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A Morawetz estimate related to almost periodic solutions and its application to Schrödinger equations

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Abstract: Under the condition $\|u_0\|_{\dot{H}^1} \leqslant \|W\|_{\dot{H}^1}$, a new type of Morawetz estimate related to almost periodic solutions was given to exclude the existence of a special minimal blowup solutions for three dimentional energy-critical focusing NLS in the radial case, where W is the ground state.

Key words: Morawetz estimate; almost periodic solutions; Schrödinger equations

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几乎周期解的 Morawetz 估计及对薛定谔方程的应用

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摘要:在初始能量小于基态能量即 $\|u_0\|_{\dot{H}^1} \leqslant \|W\|_{\dot{H}^1}$ 的条件下,给出了关于几乎周期解的一种新的 Morawetz 估计,然后排除三维径向能量临界的薛定谔方程的一个特殊极小爆破解的存在性.这里 W 为基态.

关键词:Morawetz 估计;几乎周期解;薛定谔方程

0 Introduction

In this paper, we will consider the equation

$$\left.\begin{array}{l} i\partial_t u + \Delta u = -|u|^4 u,\ (x,t) \in \mathbb{R}^3 \times \mathbb{R}; \\ u(0) = u_0 \in \dot{H}^1(\mathbb{R}^3) \end{array}\right\} \quad (1)$$

where u_0 is radial. Here, $u: \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ is a complex-valued function.

The energy

$$E(u(t)) := \int_{\mathbb{R}^3} \left[\frac{1}{2} |\nabla u(t,x)|^2 - \frac{1}{6} |u(t,x)|^6\right] dx.$$

The solutions to Eq. (1) and the energy are invariant by the scaling

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$$u(t,x)\mapsto\lambda^{\frac{1}{2}}u(\lambda^2 t,\lambda x).$$

For this reason, Eq. (1) is called energy-critical.

A function u on a non-empty time interval $I\ni 0$ is called a strong solution to (1) if it lies in the class $C_t^0\dot{H}_x^1(K\times\mathbb{R}^3)\cap L_{t,x}^{10}(K\times\mathbb{R}^3)$ for all compact $K\subset I$, and obeys the Duhamel formula

$$u(t)=e^{it\Delta}u_0+i\int_0^t e^{i(t-s)\Delta}|u(s)|^4u(s)ds.$$

u is called a maximal-lifespan solution if it cannot be extended to any strictly larger interval.

Theorem 0.1 Assume that $E(u_0)<E(W)$, $\|u_0\|_{\dot{H}^1}<\|W\|_{\dot{H}^1}$ and u_0 is radial. Then the solution u with data u_0 at $t=0$ is defined for all time and there exists $u_{0,+}$, $u_{0,-}$ in $\dot{H}^1$ such that

$$\lim_{t\to\pm\infty}\|u(t)-e^{it\Delta}u_{0,\pm}\|_{\dot{H}^1}=0 \qquad (2)$$

Theorem 0.1 was proved by Kenig and Merle by concentration-contradiction in Ref. [1]. They reduced it to the existence of minimal blowup solutions of almost periodic solutions and used the virial identity to verify that. Here, we will use Morawetz estimate to prove the similar result for almost periodic solutions.

Definition 0.2 A solution $u\in L^\infty\dot{H}_x^1(I\times\mathbb{R}^3)$ is said to be almost periodic (modulo symmetries) if there exist functions $N: I\to\mathbb{R}^+$, $x(t): I\to\mathbb{R}^3$ and $C:\mathbb{R}^+\to\mathbb{R}^+$ such that for all $t\in I$ and $\eta>0$,

$$\int_{|x-x(t)|\geqslant C(\eta)/N(t)}|\nabla u(t,x)|^2dx+\int_{|\xi|\geqslant C(\eta)N(t)}|\xi|^2|\hat{u}(t,\xi)|^2d\xi\leqslant\eta \qquad (3)$$

We refer to the function $N(t)$ as the frequency scale function for the solution u, to $x(t)$ as the spatial center function, and to $C(\eta)$ as the modulus of compactness.

Then we have the following important result:

Theorem 0.3 (Reduction to almost periodic solutions, see Refs. [1-2]) Suppose Theorem 0.1 fails. Then there exists a maximal-lifespan solution $u: I\times\mathbb{R}^3\to\mathbb{C}$ to (1) which is almost periodic and blows up both forward and backward in time in the sense that for all $t_0\in I$,

$$\int_{t_0}^{\sup I}\int_{\mathbb{R}^3}|u(t,x)|^{10}dxdt=\int_{\inf I}^{t_0}\int_{\mathbb{R}^3}|u(t,x)|^{10}dxdt=\infty.$$

The proof of Theorem 0.3 can be found in Ref. [1], but for complete details see Ref. [2]. Though Theorem 0.3 does not explicitly claim that u is a minimal counterexample, this is how it is constructed and shown to be almost periodic.

Our main result is

Theorem 0.4 There are no minimal energy blowup solutions to (1) when $N(t)\equiv 1$ such that $\int_0^{T_{\max}}N^{-1}(t)dt=\infty$.

The similar result was proved by Dodson[6] for the mass critical NLS (nonlinear Schrödinger) equations with mass below that of the ground state. Here we will prove Theorem 0.4 for the three dimensional energy-critical focusing NLS in the radial case. And the case when $N(t)$ varies is interesting.

The remainder of the paper is organized as follows. In Section 1, we will give some notations and useful lemmas. In Section 2, we will construct a Morawetz estimate that gives the contradiction

$$K=\int_0^T N^{-1}(t)dt\lesssim o(K)$$

for K large enough and its error is small. In Section 3, we will verify that the error from the truncation in frequency is small.

1 Several lemmas

We will need the Littlewood-Paley theory. Let $\phi\in C_0^\infty(\mathbb{R}^3)$, radial, supported in the ball $|x|\leqslant 2$ and $\phi(x)=1$ on the ball $|x|\leqslant 1$. Then, we can define the Littlewood-Paley projection operators

$$\widehat{P_{\leqslant N}f}(\xi):=\phi(\xi/N)\hat{f}(\xi) \qquad (4)$$

$$\widehat{P_{>N}f}(\xi):=(1-\phi(\xi/N))\hat{f}(\xi) \qquad (5)$$

$$\widehat{P_N f}(\xi):=(\phi(\xi/N)-\phi(2\xi/N))\hat{f}(\xi) \qquad (6)$$

Similarly, we can define $P_{<N}$, $P_{\geqslant N}$, and

$$P_{M<\cdot\leqslant N}:=P_{\leqslant N}-P_{\leqslant M},$$

where N and M are dyadic numbers. Sometimes, we will use $f_{\leqslant N}$ instead of $P_{\leqslant N}f$.

Theorem 1.1 (Sobolev and Bernstein

estimates)[7] For $s\geqslant 0$, $1\leqslant p\leqslant q\leqslant\infty$,

$$\|P_{\geqslant N}f\|_{L^p}\lesssim N^{-s}\||\nabla|^s P_{\geqslant N}f\|_{L^p},$$
$$\|P_{\leqslant N}|\nabla|^s f\|_{L^p}\lesssim N^s\|P_{\leqslant N}f\|_{L^p},$$
$$\|P_N|\nabla|^{\pm s}f\|_{L^p}\sim N^{\pm s}\|P_N f\|_{L^p},$$
$$\|P_{\leqslant N}f\|_{L^q}\lesssim N^{\frac{3}{p}-\frac{3}{q}}\|P_{\leqslant N}f\|_{L^p},$$
$$\|P_N f\|_{L^q}\lesssim N^{\frac{3}{p}-\frac{3}{q}}\|P_N f\|_{L^p},$$

where $X\lesssim Y$ means that there is a constant C such that $X\leqslant CY$, and $X\sim Y$ means $X\lesssim Y\lesssim X$. In addition, $X\lesssim_u Y$ means there is a constant C depending on u such that $X\leqslant C(u)Y$.

In the focusing case, the solution of Eq. (1) must satisfy the condition:

Theorem 1.2 (Energy trapping, see Ref. [1]) Let u be a solution of Eq. (1), with $t_0=0$, $u|_{t=0}=u_0$ such that for $\delta_0>0$,

$$\left.\begin{aligned}&\int|\nabla u_0|^2<\int|\nabla W|^2,\\&E(u_0)<(1-\delta_0)E(W)\end{aligned}\right\}\tag{7}$$

Let $I\ni 0$ be the maximal lifespan of solution, and $\bar{\delta}=\bar{\delta}(\delta_0,3)$. Then for each $t\in I$, we have

$$\int|\nabla u(t)|^2\leqslant(1-\bar{\delta})\int|\nabla W|^2\tag{8}$$

$$\int|\nabla u(t)|^2-|u(t)|^{2^*}\geqslant\bar{\delta}\int|\nabla u(t)|^2\tag{9}$$

$$E(u(t))\geqslant 0\tag{10}$$

where $2^*=6$.

Lemma 1.3 Let u be the solution of (1), then

$$\|u\|_{L^6}\leqslant C_3\|\nabla u\|_{L^2}\tag{11}$$

Proof We consider the elliptic equation

$$-\Delta W=|W|^4W\tag{12}$$

We have

$$\|\nabla W\|_{L^2}^2=\|W\|_{L^6}^6\tag{13}$$

So

$$C_3=\|\nabla W\|_{L^2}^{-2/3}\tag{14}$$

Then

$$\|u\|_{L^6}\leqslant\frac{1}{\|\nabla W\|_{L^2}^{2/3}}\|\nabla u\|_{L^2}\tag{15}$$

□

Lemma 1.4 Suppose $\chi\in C_0^\infty(\mathbb{R}^3)$, $\chi(x)=1$ on $|x|\leqslant\frac{R}{2}$ and $\chi(x)=0$ on $|x|\geqslant R$. u is the solution of Eq. (1). Then

$$\int|\nabla(\chi u)|^2\leqslant\int|\nabla u|^2\tag{16}$$

Proof Since $\chi\in C_0^\infty\subset\mathscr{S}$, then

$$|\chi(x)|\leqslant C_N\frac{1}{(1+|x|)^N},$$

$$|(\nabla\chi)(x)|\leqslant C_{N+1}\frac{1}{(1+|x|)^{N+1}},$$

where C_N and C_{N+1} are dependent on N, and $N=0,1,2,\cdots$.

$$\begin{aligned}&\int|\nabla(\chi(x)u)|^2dx=\\&\int_{|x|\leqslant R}|\nabla(\chi(x)u)|^2dx=\\&\int_{|x|\leqslant R/2}|\nabla u(x)|^2dx+\\&\int_{R/2\leqslant|x|\leqslant R}|\nabla(\chi u)(x)|^2dx\leqslant\\&\int_{|x|\leqslant R/2}|\nabla u(x)|^2dx+\\&2\int_{R/2\leqslant|x|\leqslant R}|\nabla\chi(x)|^2|u|^2+\chi^2|\nabla u|^2dx\leqslant\\&\int_{|x|\leqslant R/2}|\nabla u(x)|^2dx+\\&2\frac{C_{N+1}^2}{(1+R/2)^{2N+2}}\int_{R/2\leqslant|x|\leqslant R}|u|^2+\\&2\frac{C_N^2}{(1+R/2)^{2N}}\int_{R/2\leqslant|x|\leqslant R}|\nabla u|^2dx\leqslant\\&\int_{|x|\leqslant R/2}|\nabla u(x)|^2dx+\\&\left(2\frac{C_N^2}{(1+R/2)^{2N}}+8\frac{C_{N+1}^2R^2}{(1+R/2)^{2N+2}}\right)\cdot\\&\int_{R/2\leqslant|x|\leqslant R}|\nabla u|^2dx\end{aligned}\tag{17}$$

Choose $N\geqslant 2$, then fix N and choose R large enough such that

$$2\frac{C_N^2}{(1+R/2)^{2N}}+8\frac{C_{N+1}^2R^2}{(1+R/2)^{2N+2}}<1\tag{18}$$

Therefore,

$$\int|\nabla(\chi(x)u)|^2dx\leqslant\int|\nabla u|^2\tag{19}$$

□

2 Nonexistence of minimal energy blowup solutions

Let Iu denote $P_{\geqslant N}$. Suppose

$$i\partial_t Iu=-\Delta Iu-|Iu|^4Iu+\mathscr{F},$$

where $\mathscr{F}=|Iu|^4Iu-|u|^4u$. Then ignoring the Fourier truncation error $\mathscr{F}$, we only consider $i\partial_t Iu=-\Delta Iu-|Iu|^4Iu$.

Theorem 2.1 There are no minimal energy blowup solutions to (1) when $N(t)\equiv 1$ such that $\int_0^{T_{\max}} N^{-1}(t)\mathrm{d}t=\infty$.

Proof u is radial meaning $x(t)=0$. We use the Morawetz potential. Let $\psi\in C^\infty(\mathbb{R}^3)$, $\psi(x)=1$ when $|x|\leqslant R$, $\psi(x)=0$ when $|x|\geqslant 2R$ and $|\partial^\alpha\psi(x)|\lesssim\frac{1}{|x|^{|\alpha|}}$, $\forall\,\alpha\in\mathbb{N}^3$. Here R is a positive number that is to be determined. In the following pages, we write a complex function z as $z=\mathrm{Re}\,z+\mathrm{Im}\,z$, where $\mathrm{Re}\,z$ is the real part of z and $\mathrm{Im}\,z$ is the imaginary part of z.

Let

$$M(t)=\int x\psi(x)\mathrm{Im}(\overline{Iu}(x)\nabla Iu(x))\mathrm{d}x \quad (20)$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}M(t)=\int x\psi(x)\mathrm{Im}(\overline{Iu_t}(x)\nabla Iu(x))\mathrm{d}x+\int x\psi(x)\mathrm{Im}(\overline{Iu}(x)\nabla Iu_t(x))\mathrm{d}x=\mathrm{I}+\mathrm{II}.$$

Integrating by parts,

$$\begin{aligned}\mathrm{I}=&\int x\psi(x)\mathrm{Im}(\overline{(i\Delta Iu+i|Iu|^4Iu)}\nabla Iu)\mathrm{d}x=\\&-\mathrm{Re}\int x\psi(x)\Delta\overline{Iu}\nabla Iu\mathrm{d}x-\\&\mathrm{Re}\int x\psi(x)|Iu|^4\overline{Iu}\nabla Iu\mathrm{d}x=\\&\frac{1}{6}\int\nabla(x\psi(x))|Iu|^6\mathrm{d}x-\\&\frac{1}{2}\int\nabla(x\psi(x))|\nabla Iu|^2\mathrm{d}x+\\&\sum_{j,k=1}^{3}\mathrm{Re}\int\partial_k(x\psi(x))\partial_k\overline{Iu}\partial_jIu\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\mathrm{II}=&\int x\psi(x)\mathrm{Im}(\overline{Iu}\nabla Iu_t)\mathrm{d}x=\\&-\int\nabla(x\psi(x))\mathrm{Im}(\overline{Iu}Iu_t)\mathrm{d}x-\\&\int x\psi(x)\mathrm{Im}(\nabla\overline{Iu}Iu_t)\mathrm{d}x=\\&-\int\nabla(x\psi(x))\mathrm{Re}(\overline{Iu}(\Delta Iu+|Iu|^4Iu))\mathrm{d}x+\mathrm{I}=\\&-\frac{1}{2}\int\Delta(\nabla(x\psi(x)))|Iu|^2\mathrm{d}x+\\&\int\nabla(x\psi(x))|\nabla Iu|^2-\int\nabla(x\psi(x))|Iu|^6+\mathrm{I},\end{aligned}$$

then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}M(t)=&2\,\mathrm{I}+\mathrm{II}=\\&2\sum_{j,k=1}^{3}\mathrm{Re}\int\partial_k(x_j\psi(x))\partial_k\overline{Iu}\partial_jIu\mathrm{d}x-\\&\frac{2}{3}\int\nabla(x\psi(x))|Iu|^6\mathrm{d}x-\\&\frac{1}{2}\int\Delta(\nabla(x\psi(x)))|Iu|^2\mathrm{d}x.\end{aligned}$$

Let $\chi(x)\in C_0^\infty$, and $\chi(x)=1$ on $|x|\leqslant\frac{R}{2}$, $\chi(x)=0$ on $|x|\geqslant R$. Since

$$\chi\nabla Iu=\nabla(\chi Iu)-(\nabla\chi)Iu \quad (21)$$

we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}M(t)=&2\int|\nabla(\chi Iu)|^2\mathrm{d}x-2\int|\chi Iu|^6\mathrm{d}x+\\&2\int(\nabla\chi)^2|Iu|^2\mathrm{d}x+2\int(\psi-\chi^2)|\nabla Iu|^2\mathrm{d}x-\\&4\mathrm{Re}\int\nabla(\chi Iu)(\nabla\chi)\overline{Iu}\mathrm{d}x+\\&2\sum_{j,k=1}^{3}\mathrm{Re}\int(x\partial_k\psi(x))\partial_k\overline{Iu}\partial_jIu\mathrm{d}x-\\&2\int(\psi-\chi^6)|Iu|^6\mathrm{d}x-\frac{2}{3}\int x\nabla\psi(x)|Iu|^6\mathrm{d}x-\\&\frac{1}{2}\int\Delta(\nabla(x\psi(x)))|Iu|^2\mathrm{d}x.\end{aligned}$$

By Lemma 1.3, Lemma 1.4 and $\|\nabla u\|_{L^2}^2<\|\nabla W\|_{L^2}^2$,

$$\begin{aligned}&2\int|\nabla(\chi Iu)|^2\mathrm{d}x-2\int|\chi Iu|^6\mathrm{d}x\geqslant\\&2\left(1-\frac{\|\nabla Iu_0\|_{L^2}^4}{\|\nabla W\|_{L^2}^4}\right)\|\nabla(\chi Iu)\|_{L^2}^2.\end{aligned}$$

Let us make a refinement.

$$\begin{aligned}&2\int|\nabla(\chi Iu)|^2\mathrm{d}x-2\int|\chi Iu|^6\mathrm{d}x=\\&2\int|\nabla(\chi Iu)|^2\mathrm{d}x-(2+\eta)\int|\chi Iu|^6\mathrm{d}x+\\&\eta\int|\chi Iu|^6\mathrm{d}x\geqslant\\&\left[2-(2+\eta)\frac{\|\nabla u_0\|_{L^2}^4}{\|\nabla W\|_{L^2}^4}\right]\|\nabla(\chi Iu)\|_{L^2}^2+\\&\eta\int|\chi Iu|^6\mathrm{d}x.\end{aligned}$$

Choose η such that

$$2-(2+\eta)\frac{\|\nabla Iu_0\|_{L^2}^4}{\|\nabla W\|_{L^2}^4}=\eta,$$

then

$$2\int |\nabla(\chi Iu)|^2 dx - 2\int |\chi Iu|^6 dx \geqslant \eta(\|\nabla(\chi Iu)\|_{L^2}^2 + \int |\chi Iu|^6 dx) \tag{22}$$

Then

$$\frac{d}{dt}M(t) \geqslant \eta(\|\nabla(\chi Iu)\|_{L^2}^2 + \int |\chi Iu|^6 dx) - \int_{|x|>R/2} |Iu(x)|^6 dx - \frac{C(\eta)}{R(\eta)^{4/3}}\|Iu\|_{L^6}^2 - \eta\|\nabla(\chi Iu)\|_{L^2}^2 \tag{23}$$

We can choose $R(\eta)$ sufficiently large so that

$$\int_0^K \frac{d}{dt}M(t)dt \geqslant \int_0^K \eta\|\chi Iu\|_{L^6}^6 dt - K\frac{C(\eta)}{R(\eta)^{4/3}} - \int_0^K\int_{|x|>R/2} |Iu(x)|^6 dxdt \gtrsim_\eta K \tag{24}$$

From the definition of almost periodic solution,

$$M(t) = \int x\psi(x)\mathrm{Im}(\overline{Iu}(x)\nabla Iu(x))dx \lesssim Ro(K) \tag{25}$$

For K sufficiently large, this gives a contradiction.

3　The estimates of Fourier truncation errors

In previous sections, we ignored the Fourier truncation errors. Now, we will show that the errors can be controlled by $o(K)$. First, we need some restrictions on these parameters. We choose N small enough and R sufficiently large, so that given $\eta = \eta(u)$,

$$\int_{\mathbb{R}^3} |\nabla u_{lo}(t,x)|^2 dx + \int_{\mathbb{R}^3} |Nu_{hi}(t,x)|^2 dx + \int_{|x|>\frac{R}{2}} |\nabla u_{hi}(t,x)|^2 dx < \eta^2 \tag{26}$$

uniformly for $0 \leqslant t < T_{\max}$ where $u_{hi} := u_{>N} = P_{>N}$, and $u_{lo} := u - u_{hi}$. This observation follows immediately from the fact that u is almost periodic modulo symmetries and $N(t) \geqslant 1$.

Lemma 3.1 (A priori bounds, see Ref.[3]) For all $\frac{2}{q} + \frac{3}{r} = \frac{3}{2}$ with $2 \leqslant q \leqslant \infty$ and any $s < 1 - \frac{3}{q}$,

$$\|\nabla u_{lo}\|_{L_t^q L_x^r} + \|N^{1-s}|\nabla|^s u_{hi}\|_{L_t^q L_x^r} \lesssim_u (1 + N^3 K)^{1/q} \tag{27}$$

Under the hypothesis of (26),

$$\|u_{lo}\|_{L_t^4 L_x^\infty} \lesssim_u \eta^{1/2}(1 + N^3 K)^{1/4} \tag{28}$$

In the proof, we will use the fact $N \leqslant 1$[3]. Let

$$\mathscr{F} = |u_{hi}|^4 u_{hi} - |u|^4 u$$

and

$$\{\mathscr{F}, u_{hi}\}_p = \mathrm{Re}(\mathscr{F}\nabla\overline{u_{hi}} - u_{hi}\nabla\overline{\mathscr{F}}).$$

Proposition 3.2　For any $\varepsilon \in (0,1]$,

$$\left|\int_0^T\int_{\mathbb{R}^3} \psi(x)x\cdot\{\mathscr{F}, u_{hi}\}_p dxdt\right| \lesssim_u \varepsilon\int_I\int_{\mathbb{R}^3} |u_{hi}(t,x)|^6 dxdt + \eta\|u_{hi}\|_{L_t^4 L_x^3}^2 + (\varepsilon^{-1}\eta + \eta)(N^{-3} + K) \tag{29}$$

Proof　First, we know that

$$\{F(\phi), \phi\} = \frac{2}{3}\nabla|\phi|^6,$$

and

$$\{\mathscr{F}, u_{hi}\} = \nabla\sum_{j=1}^5 \emptyset(u_{hi}^j u_{lo}^{6-j}) + \emptyset(u^2 u_{hi} u_{lo}^2 \nabla u_{lo}) + \emptyset(u^3 u_{hi}^2 \nabla u_{lo}) + \nabla\emptyset(u_{hi}P_{lo}F(u)) + \emptyset(u_{hi}\nabla P_{lo}F(u)) \tag{30}$$

We first estimate the first term of (30). Integrating by parts and using

$$\sum_{j=1}^5 |u_{hi}|^j |u_{lo}|^{6-j} \lesssim \varepsilon|u_{hi}|^6 + \varepsilon^{-1}|u_{lo}|^2|u_{hi}|[|u_{hi}| + |u_{lo}|]^3,$$

we have that

$$\varepsilon\int_I\int_{\mathbb{R}^3} \nabla(\psi(x)x)|u_{hi}(t,x)|^6 dxdt \lesssim \varepsilon\int_I\int_{\mathbb{R}^3} |u_{hi}(t,x)|^6 dxdt$$

and

$$\varepsilon^{-1}\int_I\int_{\mathbb{R}^3} \nabla(\psi(x)x)|u_{lo}|^2|u_{hi}|[|u_{hi}| + |u_{lo}|]^3 = \varepsilon^{-1}\int_I\int_{\mathbb{R}^3} \nabla(\psi(x)x)(|u_{lo}|^2|u_{hi}|^2 + |u_{lo}|^3|u_{hi}|)\cdot[|u_{hi}| + |u_{lo}|]^2 dxdt \lesssim$$
$$\varepsilon^{-1}\|u_{hi}\|_{L_t^4 L_x^3}^2\|u_{lo}\|_{L_t^4 L_x^\infty}^2\|u\|_{L_t^\infty L_x^6}^2 + \varepsilon^{-1}\|u_{hi}\|_{L_t^4 L_x^3}\|u_{lo}\|_{L_t^4 L_x^\infty}^2\|u\|_{L_t^\infty L_x^6}^2\|u_{lo}\|_{L_t^4 L_x^3}^2 \lesssim \varepsilon^{-1}\eta(N^{-3} + K),$$

where we have used the Hölder inequality. Now, we use Lemma 3.1 to estimate the second term of (30).

$$\|\emptyset(u^2 u_{hi} u_{lo}^2 \nabla u_{lo})\|_{L_{t,x}^1} \lesssim \|\emptyset(u^2 u_{hi} u_{lo}^2 \nabla u_{lo})\|_{L_{t,x}^1} \lesssim \|u_{hi}\|_{L_t^\infty L_x^2}\|u_{lo}\|_{L_t^4 L_x^\infty}^2\|\nabla u_{lo}\|_{L_t^2 L_x^6}\|u\|_{L_t^\infty L_x^6}^2 \lesssim_u \eta(N^{-3} + K).$$

We use Bernstein and Lemma 3.1 to control the third term in (30),

$$\| Ø(u^3 u_{hi}^2 \nabla u_{lo}) \|_{L^1_{t,x}} \lesssim \| Ø(u^3 u_{hi}^2 \nabla u_{lo}) \|_{L^1_{t,x}} \lesssim$$
$$\| u_{hi} \|^2_{L^4_{t,x}} \| \nabla u_{lo} \|_{L^4_t L^\infty_x} \| u \|_{L^4_t L^6_x} \| u \|^2_{L^\infty_t L^6_x} \lesssim$$
$$N \| u_{hi} \|^2_{L^4_{t,x}} \| u_{lo} \|_{L^4_t L^\infty_x} \| u_{lo} \|_{L^4_t L^6_x} \| u \|^2_{L^\infty_t L^6_x} \lesssim$$
$$N^{1/2} \| u_{hi} \|^2_{L^4_{t,x}} \| u_{lo} \|_{L^4_t L^\infty_x} \| \nabla u_{lo} \|_{L^4_t L^3_x} \| u \|^2_{L^\infty_t L^6_x} \lesssim$$
$$N^{1/2} \| u_{hi} \|^2_{L^4_{t,x}} \| u_{lo} \|^2_{L^\infty_t L^6_x} \lesssim$$
$$[N^2 \| u_{hi} \|^4_{L^4_{t,x}} + N^{-1} \eta(1 + N^3 K)] \| u_{lo} \|^2_{L^\infty_t L^6_x} \lesssim_u$$
$$\eta(\| u_{hi} \|^4_{L^4_{t,x}} + N^{-3} + K).$$

Finally, we will use the same method as used in Ref.[3] to control the fourth and the fifth terms, and we have

$$\| \nabla Ø(u_{hi} P_{lo} F(u)) \|_{L^1_{t,x}} \lesssim$$
$$\| \, |\nabla|^{-1} u_{hi} \|_{L^2_t L^6_x} (\| P_{lo} F(u) \|_{L^2_t L^{6/5}_x} +$$
$$\| \nabla P_{lo} F(u) \|_{L^2_t L^{6/5}_x}) \lesssim$$
$$N^{-2} (1 + N^3 K)^{1/2} (\eta(1 + N^3 K)^{1/2} +$$
$$\| u_{hi} \|_{L^4_{t,x}} N^{-1/4} (1 + N^3 K)^{1/4}) \| u \|^3_{L^\infty_t L^6_x} \lesssim$$
$$N^{-2} (1 + N^3 K)^{1/2} (\eta(1 + N^3 K)^{1/2} + N^{1/2} \| u_{hi} \|^2_{L^4_{t,x}} +$$
$$N^{-1} (1 + N^3 K)^{1/2}) \| u \|^3_{L^\infty_t L^6_x} \lesssim$$
$$\eta(N^{-3} + K) +$$
$$N^{-3/2} \| u_{hi} \|^2_{L^4_{t,x}} (1 + N^3 K)^{1/2} \| u_{lo} \|^3_{L^\infty_t L^6_x} \lesssim$$
$$\eta(\| u_{hi} \|^4_{L^4_{t,x}} + N^{-3} + K),$$

and

$$\| Ø(u_{hi} \nabla P_{lo} F(u)) \|_{L^1_{t,x}} \lesssim$$
$$\| \, |\nabla|^{-1} u_{hi} \|_{L^2_t L^6_x} \cdot$$
$$(\| \Delta P_{lo} F(u) \|_{L^2_t L^{6/5}_x} + \| \nabla P_{lo} F(u) \|_{L^2_t L^{6/5}_x}) \lesssim$$
$$\| \, |\nabla|^{-1} u_{hi} \|_{L^2_t L^6_x} (\| \Delta P_{lo} F(u_{lo}) \|_{L^2_t L^{6/5}_x} +$$
$$\| \Delta P_{lo} Ø(u_{hi} u_{lo}^2 u^2) \|_{L^2_t L^{6/5}_x} +$$
$$\| \Delta P_{lo} Ø(u_{hi}^2 u^3) \|_{L^2_t L^{6/5}_x} + \| \nabla P_{lo} F(u) \|_{L^2_t L^{6/5}_x}) \lesssim$$
$$\eta(\| u_{hi} \|^4_{L^4_{t,x}} + N^{-3} + K).$$

The proof now is complete. □

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