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Ricci flow on surfaces with Gaussian curvature of initial metrics unbounded below

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Abstract: The existence of Ricci flow on 2-dimension open manifolds with the Gaussian curvature of initial metrics unbounded below was proved. The initial metric can be either complete or incomplete.

Key words: Riemannian metric; Guassian curvature; Ricci flow

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Gauss 曲率具下界的 Ricci 流

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摘要:证明了一个 2 维流形上,如果初始 Riemann 度量的 Gauss 曲率有下界,则不论度量是否完备,它的 Ricci 流存在.

关键词:Riemann 度量;Gauss 曲率;Ricci 流

0 Introduction

Since Hamilton's introduction, the Ricci flow equation

$$\frac{\partial g_{ij}}{\partial t} = -2Rc_{ij}(g) \tag{1}$$

in Ref. [1], it has proved to be a powerful method in researching differential geometry problems, and has produced lots of beautiful results, such as the proof of the Pioncare conjecture by Perelman.

In 1989, Shi proved the following existence theorem on a noncompact manifold in Ref. [2], which led to a lot of research on the Ricci flow equation itself.

Theorem 0.1 Let $(\mathscr{M}, g)$ be an n-dimensional smooth complete noncompact Riemannian manifold with its Riemannian curvature tensor Rm satisfying

$$|Rm| \leqslant k_0, \ on\ \mathscr{M}, \tag{2}$$

where $0 < k_0 < +\infty$ is a constant. Then there exists a constant $T(n, k_0) > 0$ depending only on n and k_0 such that the evolution equation

$$\left.\begin{aligned} &\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) \text{ on } \mathscr{M}, \\ &g_{ij}(x,0) = g_{ij}(x), \forall x \in \mathscr{M} \end{aligned}\right\} \tag{3}$$

has a smooth solution $g_{ij}(x,t) > 0$ for a short time $0 \leqslant t \leqslant T(n, k_0)$, and satisfies the following

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estimates: For any integer $m \geqslant 0$, there exist constants $C_m > 0$ depending only on n, m and k_0 such that

$$\sup_{x\in\mathscr{M}} |\nabla^m Rm(x,t)|^2 \leqslant C_m/t^m,\ 0 < t \leqslant T(n,k_0). \tag{4}$$

In this paper, we consider the existence problem of the Ricci flow on a 2-dimensional noncompact manifold with the Gaussian curvature of the initial metric bounded above only. Under this dimension, as $Ric = Kg$, where K is the Gaussian curvature, the Ricci flow equation turns out to be

$$\frac{\partial g_{ij}}{\partial t} = -2Kg_{ij}, \tag{5}$$

we have more interesting literatures compared to higher dimensions.

We mainly use the pseudolocality theorem and the maximal principle theorem to prove the existence theorem below.

Theorem 0.2 Let $\mathscr{M}$ be an open surface equipped with a smooth metric $\bar{g}$, and the Gaussian curvature be bounded above only. Then, there exists a constant $T > 0$ depending only on the supremum of $K(\bar{g})$, such that a smooth Ricci flow $g(t)$ exists on $\mathscr{M}$ for $t\in[0,T]$, $g(0)=\bar{g}$, and the Gaussian curvature $K(g(t))$ is bounded for $t\in[t_0, T]$, with any $t_0>0$.

Actually, it is Topping who first considered the existence problem of Ricci flow with incomplete initial metric, as can be seen in Ref. [3]. However, Topping's proof is not complete in detail, especially in making use of the Maximal principle for noncompact domains. Besides, Topping used Hamilton's compactness theorem in his proof, which made his proof very lengthy.

In this paper, we give a detailed and complete proof for the existence theorem in dimension 2. As we just use the pseudolocality theorem together with the maximal principle, our proof is more direct.

Since the proof of the pseudolocality theorem on complete noncompact manifolds that we need is not found in any references yet, we give a simple proof in the last section for completeness.

1 The construction of Ricci flows with initial metrics converging

Let $\mathscr{M}$ be an open surface equipped with a smooth metric $\bar{g}$, and the Gaussian curvature of $\bar{g}$ is bounded above only.

In this section, we first construct a sequence of metrics $g_i(0)$ converging to $\bar{g}$ in the Cheeger-Gromov sense, the definition of which can be seen in Chapter 5 of Ref. [4].

For convenience, we define

$$\overline{K} = \max\{\sup_{\mathscr{M}} K(\bar{g}), 0\}$$

in this paper.

We take a sequence of subdomains $\mathscr{M}_i \subset\subset \mathscr{M}$ with smooth boundaries, so that $\mathscr{M}_i \subset\subset \mathscr{M}_{i+1}$, and $\mathscr{M}_i$ exhaust $\mathscr{M}$ as $i\to\infty$. Then We have the following lemma, which is due to Topping's work in Ref. [3].

Lemma 1.1 For each i, We can construct a complete Riemannian metric $g_i(0)$ on $\mathscr{M}_{i+1}$, with curvature bounded above by $\overline{K}$ and bounded below by some constant C_i depending on i, and each $g_i(0)$ agrees with $\bar{g}$ on $\mathscr{M}_i$.

Proof The proof is the same as Topping's, and hence omitted here. □

For each Riemannian manifold $(\mathscr{M}_{i+1}, g_i(0))$ constructed above, by Shi's Ricci flow existence Theorem 0.1, there exists a Ricci flow $g_i(t)$ on a maximal time interval $[0, T_i)$. The Gaussian curvature $K_i(t)$ is bounded on any time interval compactly contained in $[0,T_i)$, but blows up when $t\to T_i$.

Now, we have the following maximum principle Theorem 1.2 for Ricci flow on complete noncompact manifolds. With this theorem, we can prove Lemma 1.3 below, which shows that there exists a uniform time interval $[0,T]$, such that all $g_i(t)$ exist on it. We refer the proof of Theorem 1.2 to Corollary 7.45 in Ref. [5].

Theorem 1.2 Let $(\mathscr{M}, g(t))$, $t\in[0,T]$, be a complete solution of the Ricci flow with bounded curvature tensor and let $p\in\mathscr{M}$. Suppose ϕ is a smooth function satisfying $\phi(0)\leqslant 0$. Suppose there exist some $C>0$ and $A_1<\infty$, such that

$$\frac{\partial}{\partial t}\phi\leqslant\Delta\phi+C\phi,\ \text{whenever}\ \phi(x,t)>0,$$

and

$$\phi(x,t)\leqslant e^{A_1(d_{g(t)}(x,p)+1)}$$

for all (x,t). Then $\phi(x,t)\leqslant 0$ for all $x\in\mathscr{M}$ and $t\in[0,T]$.

Lemma 1.3 There exists a uniform time interval $[0,T]$, such that all $g_i(t)$ exist on it. Furthermore, for any $t_0>0$, $K(g_i(t))$ have a uniform bound $\left[\frac{1}{-2t_0},\frac{1}{\overline{K}^{-1}-2T}\right]$ when $t\in[t_0,T]$.

Proof Under Ricci flow, the Gaussian curvature obeys the equation

$$\frac{\partial K}{\partial t}=\Delta K+2K^2. \tag{6}$$

We simply note K_i for $K(g_i(t))$, Δ_i for Laplacian with respect to $g_i(t)$ and define $\phi_i(x,t):=K_i-\frac{1}{\overline{K}^{-1}-2t}$. Then we have $\phi_i(x,0)=K_i(0)-\overline{K}\leqslant 0$. Whenever $\phi_i(x,t)>0$, we see that

$$\begin{aligned}\frac{\partial\phi_i(x,t)}{\partial t}&=\frac{\partial K_i}{\partial t}-\frac{2}{(\overline{K}^{-1}-2t)^2}=\\ &\Delta_iK_i+2K_i^2-\frac{2}{(\overline{K}^{-1}-2t)^2}=\\ &\Delta_i\left(K_i-\frac{1}{\overline{K}^{-1}-2t}\right)+\\ &2\left(K_i+\frac{1}{\overline{K}^{-1}-2t}\right)\left(K_i-\frac{1}{\overline{K}^{-1}-2t}\right).\end{aligned} \tag{7}$$

When $t\leqslant\bar{t}_i<\frac{\overline{K}^{-1}}{2}$, we have

$$K_i+\frac{1}{\overline{K}^{-1}-2t}\leqslant\overline{K}+\frac{1}{\overline{K}^{-1}-2\bar{t}_i}. \tag{8}$$

Since $\phi_i(x,t)=K_i-\frac{1}{\overline{K}^{-1}-2t}$ is bounded for $t\in[0,\bar{t}_i]$, we can apply Theorem 1.2 to yield that, for $t\in\left[0,\frac{\overline{K}^{-1}}{2}\right)$,

$$K_i(x,t)\leqslant\frac{1}{\overline{K}^{-1}-2t}. \tag{9}$$

On the other hand, let $\underline{K}_i:=\inf_{\mathscr{M}_{i+1}}K_i(0,x)$, which we know to be nonpositive, and may tend to $-\infty$ as $i\to\infty$. We define $\varphi_i(x,t):=\frac{1}{\underline{K}_i^{-1}-2t}-K_i(t)$. As $\varphi_i(x,0)\leqslant 0$ and $\varphi_i(x,t)$ are bounded when $t\in[0,\bar{t}_i]$, we may use again the maximum principle to $\varphi_i(x,t)$, and deduce that

$$\frac{1}{\underline{K}_i^{-1}-2t}\leqslant K_i. \tag{10}$$

Combining (9) and (10), we have

$$\frac{1}{\underline{K}_i^{-1}-2t}\leqslant K_i\leqslant\frac{1}{\overline{K}^{-1}-2t}. \tag{11}$$

From inequality (11), we can see that for all $t\in[t_0,T]$ with $t_0>0$ and $T<\frac{\overline{K}^{-1}}{2}$, $K_i(t)\in\left[\frac{1}{-2t_0},\frac{1}{\overline{K}^{-1}-2T}\right]$. Consequently there exists an i-independent time interval $[0,T]$, $T<\frac{\overline{K}^{-1}}{2}$, such that all $g_i(t)$ exist on $[0,T]$. □

Remark 1.4 For general dimensions, we note that the scalar curvature obeys the inequality

$$\frac{\partial R}{\partial t}\geqslant\Delta R+\frac{2}{n}R^2, \tag{12}$$

and the equality could be valid only for dimension 2. Thus we seem unable to get a uniform time interval $[0,T]$ for any dimension higher than 2.

2 Pseudolocality and the proof of Theorem 0.2

In this section, we will show that on any subdomain $\Sigma\subset\mathscr{M}$, there exists a subsequence of $g_i(t)$ that converges uniformly on $\Sigma\times[0,T]$ to some Ricci flow $g(t)$, and finish the proof of Theorem 0.2.

Before doing this, we state some known results, which are all needed in the proof of Theorem 0.2.

The first one is the peudolocality theorem below for complete curvature-bounded Ricci flow on noncompact manifolds. A simple proof of this theorem is arranged in Section 3.

Theorem 2.1 For each $n\in\mathbb{N}$, there exists $\varepsilon>0$, depending only on n, such that for all $r_0>0$, if $g(t)$ is a complete curvature-bounded Ricci flow for $t\in[0,T]$, where $0<T\leqslant(\varepsilon r_0)^2$, on an n-dimensional manifold $\mathscr{M}$ containing some point x_0, and if, with respect to $g(0)$,

①

$$|Rm|_{g(0)}\leqslant r_0^{-2} \text{ on } B_{g(0)}(x_0,r_0),$$

②

$$\frac{\mathrm{Vol}(B_{g(0)}(x_0,r_0))}{r_o^n}\geqslant(1-\varepsilon)\omega_n,$$

then,

$$|Rm|(x,t)\leqslant(\varepsilon r_0)^{-2},$$

for $t\in[0,T]$ and $\mathrm{dist}_{g(t)}(x,x_0)<\varepsilon r_0$.

The following remark advanced by Topping is important, though the proof is only a direct computation.

Remark 2.2 If $|Rm|(x,t)\leqslant(\varepsilon r_0)^{-2}$ when $0\leqslant t\leqslant(\varepsilon r_0)^2$, $x\in B_{g(0)}(x,\varepsilon r_0)$, then the conclusion remains true for $\mathrm{dist}_{g(0)}(x,x_0)<\varepsilon r_0$, by a slight reduction of ε. Actually, we can compute from the Ricci flow equation to see that if $|Rm|_{g(t)}\leqslant M$ for some $M\in\mathbb{R}^+$, $t\in[0,T]$, then

$$e^{-C(n)Mt}\mathrm{dist}_{g(0)}(x,x_0)\leqslant\mathrm{dist}_{g(t)}(x,x_0)\leqslant e^{C(n)Mt}\mathrm{dist}_{g(t)}(x,x_0),$$

with $t\in[0,T]$.

We also need the following theorem to control all orders of derivatives of the Riemannian curvature tensor, which is actually an improved version of the estimates in Theorem 0.1. The proof of this theorem can be seen in Chapter 3 in Ref.[4].

Theorem 2.3 Fix the dimension n of the Ricci flow under consideration. Let $K<\infty$ and $\alpha>0$ be given positive constants. Fix an integer $l\geqslant 0$. Then for each integer $k\geqslant 0$ and for each $r>0$ there is a constant $C'_{k,l}=C'_{k,l}(K,\alpha,r,n)$ such that the following holds. Let $(U,g(t))$, $0\leqslant t\leqslant T$, be a Ricci flow with $T\leqslant\alpha/K$. Fix $p\in U$ and suppose that the metric ball $B(p,0,r)$ has compact closure in U. Suppose that

$$|Rm(x,t)|\leqslant K, \forall(x,t)\in U\times[0,T], \tag{13}$$

$$|\nabla^{\beta}Rm(x,0)|\leqslant K, \forall x\in U \text{ and } \beta\leqslant l. \tag{14}$$

Then

$$|\nabla^k Rm(y,t)|\leqslant\frac{C'_{k,l}}{t^{\max\{k-l,0\}/2}} \tag{15}$$

for all $y\in B(p,0,r/2)$ and all $t\in(0,T]$. In particular, if $k\leqslant l$, then for $y\in B(p,0,r/2)$ and $t\in(0,T]$ we have

$$|\nabla^k Rm(y,t)|\leqslant C'_{k,l}. \tag{16}$$

Furthermore, we need the following lemma, which gives the uniform bounds of all orders of derivatives of the metrics. For details of the lemma, we refer to Lemma 1.4 in Ref.[6].

Lemma 2.4 Let $\mathscr{M}$ be a Riemannian manifold with metric G, Ω a compact subset of $\mathscr{M}$, and G_k a collection of solutions to the Ricci flow defined on neighborhoods of $\Omega\times[\beta,\varphi]$ with the time interval $[\beta,\varphi]$ containing $t=0$. Let D denote covariant derivative with respect to G and $|\ |$ the length of a tensor with respect to G, while D_k and $|\ |_k$ are the same for G_k. Suppose that

(Ⅰ) the metrics G_k are all uniformly equivalent to G at $t=0$ on K, so that

$$cG\leqslant G_k\leqslant CG$$

for some constants $0<c,C<\infty$ independent of k;

(Ⅱ) the covariant derivative of the metrics G_k with respect to the metric G are all uniformly bounded at $t=0$ on Ω, so that

$$|D^p G_k|\leqslant C_p$$

for some constant $C_p<\infty$ independent of k for $p=0,1,2,\cdots,l$;

(Ⅲ) the covariant derivatives of the curvature tensors Rm_k of the metrics G_k are uniformly bounded with respect to G_k on $\Omega\times[\beta,\varphi]$ so that

$$|D_k{}^p Rm_k|_k\leqslant C'_p$$

for some constants C'_p independent of k for $p=0,1,2\cdots,l$.

Then the metrics G_k are uniformly bounded with respect to G on $\Omega\times[\beta,\varphi]$, so that

$$\tilde{c}G\leqslant G_k\leqslant\tilde{C}G$$

for some constant $\tilde{c}$ and $\tilde{C}$ independent of k, and the covariant derivatives of the metrics G_k with respect to metric G are uniformly bounded on $\Omega\times[\beta,\varphi]$, so that

$$|D^p G_k| \leqslant \widetilde{C}_p$$

for some constant $\widetilde{C}_p$ independent of k with $\tilde{c}, \widetilde{C}$ and $\widetilde{C}_p$ depending on c, C, C_p and C'_p and the dimension, for $p=0,1,2\cdots,l$.

Proof of Theorem 0.2 We firstly fix an arbitrary $\Sigma \subset\subset \mathscr{M}$. Then there exists a positive integer $i_\Sigma > 0$, so that $g_i(0)|_\Sigma = \bar{g}$ when $i \geqslant i_\Sigma$. We assume that integer i below is taken larger than i_Σ. Then, for any point $x_1 \in \Sigma$, as soon as the constant $r(x_1)$ is small enough, we have $|K_i(x,0)| = |K_{\bar{g}}(x,0)| \leqslant r(x_1)^{-2}$ for all $x \in B_{\bar{g}}(x_1, r(x_1))$. The pseudolocality Theorem 2.1 implies that there exists some $\varepsilon > 0$, so that

$$|K_i(x,t)| \leqslant (\varepsilon r(x_1))^{-2},$$
$$\forall (x,t) \in B_{\bar{g}}(x_1, \varepsilon r(x_1)) \times [0, (\varepsilon r(x_1))^2].$$

Since Σ is a fixed relatively compact subset, we can choose finite many $B_{\bar{g}}(x_k, \varepsilon r(x_k))$ covering Σ. Let $r(\Sigma) := \inf_k r(x_k)$, then

$$|K_i(x,t)| \leqslant (\varepsilon r(\Sigma))^{-2},$$
$$\forall (x,t) \in \Sigma \times [0, (\varepsilon r(\Sigma))^2].$$

On the other hand, due to Lemma 1.3, for any $(x,t) \in \Sigma \times [(\varepsilon r(\Sigma))^2, T]$, we have

$$\frac{-1}{2(\varepsilon r(\Sigma))^2} \leqslant K_i(x,t) \leqslant \frac{1}{\overline{K}^{-1} - 2T}.$$

Hence for any $(x,t) \in \Sigma \times [0,T]$, we have

$$|K_i(x,t)| \leqslant \max\left\{\frac{1}{(\varepsilon r(\Sigma))^2}, \frac{1}{\overline{K}^{-1} - 2T}\right\}. \tag{17}$$

Then, from the Ricci flow equations (5) and (17), we can deduce by a direct computation that there exists a uniform constant $C = C\left\{\frac{1}{(r(\Sigma))^2}, \overline{K}, T\right\} > 0$, such that for $i > i_\Sigma$,

$$e^{-Ct}\bar{g} \leqslant g_i(t) \leqslant e^{Ct}\bar{g},\ \forall (x,t) \in \Sigma \times [0,T]. \tag{18}$$

Especially,

$$e^{-CT}\bar{g} \leqslant g_i(t) \leqslant e^{CT}\bar{g}, \forall (x,t) \in \Sigma \times [0,T]. \tag{19}$$

We arbitrarily take an $l > 0$. Since condition (14) holds on any $\overline{B}_{\bar{g}}(x, r(\Sigma)) \subset \mathscr{M}$ for all integers $k \leqslant l$, and by inequality (17), we apply Theorem 2.3 to obtain

$$|\nabla^k K_i(x,t)| \leqslant C'_{k,l},\ k = 1,2,\cdots,l, \tag{20}$$

and constant $C'_{k,l}$ is i-independent.

Take $\Sigma^o \subset\subset \Sigma$ with $\mathrm{dist}_{\bar{g}}(\partial\Sigma^o, \partial\Sigma) = \frac{2r(\Sigma)}{3}$ and cover it with finite many $B_g(x, r(\Sigma))$. As $C'_{k,l}$ is a uniform bound, the inequality (20) holds on Σ^o.

Now, according to Lemma 2.4 and the Arzela-Ascoli lemma, there exist a subsequence of $g_i(t)$ on $\Sigma^o \times [0,T]$, which is also denoted by $g_i(t)$, converging uniformly together with their derivatives of orders lower than l, to a limit metric $g(t)$. For the uniformity of the convergence, we surely have $g(0) = \bar{g}$.

As Σ is taken arbitrarily and T is independent of Σ, we can use the standard diagonal process to get a global solution $g(t)$ for $(x,t) \in \mathscr{M} \times [0,T]$, while Σ exhausts $\mathscr{M}$. The limit metric $g(t)$ is also a Ricci flow, and satisfies the condition that $g(0) = \bar{g}$.

As the integer l above can be taken as any positive integer, we see that $g(t)$ is actually a smooth solution.

In the end, the uniform bound claimed in the theorem for $t \in [t_0, T]$ with $t_0 > 0$ has actually been assured by Lemma 1.3. Hence we complete the proof of Theorem 0.2. □

Corollary 2.5 The Ricci flow $g(t)$ constructed above is complete, if the initial metric $\bar{g}$ is assumed to be complete.

Proof Actually, we can compute directly from the Ricci flow equation (5) to see that

$$g(x,t) = e^{-2\int_0^t K(x,s)\mathrm{d}s}\bar{g}. \tag{21}$$

As $K(t) \leqslant \frac{1}{\overline{K}^{-1} - 2t}$ for $t \in [0,T]$, we have

$$g(x,t) \geqslant (1 - 2t\overline{K})\bar{g}, t \in [0,T], T < 1/2\overline{K}.$$

Since the initial metric $\bar{g}$ is complete, $g(t)$ is complete for $t \in [0,T]$, too. □

3 Logarithmic Sobolev inequality on Euclidean space and the pseudolocality theorem

In this section, we give a simple proof of the pseudolocality theorem. We first state Gross L's logarithmic Sobolev inequality on Euclidean space[7]247.

Theorem 3.1 For any nonnegative function $\phi\in W^{1,2}(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n}\phi^2\log\phi\,dv\leqslant\int_{\mathbb{R}^n}|\nabla\phi|^2dv+\frac{1}{2}\int_{\mathbb{R}^n}\phi^2dv\cdot\log\left(\int_{\mathbb{R}^n}\phi^2dv\right),$$

where $dv:=(2\pi)^{-\frac{n}{2}}e^{-\frac{|x|^2}{2}}dx$.

For an application, we have the following corollary.

Corollary 3.2 If $\int_{\mathbb{R}^n}(4\pi\tau)^{-\frac{n}{2}}e^{-f}dx=1$, with τ being a positive scale-factor, then we have

$$\int_{\mathbb{R}^n}(\tau|\nabla f|^2+f-n)(4\pi\tau)^{-\frac{n}{2}}e^{-f}dx\geqslant 0. \tag{22}$$

In particular, taking $\tau=\frac{1}{2}$, we have

$$\int_{\mathbb{R}^n}\left(\frac{1}{2}|\nabla f|^2+f-n\right)(4\pi\tau)^{-\frac{n}{2}}e^{-f}dx\geqslant 0, \tag{23}$$

with $\int_{\mathbb{R}^n}(2\pi)^{-\frac{n}{2}}e^{-f}dx=1$.

By scaling, we only need to show Theorem 2.1 is true for $r_0=1$. To show this, we need the following pseudolocality theorem on complete noncompact manifolds, which is due to Theorem 8.1 in Ref.[8].

Lemma 3.3 Let n be fixed. There exist $\delta,\varepsilon>0$ with the following property:

Suppose $g(x,t)$ is a smooth complete noncompact solution of the Ricci flow with bounded curvature on $\mathscr{M}^n\times[0,\varepsilon^2]$. Suppose at some point $x_0\in\mathscr{M}$ the isomperimetric constant in $B_0(x_0,1)$ is larger than $(1-\delta)c_n$, where c_n is the isoperimetric constant of $\mathbb{R}^n$, and $R(x,0)\geqslant -1$ for all $x\in B_0(x_0,1)$. Then $|Rm|(x,t)\leqslant t^{-1}+\varepsilon^{-2}$ for $0<t\leqslant\varepsilon^2$ and $x\in B_t(x_0,\varepsilon)$.

Proof of Theorem 2.1 Actually, the proof of Theorem 2.1 is a slight modification of the proof of Lemma 3.3. We sketch the proof for completeness.

Firstly, it is obvious that the lemma still holds if we reduce ε to make it smaller than δ and replace δ by ε.

By contradiction, we assume that Theorem 2.1 is not correct. Then for any $\varepsilon_i>0$, we can find $(\mathscr{M}_i,g_i(t))$ satisfying the following conditions:

(Ⅰ) $g_i(t)$ is a smooth solution of the Ricci flow on $[0,\varepsilon_i^2]$ with bounded $|\nabla^kRm|$ on $\mathscr{M}_i\times[0,\varepsilon_i^2]$ for all $k\geqslant 0$.

(Ⅱ) The isomperimetric constant in $B_0^{(i)}(p_i,1)$ is lager than $(1-\delta_i)c_n$.

(Ⅲ) There exits $0<t_i\leqslant\varepsilon_i^2$, and $x_i\in B_0^{(i)}(p_i,\varepsilon_i)$ and $|Rm_i|(x_i,t_i)>\varepsilon_i^2$.

Let $\Lambda_i=\frac{1}{100n\varepsilon_i}$. We can follow what Chau-Tam-Yu did in Ref.[8] to find proper $\bar{x}_i$, $\bar{t}_i$. Denote $u_i=(-4\pi t)^{-\frac{n}{2}}e^{-f_i}$ to be the solutions to equation $(-\partial_t-\Delta+R)u_i=0$, which satisfies $\lim\limits_{t\to 0}u_i(x,t)=\delta_{\bar{x}_i}(x)$. We take $\tilde{u}_i=h_iu_i$ and $\tilde{f}_i=f_i-\log h_i$, where h_i is a cut-off function with its support set contained in $B(\bar{x}_i,\sqrt{\bar{t}_i})$. Then, following what Chau-Tam-Yu did in Ref.[8], we also obtain the inequality at $t=0$ that

$$\int_{\mathscr{M}_i}[-\bar{t}_i|\nabla_i\tilde{f}_i|^2-\tilde{f}_i+n]\tilde{u}_idV_i\geqslant C>0, \tag{24}$$

for big enough i.

Only in the last step, as we have the condition ① in Theorem 2.1, can we make a simplification of the proof of Lemma 3.3.

We scale metrics $g_i(t)$ by the factor $\frac{1}{2\bar{t}_i}$, i.e. let $\hat{g}_i(s)=\frac{1}{2\bar{t}_i}g_i(2\bar{t}_is)$, $s\in\left[0,\frac{1}{2}\right]$. Then for $(\mathscr{N}_i,\hat{g}_i(0),p_i)$, with $\mathscr{N}_i\triangleq B_0\left(p_i,\frac{1}{\sqrt{2\bar{t}_i}}\right)$ we have

$$\left.\begin{aligned}&|\widehat{Rm}_i|(x,0)\leqslant 2\bar{t}_i\to 0,\ \text{as } i\to\infty,\\&\mathrm{Vol}\,B_0\left(p_i,\frac{1}{\sqrt{2\bar{t}_i}}\right)\geqslant(1-\varepsilon_i)\omega_n\left(\frac{1}{\sqrt{2\bar{t}_i}}\right)^n.\end{aligned}\right\} \tag{25}$$

The two conditions above imply a uniform lower bound of the injective radius at p_i with $s=0$. By Hamilton's compactness theorem[9-10], we can take a subsequence of $(\mathcal{N}_i, \hat{g}_i(0), p_i)$ that converges to a manifold isometric to $\mathbb{R}^n$.

On the other hand, if we put $\hat{g}_i = \frac{1}{2\bar{t}_i}\tilde{g}_i$, $\hat{u}_i = (2\bar{t}_i)^{\frac{n}{2}}\hat{u}_i$, and define $\hat{f}_i$ by $\hat{u}_i = (2\pi)^{-\frac{n}{2}} e^{-\hat{f}_i}$. Then we have $\lim\limits_{i\to\infty}\int_{\mathcal{M}_i} \hat{u}_i \mathrm{d}\hat{V}_i = 1$, and the inequality (24) turns out to be

$$\int_{\mathcal{N}_i} \left(-\frac{1}{2} \mid \nabla \hat{f}_i \mid^2 - \hat{f}_i + n\right) \hat{u}_i \mathrm{d}\hat{V}_i \geqslant C > 0. \tag{26}$$

Let i go to $+\infty$, we then have

$$\int_{\mathbb{R}^n} \left(-\frac{1}{2} \mid \nabla \hat{f} \mid^2 - \hat{f} + n\right) \hat{u} \mathrm{d}\hat{V} \geqslant C > 0. \tag{27}$$

This inequality contradicts Corollary 3.2. We complete the proof of Theorem 2.1. □

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