, j = 0,1,\cdots)$ possesses boundary layer behavior. Then we have the first and second boundary corrective terms V and W in $\bar{\Omega}$.

Thus we can construct the following formal asymptotic expansion of the solution u for the original problem (1), (2):

$$u \sim \sum_{i,j=0}^{M} U_{ij}\varepsilon^i\mu^j + \sum_{i=0}^{M}\sum_{j=0}^{M+2} v_{ij}\varepsilon^i\sigma^j + \sum_{i,j=0}^{M+2} w_{ij}\nu^i\mu^j + O(\lambda),\ 0 < \varepsilon, \mu, \sigma \ll 1 \tag{27}$$

where $\lambda = \max(\varepsilon^M\mu^{M+1}, \varepsilon^{M+1}\mu^M, \varepsilon^M\sigma^{M+1}, \varepsilon^{M+1}\sigma^M, \nu^M\mu^{M+1}, \nu^{M+1}\mu^M)$.

We have the theorem as follows:

Theorem 4.1 Under the hypotheses [H_1]~[H_4], there exists a solution u of the singularly perturbed boundary value problem (1), (2) for the elliptic equation of higher order with two parameters and solution u is a solution of the uniformly valid asymptotic expansion (27) for ε, μ on $\bar{\Omega}$.

Proof We first construct the auxiliary functions α and β:

$$\alpha = Y_M - \delta\lambda,\ \beta = Y_M + \delta\lambda \tag{28}$$

where δ is a positive constant large enough, which will be decided below, and

$$Y_m = \sum_{i,j=0}^{M} U_{ij}\varepsilon^i\mu^j + \sum_{i=0}^{M}\sum_{j=0}^{M+2}\nu_{ij}\varepsilon^i\sigma^j + \sum_{i,j=0}^{M+2} w_{ij}\nu^i\mu^j.$$

Obviously, we have

$$\alpha \leqslant \beta,\ x \in \Omega \tag{29}$$

$$\frac{\partial^l \alpha}{\partial n^l} \leqslant g_l(x),\ l = 0,1,\cdots,m-1, x \in \partial\Omega \tag{30}$$

$$\frac{\partial^l \beta}{\partial n^l} \geqslant g_l(x),\ l = 0,1,\cdots,m-1, x \in \partial\Omega \tag{31}$$

Now we prove that

$$\varepsilon^{2(m-1)}L^m\alpha + \mu^{2(k-1)}L^k\alpha + L\alpha - f(x,\alpha,T\alpha,\varepsilon,\mu) \geqslant 0,\ x \in \Omega \tag{32}$$

$$\varepsilon^{2(m-1)}L^m\beta + \mu^{2(k-1)}L^k\beta + L\beta - f(x,\beta,T\beta,\varepsilon,\mu) \leqslant 0,\ x \in \Omega \tag{33}$$

In order to prove Eq. (32), we need to consider three cases: (ⅰ) $0 \leqslant \rho \leqslant \frac{1}{3}\rho_0$; (ⅱ) $\frac{1}{3}\rho_0 < \rho \leqslant \frac{2}{3}\rho_0$; (ⅲ) $\frac{2}{3}\rho_0 < \rho$.

In case (ⅰ), from the hypotheses for ε,μ small enough, there is a positive constant K, such that

$$\begin{aligned}
&\varepsilon^{2(m-1)}L^m\alpha + \mu^{2(k-1)}L^k\alpha + L\alpha - f(x,\alpha,T\alpha,\varepsilon,\mu) = \\
&\quad \varepsilon^{2(m-1)}L^m Y_m + \mu^{2(k-1)}L^k Y_m + LY_m - f(x, Y_m - \delta\lambda, T(Y_m - \delta\lambda),\varepsilon,\mu) = \\
&\quad \varepsilon^{2(m-1)}L^m Y_m + \mu^{2(k-1)}L^k Y_m + LY_m - f(x, Y_m T_1 Y_m,\varepsilon,\mu) + \\
&\quad [f(x,Y_m T Y_m,\varepsilon,\mu) - f(x, Y_m - \delta\lambda, T(Y_m-\delta\lambda),\varepsilon,\mu)] \geqslant \\
&\quad LU_0 - f(x,U_0,TU_0,0,0) + \sum_{i,j=0,i+j\neq 0}^{M}[LU_{ij} - f_u(x,U_{00},TU_{00},0,0)U_{ij} - f_{Tu}(x,U_{00},TU_{00},0,0)TU_{ij} + \\
&\quad L^m U_{(i-2m+2)j} + L^k U_{i(j-2k+2)} - F_{ij}]\varepsilon^i\mu^j + \frac{\partial^{2k} v_{00}}{\partial\tau^{2k}} + \frac{\partial^2 v_{00}}{\partial\tau^2} + \\
&\quad \sum_{\substack{i=0\\ i+j\neq 0}}^{M}\sum_{j=0}^{M+2}\Big[\frac{\partial^{2k} v_{ij}}{\partial\tau^{2k}} + \frac{\partial^2 v_{ij}}{\partial\tau^2} - f_u(0,\phi,U_{00}+v_{00},T(U_{00}+v_{00}),0,0)v_{i(j-2)} + \\
&\quad f_{Tu}(0,\phi,U_{00}+v_{00},T(U_{00}+v_{00}),0,0)Tv_{i(j-2)} + G_{ij}\Big]\varepsilon^i\sigma^j + \frac{\partial^{2m} w_{00}}{\partial\eta^{2m}} + \frac{\partial^{2k} w_{00}}{\partial\eta^{2k}} + \\
&\quad \sum_{i,j=0,i+j\neq 0}^{M+2}\Big[\frac{\partial^{2m} w_{ij}}{\partial\eta^{2m}} + \frac{\partial^{2k} w_{ij}}{\partial\eta^{2k}} - f(0,U_{00}+w_{00},T(U_{00}+w_{00}),0,0)w_{(i-2m)(j-2m)} - \\
&\quad f(0,U_{00}+w_{00},T(U_{00}+w_{00}),0,0)w_{(i-2m)(j-2m)} + \overline{G}_{ij}\Big]\nu^i\mu^j - K\lambda + N\delta\lambda = (N\delta - K)\lambda.
\end{aligned}$$

Now we select $\delta \geqslant K/N$. Then the inequality (32) is proved.

For cases (ⅰ) and (ⅲ), we can also prove inequality (32) using the same method.

Analogously, we can prove inequality (33). Thus from Eqs. (29)～(33) and the theory of differential inequalities, we obtain

$$\alpha(x,\varepsilon,\mu) \leqslant u(x,\varepsilon,\mu) \leqslant \beta(x,\varepsilon,\mu),\ x \in \overline{\Omega}.$$

Thus from Eq. (28), the Eq. (27) is proved. The proof of the theorem is completed. □

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