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The distance signless Laplacian spectral radius of trees with $n-3$ pendent vertices

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Abstract: The distance signless Laplacian spectral radius of a connected graph G is the spectral radius of the distance signless Laplacian matrix of G, defined as $\mathcal{Q}(G)=Tr(G)+D(G)$, where $Tr(G)$ is the diagonal matrix of vertex transmissions of G, and $D(G)$ is the distance matrix of G. It was investigated that the minimum of the distance signless Laplacian spectral radius among all trees with $n-3$ pendent vertices, and characterized that the unique tree whose distance signless Laplacian spectral radius is the maximum (minimum) among some trees with $n-3$ pendent vertices.

Key words: graph; tree; distance signless Laplacian matrix; distance signless Laplacian spectral radius

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具有 $n-3$ 个悬挂点的树的距离无符号拉普拉斯谱半径

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摘要：一个连通图 G 的距离无符号拉普拉斯谱半径是 G 的距离无符号拉普拉斯矩阵的谱半径. G 的距离无符号拉普拉斯矩阵定义为 $\mathcal{Q}(G)=Tr(G)+D(G)$，这里 $Tr(G)$ 是 G 的顶点传递的对角阵，且 $D(G)$ 是 G 的距离矩阵. 研究了所有 n 阶具有 $n-3$ 个悬挂点的树的距离无符号拉普拉斯谱半径的极小值，并刻画了一类 n 阶具有 $n-3$ 个悬挂点的树的距离无符号拉普拉斯谱半径的极大值与极小值.

关键词：图；树；距离无符号拉普拉斯矩阵；距离无符号拉普拉斯谱半径

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0 Introduction

In this paper, we consider finite simple graphs. Let $G=(V, E)$ be a simple connected graph with vertex set $V=V(G)=\{v_1, v_2, \cdots, v_n\}$ and edge set $E=E(G)$. For $u, v\in V(G)$, the distance between u and v, denoted by $d_G(u,v)$ or d_{uv}, is the length of a shortest path connecting them in G. For $u\in V(G)$, the transmission of u, denoted by $Tr_G(u)$, is the sum of distance from u to all other vertices of G. The distance matrix of G is the $n\times n$ matrix $D(G)=(d_{v_iv_j})$. For $1\leqslant i\leqslant n$, $Tr_G(v_i)$ is just the i-th row sum of $D(G)$. Let $Tr(G)=\mathrm{diag}(Tr_G(v_1), Tr_G(v_2), \cdots, Tr_G(v_n))$ be the diagonal matrix of vertex transmissions of G. The distance signless Laplacian matrix of G is the $n\times n$ matrix defined as $\mathcal{Q}(G)=Tr(G)+D(G)$. The distance signless Laplacian spectral radius of G, denoted by $\rho(G)$, is the spectral radius of $\mathcal{Q}(G)$. By the Perron-Frobenius theorem, there is a unique normalized positive eigenvector of $\mathcal{Q}(G)$ corresponding to $\rho(G)$, which is called the (distance signless Laplacian) principal eigenvector of G.

The distance spectral radius (the spectral radius of the distance matrix) of a connected graph has been studied extensively, see Refs. [1-6]. However, distance signless Laplacian spectral radius of a connected graph was proposed only recently in Ref. [7] and received little attention. Xing et al. did a further study in Refs. [8-9]. One of the conclusions of Ref. [8] is that the tree with minimum distance signless Laplacian spectral radius among the trees of n-vertex with $n-2$ pendent vertices was determined. This has motivated researchers to consider the trees of n-vertex with distance $n-3$ pendent vertices. In fact, a tree of n-vertex which has $n-3$ pendent vertices must be a tree with diameter 4.

Denote by $D(a,b,c)$ the tree of n-vertex with $n-3$ pendent vertices, and three vertices pendent a vertices, b vertices, c vertices respectively, $a+b+c=n-3$, $a,c\geqslant 1$, $b\geqslant 0$, see Fig. 1.

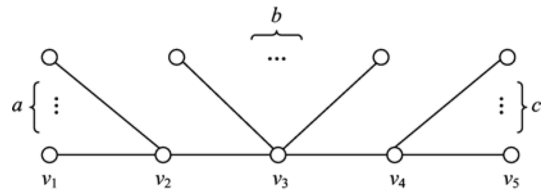

Fig. 1 The graph $D(a,b,c)$

In this paper, we determine the graph with minimum distance signless Laplacian spectral radius in

$\mathscr{D}(a,b,c)=\{D(a,b,c), a+b+c=n-3,\ a,c\geqslant 1,\ b\geqslant 0\}$,

and give the graph with maximum (minimum) distance signless Laplacian spectral radius in $\mathscr{D}(a,0,c)=\{D(a,0,c), a+c=n-3,\ a,c\geqslant 1\}$.

1 Main results

We begin with some definitions. Let G be a connected graph with $V(G)=\{v_1, v_2, \cdots, v_n\}$. A column vector $X=(X_1, X_2, \cdots, X_n)^{\mathrm{T}}\in R^n$ can be called a function defined on G, if there is a 1-1 map φ from $V(G)$ to the entries of X; $X(v_i)$ can be simplified as X_i for $i=1,2,\cdots,n$. $X(v_i)$ is also called the value of v_i given by X, $i=1,2,\cdots,n$. If X is an eigenvector of $\mathcal{Q}(G)$, then X is defined naturally on G, i. e. X_i is the entry of X corresponding to the vertex v_i, $i=1,2,\cdots,n$.

One can find that

$$X^{\mathrm{T}}\mathcal{Q}(G)X=\sum_{\{u,v\}\subset V(G)} d_{uv}(X_u+X_v)^2 \qquad (1)$$

and λ is an eigenvalue of $\mathcal{Q}(G)$ corresponding to the eigenvector X if and only if $X\neq 0$ and for each $v\in V(G)$

$$\lambda X(v)=\sum_{u\in V(G)} d_{uv}(X_u+X_v) \qquad (2)$$

Eq. (2) is called (λ, X)-eigenequation of G. In addition, for an arbitrary unit vector $X\in R^n$,

$$\rho(G)\geqslant X^{\mathrm{T}}\mathcal{Q}(G)X \qquad (3)$$

with equality if and only if X is the principal eigenvector of $\mathcal{Q}(G)$, see Ref. [10].

For a connected graph G, let $Tr_{\max}(G)$ be the maximal vertex transmission of G. We have the following result.

Lemma 1.1[8] Let G be a connected graph.

Then

$$Tr_{\max}(G) < \rho(G) \leqslant 2Tr_{\max}(G).$$

Theorem 1.2 $\rho(G(a,b,c)) \geqslant \rho(G(1,n-5,1))$ with equality if and only if

$$G(a,b,c) = G(1,n-5,1).$$

Proof Let G denote $G(a,b,c)$, and G' denote $G(1,n-5,1)$. Let $P = v_1v_2v_3v_4v_5$ be the path of length 4 in G, see Fig. 1. Let $u_{21}, u_{22}, \cdots, u_{2(a-1)}$ be the pendent neighbors of v_2 other than v_1 in G. Let $u_{31}, u_{32}, \cdots, u_{3b}$ be the pendent neighbors of v_3 in G. Let $u_{41}, u_{42}, \cdots, u_{4(c-1)}$ be the pendent neighbors of v_4 other than v_5 in G. Then G' may be obtainable from G by deleting the edges v_2u_{21}, v_2u_{22}, $\cdots$, $v_2u_{2(a-1)}$, v_4u_{41}, v_4u_{42}, $\cdots$, $v_4u_{4(c-1)}$, and adding the edges v_3u_{21}, v_3u_{22}, $\cdots$, $v_3u_{2(a-1)}$, v_3u_{41}, v_3u_{42}, $\cdots$, $v_3u_{4(c-1)}$. Let

$A_0 = \{v_1, v_2, v_3, v_4, v_5\}$, $A_1 = \{u_{21}, u_{22}, \cdots, u_{2(a-1)}\}$, $A_2 = \{u_{31}, u_{32}, \cdots, u_{3b}\}$, $A_3 = \{u_{41}, u_{42}, \cdots, u_{4(c-1)}\}$, and if $a=1$, $A_1 = \emptyset$; if $b=0$, $A_2=\emptyset$; if $c=1$, $A_3=\emptyset$. Obviously, $V(G) = V(G') = \bigcup_{i=0}^{3} A_i$.

Let X be the principal eigenvector of G'. By symmetry, we may suppose that

$$X(v_1) = X(v_5) = X_1,\ X(v_2) = X(v_4) = X_2,$$
$$X(u_{21}) = X(u_{22}) = \cdots = X(u_{2(a-1)}) = X(u_{31}) = X(u_{32}) = \cdots X(u_{3b}) = X(u_{41}) = X(u_{42}) = \cdots = X(u_{4(c-1)}) = X_4,$$
$$X(v_3) = X_3.$$

For $u, v \in A_i (i=0,1,2,3)$,

$$d_G(u,v) - d_{G'}(u,v) = 0.$$

For $u \in A_1$, if $v \in A_2$, $d_G(u,v) - d_{G'}(u,v) = 1$; if $v \in A_3$, $d_G(u,v) - d_{G'}(u,v) = 2$.

For $u \in A_2$, $v \in A_3$,

$$d_G(u,v) - d_{G'}(u,v) = 1.$$

For $u \in A_0$, $v \in A_2$,

$$d_G(u,v) - d_{G'}(u,v) = 0.$$

For $u = v_1$, if $v \in A_1$, $d_G(u,v) - d_{G'}(u,v) = -1$; if $v \in A_3$, $d_G(u,v) - d_{G'}(u,v) = 1$.

For $u = v_2$, if $v \in A_1$, $d_G(u,v) - d_{G'}(u,v) = -1$; if $v \in A_3$, $d_G(u,v) - d_{G'}(u,v) = 1$.

For $u = v_3$, if $v \in A_1$, $d_G(u,v) - d_{G'}(u,v) = 1$; if $v \in A_3$, $d_G(u,v) - d_{G'}(u,v) = 1$.

For $u = v_4$, if $v \in A_1$, $d_G(u,v) - d_{G'}(u,v) = 1$; if $v \in A_3$, $d_G(u,v) - d_{G'}(u,v) = -1$.

For $u = v_5$, if $v \in A_1$, $d_G(u,v) - d_{G'}(u,v) = 1$; if $v \in A_3$, $d_G(u,v) - d_{G'}(u,v) = -1$.

Then, by the equation (1) and the inequality (3)

$$\rho(G) - \rho(G') \geqslant X^T \mathcal{Q}(G) X - X^T \mathcal{Q}(G') X =$$
$$X^T(\mathcal{Q}(G) - \mathcal{Q}(G'))X =$$
$$\sum_{\{u,v\} \subset V(G)} (d_G(u,v) - d_{G'}(u,v))(X(u) + X(v))^2 =$$
$$(4(a-1)b + 8(a-1)(c-1) + 4b(c-1) + a + c - 2)X_4^2 + (a+c-2)X_3^2 + 2(a+c-2)X_3X_4.$$

Because $a \geqslant 1$, $b \geqslant 0$, $c \geqslant 1$, and $X_3 > 0$, $X_4 > 0$, we have $\rho(G) - \rho(G') \geqslant 0$, with equality if and only if $a = c = 1$, namely $G = G'$, or $G(a,b,c) = G(1,n-5,1)$. □

Lemma 1.3 For $2 \leqslant a \leqslant \left\lfloor \frac{n-3}{2} \right\rfloor$, we have $\rho(G(a,0,c)) > \rho(G(a-1,0,c+1))$.

Proof Let $P = v_1v_2v_3v_4v_5$ be the path of length 4 in $G(a,0,c)$, see Fig. 1. Let $u_{21}, u_{22}, \cdots, u_{2(a-1)}$ be the pendent neighbors of v_2 other than v_1 in $G(a,0,c)$. Let $u_{41}, u_{42}, \cdots, u_{4(c-1)}$ be the pendent neighbors of v_4 other than v_5 in $G(a,0,c)$.

Let X be the principal eigenvector of $G(a,0,c)$. By symmetry, we may suppose that

$$X(v_1) = X(u_{21}) = X(u_{22}) = \cdots = X(u_{2(a-1)}) = X_1,$$
$$X(v_2) = X_2,\ X(v_3) = X_3,\ X(v_4) = X_4,$$
$$X(v_5) = X(u_{41}) = X(u_{42}) = \cdots = X(u_{4(c-1)}) = X_5.$$

For a fix n, let $\rho_a = \rho(G(a,0,c))$. Then by (ρ_a, X)-eigenequations of $G(a,0,c)$, we have

$$\begin{cases} \rho_a X_1 = 2(a-1)(X_1+X_1) + (X_1+X_2) + 2(X_1+X_3) + 3(X_1+X_4) + 4(n-a-3)(X_1+X_5), \\ \rho_a X_2 = a(X_2+X_1) + (X_2+X_3) + 2(X_2+X_4) + 3(n-a-3)(X_2+X_5), \\ \rho_a X_3 = 2a(X_3+X_1) + (X_3+X_2) + (X_3+X_4) + 2(n-a-3)(X_3+X_5), \\ \rho_a X_4 = 3a(X_4+X_1) + 2(X_4+X_2) + (X_4+X_3) + (n-a-3)(X_4+X_5), \\ \rho_a X_5 = 4a(X_5+X_1) + 3(X_5+X_2) + 2(X_5+X_3) + (X_5+X_4) + 2(n-a-4)(X_5+X_5) \end{cases} \tag{4}$$

Transform (4) into a matrix equation $(B-\rho_a I)X'=0$, where $X'=(X_1, X_2, X_3, X_4, X_5)^T$ and

$$B=\begin{bmatrix} 4n-10 & 1 & 2 & 3 & 4n-4a-12 \\ a & 3n-2a-6 & 1 & 2 & 3n-3a-9 \\ 2a & 1 & 2n-4 & 1 & 2n-2a-6 \\ 3a & 2 & 1 & n+2a & n-a-3 \\ 4a & 3 & 2 & 1 & 4n-10 \end{bmatrix}.$$

And thus ρ_a is the largest root of the equation

$$\begin{aligned}&\rho^5-(14n-30)\rho^4+\\&(12a^2-12an+36a+75n^2-328n+360)\rho^3+\\&(-56a^2n+112a^2+56an^2-280an+\\&336a-190n^3+1260n^2-2786n+2048)\rho^2+\\&(-64a^4+128a^3n-384a^3-16a^2n^2+336a^2n-\\&816a^2-48an^3+192an^2+96an-720a+\\&224n^4-1964n^3+6372n^2-8972n+4544)\rho+\\&128a^4n-384a^4-256a^3n^2+1536a^3n-\\&2304a^3+160a^2n^3-1696a^2n^2+5728a^2n-\\&6208a^2-32an^4+640an^3-3904an^2+\\&9568an-8256a-96n^5+1008n^4-4032n^3+\\&7360n^2-5456n+640=0.\end{aligned}$$

Let $f_a(\rho)$ be the left side of the above equation. For $2\leqslant a\leqslant\left\lfloor\frac{n-3}{2}\right\rfloor$, we have $n\geqslant 2a+3$, and

$$\begin{aligned}&f_a(\rho)-f_{a-1}(\rho)=\\&\quad -4(2a+2-n)\cdot(-3\rho^3+(14n-28)\rho^2+\\&\quad (32a^2-32an+64a-12n^2+28n+28)\rho-\\&\quad 64a^2n+192a^2+64an^2-320an+\\&\quad 384a-8n^3+104n^2-408n+496).\end{aligned}$$

On the other hand, by Lemma 1.1, we have

$$4n-2a-8=Tr_{\max}(D(a,0,c))<\rho_a\leqslant 2Tr_{\max}(D(a,0,c))=8n-4a-16.$$

Let

$$g_a(\rho)=f_a(\rho)-f_{a-1}(\rho)=-4(2a+2-n)g_{a1}(\rho).$$

Because $n\geqslant 2a+3>2a+2$, we have

$$-4(2a+2-n)>0.$$

$$\begin{aligned}g_{a1}(\rho)=&-3\rho^3+(14n-28)\rho^2+\\&(32a^2-32an+64a-12n^2+28n+28)\rho-\\&64a^2n+192a^2+64an^2-320an+\\&384a-8n^3+104n^2-408n+496,\end{aligned}$$

$$\begin{aligned}g'_{a1}(\rho)=&-9\rho^2+(28n-56)\rho+32a^2-\\&32an+64a-12n^2+28n+28,\end{aligned}$$

$$g''_{a1}(\rho)=-18\rho+28n-56.$$

Because $g''_{a1}(\rho)$ is strictly decreasing with respect to ρ and $n\geqslant 2a+3$, we have

$$\begin{aligned}g''_{a1}(\rho)<g''_{a1}(4n-2a-8)=&\\-44n+36a+88\leqslant&\\-44(2a+3)+36a+88<0&\end{aligned}$$

when $\rho>4n-2a-8$.

So, $g'_{a1}(\rho)$ is strictly decreasing with respect to ρ when $\rho>4n-2a-8$.

Thus, if $\rho>4n-2a-8$, we have

$$\begin{aligned}&g'_{a1}(\rho)<g'_{a1}(4n-2a-8)=\\&\quad -44n^2+(56a+156)n-4a^2-112a-100.\end{aligned}$$

Let

$$g(x)=-44x^2+(56a+156)x-4a^2-112a-100,$$

we find that $g(x)$ is strictly decreasing with respect to x for $x>\frac{14a+39}{22}$.

So,

$$\begin{aligned}&g'_{a1}(4n-2a-8)=g(n)\leqslant g(2a+3)=\\&\quad -66a^2-160a-28<0,\end{aligned}$$

for $n\geqslant 2a+3\geqslant\frac{14a+39}{22}$. Then, if $\rho>4n-2a-8$, we have

$$g'_{a1}(\rho)<g'_{a1}(4n-2a-8)<0,$$

and then $g_{a1}(\rho)$ is strictly decreasing with respect to ρ when $\rho>4n-2a-8$.

So, if $\rho>4n-2a-8$, we have

$$\begin{aligned}g_{a1}(\rho)<\ &g_{a1}(4n-2a-8)=\\&-24n^3+(24a+120)n^2+\\&(40a^2-120a-136)n-40a^3-\\&16a^2+72a+16.\end{aligned}$$

Let

$$\begin{aligned}f(x)=&-24x^3+(24a+120)x^2+\\&(40a^2-120a-136)x-40a^3-\\&16a^2+72a+16,\end{aligned}$$

then

$$f'(x)=-72x^2+(48a+240)x+40a^2-120a-136.$$

Let $f'(x)=0$, we have two values x_1 and x_2, such that $f'(x_1)=f'(x_2)=0$. In fact,

$$x_1<x_2=\frac{a+5}{3}+\frac{\sqrt{13824a^2-11520a+18432}}{72\times 2}<$$

$2a+3$,

and $f'(x)$ is strictly decreasing with respect to x for $x>x_2$.

Thus,

$$f'(x)<f'(x_2)=0,$$

for $x>x_2$. Namely, $f(x)$ is strictly decreasing with respect to x for $x>x_2$.

So,

$$g_{a1}(4n-2a-8)=f(n)\leqslant f(2a+3)= -56a^3-232a^2-200a+40<0,$$

for $n\geqslant 2a+3>x_2$.

Then, if $\rho>4n-2a-8$, we have

$$g_{a1}(\rho)<g_{a1}(4n-2a-8)<0,$$

and

$$g_a(\rho)=f_a(\rho)-f_{a-1}(\rho)= -4(2a+2-n)g_{a1}(\rho)<0.$$

So, if $\rho>4n-2a-8$, we have $f_a(\rho)<f_{a-1}(\rho)$. Because $\rho_a>4n-2a-8$, then if $\rho>\rho_a$, we have $f_a(\rho)<f_{a-1}(\rho)$, which implies that $\rho_a>\rho_{a-1}$, i. e.,

$$\rho(D(a,0,c))>\rho(D(a-1,0,c+1)). \qquad \square$$

According to Lemma 1.3, we have the following result.

Theorem 1.4 For $1\leqslant a\leqslant\left\lfloor\frac{n-3}{2}\right\rfloor$, we have $\rho(G(a,0,c))\geqslant\rho(G(1,0,n-4))$ with equality if and only if $G(a,0,c)=G(1,0,n-4)$, and

$$\rho(G(a,0,c))\leqslant\rho\left(G\left(\left\lfloor\frac{n-3}{2}\right\rfloor,0,\left\lceil\frac{n-3}{2}\right\rceil\right)\right)$$

with equality if and only if

$$G(a,0,c)=G\left(\left\lfloor\frac{n-3}{2}\right\rfloor,0,\left\lceil\frac{n-3}{2}\right\rceil\right).$$

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