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Equivalent conditions of Devaney chaos on the hyperspace

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Abstract: Let T be a continuous self-map of a compact metric space X. The transformation T induces naturally a continuous self-map T_K on the hyperspace $K(X)$ of all non-empty closed subsets of X. It is shown that the system $(K(X), T_K)$ is Devaney chaotic if and only if $(K(X), T_K)$ is an HY-system if and only if (X, T) is an HY-system, where a system is called an HY-system if it is totally transitive and has dense small periodic sets.

Key words: hyperspace; Devaney chaos; HY-system

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超空间系统为 Devaney 混沌的等价条件

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摘要：设 T 是紧致度量空间 X 上的一个连续自映射. 映射 T 自然诱导了由 X 所有非空闭子集组成的超空间 $K(X)$ 上的一个连续自映射 T_K. 证明了系统 $(K(X), T_K)$ 为 Devaney 混沌的当且仅当 $(K(X), T_K)$ 为一个 HY 系统当且仅当 (X, T) 为一个 HY 系统，其中，称一个系统为 HY 系统如果它是完全传递的和具有稠密的小周期集.

关键词：超空间；Devaney 混沌；HY 系统

0 Introduction

Throughout this paper, by a topological dynamical system (X, T) we mean a compact metric space X with a continuous map T from X into itself; the metric on X is denoted by d.

Let $K(X)$ be the hyperspace on X, i.e., the space of non-empty closed subsets of X equipped with the Hausdorff metric d_H defined by

$$d_H(A, B) = \max\{\max_{x\in A}\min_{y\in B} d(x, y), \max_{y\in B}\min_{x\in A} d(x, y)\} \quad \text{for} \quad A, B \in K(X).$$

The transformation T induces naturally a continuous self-map T_K on the hyperspace $K(X)$ defined by

$$T_K(C) = TC, \quad \text{for} \quad C \in K(X).$$

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Then $(K(X), T_K)$ is also a topological dynamical system.

In 1975, Bauer et al.[1] initiated the study on the connection between the properties of (X, T) and $(K(X), T_K)$. They considered such properties as distality, transitivity and mixing property. This leads to a natural question: if one of the systems (X, T) and $(K(X), T_K)$ is chaotic in some sense, how about the other? This question attracted a lot of attention, see, e. g., Refs. [2-4] and references therein, and many partial answers were obtained.

One of the most popular definitions of chaos is Devaney chaos introduced in Ref. [5] (see Section 1). The aim of this paper is to obtain some equivalent conditions of Devaney chaos on the hyperspace. The main result is the following:

Theorem 0.1 Let (X, T) be a dynamical system with X being infinite. Then the following conditions are equivalent:

① $(K(X), T_K)$ is Devaney chaotic;

② $(K(X), T_K)$ is an HY-system;

③ (X, T) is an HY-system.

1 Preliminaries

1.1 Basic definitions and notations

Let $\mathbb{N}$ denote the set of positive integers. Through out the rest of this paper X denotes an infinite compact metric space with a metric d.

Let (X, T) be a dynamical system. We say that $x \in X$ is a periodic point if $T^n x = x$ for some $n \in \mathbb{N}$. The set of all periodic points of the system (X, T) is denoted as $Per(X, T)$.

Definition 1.1 Let (X, T) be a dynamical system. The system (X, T) is called

① transitive if for any two non-empty open subsets U and V of X, there exists $n \in \mathbb{N}$ such that $T^n U \cap V \neq \emptyset$;

② totally transitive if for each $n \in \mathbb{N}$, (X, T^n) is transitive;

③ weakly mixing if the product system $(X \times X, T \times T)$ is transitive;

It is not hard to see that every weakly mixing system is totally transitive.

Definition 1.2 Let (X, T) be a dynamical system. We say that (X, T) has sensitive dependence on initial conditions if there exists a constant $\delta > 0$ such that for every point $x \in X$ and every neighborhood U of x, there exists a point $y \in U$ and a positive integer n such that $d(T^n x, T^n y) > \delta$.

Definition 1.3 Let (X, T) be a dynamical system. The system (X, T) is called Devaney chaotic if it satisfies the following conditions:

① it is transitive;

② periodic points are dense;

③ it has sensitive dependence on initial conditions.

It is well known that if X is infinite then the third condition of Devaney chaos is redundant (see, e. g., Ref. [6]). Since we will restrict our attention to infinite compact metric spaces we will say that (X, T) is Devaney chaotic if it is transitive and has dense periodic points.

Definition 1.4 Let (X, T) be a dynamical system. We say that (X, T) has dense small periodic sets if for any non-empty open subset U of X there exists a closed subset Y of U and $k \in \mathbb{N}$ such that $T^k Y \subset Y$.

Definition 1.5 Let (X, T) be a dynamical system. The system (X, T) is called

① an F-system if it is totally transitive and has a dense set of periodic points.

② an HY-system if it is totally transitive and has dense small periodic sets.

In Ref. [7] Furstenberg showed that every F-system is weakly mixing and disjoint from any minimal system. Recently, Huang et al.[8] showed that a system which is totally transitive and has dense small periodic sets is also weakly mixing and disjoint from any minimal system. For this reason, such a system is called an HY-system in Ref. [9]. It should be noticed that Ref. [9] characterized HY-system by transitive points via the family of weakly thick sets. Clearly, every F-system is also an HY-system, but there exists an HY-system without periodic points[8].

1.2 Hyperspaces

Let $K(X)$ be the hyperspace of X, i.e., the space of non-empty closed subset of X equipped with the Hausdorff metric d_H. Then $K(X)$ is also a compact mertic space. The following family

$$\{\langle U_1,\cdots,U_n\rangle: U_1,\cdots,U_n \text{ are non-empty open subsets of } X,\ n\in\mathbb{N}\}$$

forms a basis for a topology of $K(X)$ called the Vietoris topology, where

$$\langle S_1,\cdots,S_n\rangle \doteq \{A\in K(X): A\subset\bigcup_{i=1}^{n} S_i \text{ and } A\cap S_i\neq\emptyset \text{ for each } i=1,\cdots,n\}$$

is defined for arbitrary non-empty subsets $S_1,\cdots,S_n\subset X$. It is not hard to see that the Hausdorff topology (the topology induced by the Hausdorff metric d_H) and the Vietoris topology for $K(X)$ coincide. For more details on hyperspaces see Ref. [10].

2 Proof of the main result

First, we need the following useful Lemma.

Lemma 2.1[2] Let (X,T) be a dynamical system. Then the following conditions are equivalent:

① $(K(X),T_K)$ is weakly mixing;

② $(K(X),T_K)$ is transitive;

③ (X,T) is weakly mixing.

Now we turn to the proof of the main result of this paper.

Proof of Theorem 0.1

①⇒② follows from the definitions and Lemma 2.1.

②⇒③ Assume that $(K(X),T_K)$ is an HY-system. By Lemma 2.1, (X,T) is weakly mixing. Then we only need to show that (X,T) has dense small periodic sets. Let U be a non-empty open subset of X. Then $\langle U\rangle=\{A\in K(X): A\subset U\}$ is an open subset of $K(X)$. Since $(K(X),T_K)$ has dense small periodic sets, there exists a closed subset $\mathscr{A}\subset\langle U\rangle$ and $k\in\mathbb{N}$ such that $(T_K)^k\mathscr{A}\subset\mathscr{A}$. Let $Y=\bigcup\{A: A\in\mathscr{A}\}$. We want to show that Y is a closed subset of X. Let x_n be a sequence of points in Y and converge to x. Then for every $n\in\mathbb{N}$ there exists an $A_n\in\mathscr{A}$ such that $x_n\in A_n$. Since $\mathscr{A}$ is a closed subset of $K(X)$ and $K(X)$ is compact, without loss of generality, we can assume that sequence A_n converges to $A\in\mathscr{A}$ in the hyperspace $K(X)$. By the definition of Hausdorff metric, it is not hard to check that $x\in A$. Then $x\in Y$, which implies that Y is a closed subset of X. Clearly, $Y\subset U$ and $T^kY\subset Y$. Thus, (X,T) has dense small periodic sets.

③⇒① Assume that (X,T) is an HY-system. Then (X,T) is weakly mixing, and by Lemma 2.1 $(K(X),T_K)$ is also weakly mixing. So what remains to be done is to show that $(K(X),T_K)$ has a dense set of periodic points. Let $\mathscr{U}$ be a non-empty open subset of $K(X)$. Then there are open subsets $U_1,\cdots,U_n$ of X such that $\langle U_1,\cdots,U_n\rangle\subset\mathscr{U}$. Since (X,T) has dense small periodic sets, there exist $Y_1,\cdots,Y_n$ and $k_1,\cdots,k_n$ such that $Y_i\subset U_i$ and $T^{k_i}Y_i\subset Y_i$ for $i=1,\cdots,n$. For each $i=1,\cdots,n$, choose a closed subset Z_i of Y_i with $T^{k_i}Z_i=Z_i$. Let $Z=\bigcup_{i=1}^{n}Z_i$ and $k=k_1\times\cdots\times k_n$. Then

$$Z\in\langle U_1,\cdots,U_n\rangle \text{ and } (T_K)^kZ=Z.$$

Therefore, $(K(X),T_K)$ is chaotic in the Devaney sense. □

Ref. [11] strengthened the notion of Devaney chaos in a sense to the extreme, and introduced exact Devaney chaos. Recall that a system (X,T) is called topologically exact if for any non-empty open subset U of X there exists an $n\in\mathbb{N}$ such that $T^nU=X$. The system (X,T) is called exactly Devaney chaotic if it is topologically exact with dense periodic points.

Lemma 2.2[3] Let (X,T) be a dynamical system. Then $(K(X),T_K)$ is topologically exact if and only if (X,T) is topologically exact.

Corollary 2.3 Let (X,T) be a dynamical system. Then $(K(X),T_K)$ is exactly Devaney chaotic if and only if (X,T) is a topologically exact HY-system.

Proof It follows from Theorem 0.1 and Lemma 2.2. □

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