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On weakly Π-embedded subgroups of finite groups

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Abstract: Let G be a finite group and H a subgroup of G. H is called weakly Π-embedded in G if there exists a subgroup pair (T, S), where T is a quasinormal subgroup of G containing H_G and $S/H_G \leqslant H/H_G$ satisfies Π-property in G/H_G, such that $|G:HT|$ is a power of a prime and $(H \cap T)/H_G \leqslant S/H_G$. Here weakly Π-embedded subgroups were used to explore the structure of finite groups. In particular, new criterions of hypercyclically embedded subgroups were obtained.

Key words: Sylow subgroup; weakly Π-embedded subgroup; p-nilpotent; hypercyclically embedded subgroups

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有限群的弱 Π 嵌入子群

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摘要:令 G 是1个有限群,且 H 是 G 的1个子群. 子群 H 称为在 G 中是弱 Π 嵌入的,如果存在 G 的1个子群对 (T, S) 使得 $|G:HT|$ 是某个素数的方幂,且 $(H\cap T)/H_G \leqslant S/H_G$,其中 T 是 G 的1个包含 H_G 的拟正规子群且 $S/H_G \leqslant H/H_G$ 是 G/H_G 的1个满足 Π 性质的子群. 这里利用弱 Π 嵌入子群研究有限群的结构. 特别地,得到了子群是超循环嵌入的新判断准则.

关键词:Sylow 子群;弱 Π 嵌入子群;p 幂零;超循环嵌入子群

0 Introduction

Throughout this paper, all groups are finite and G denotes a finite group. All unexplained notation and terminology are standard, as in Refs. [1-3].

The embedding properties of subgroups are important tools to explore finite groups. The question is to study their influences on the structure of finite groups. One of the important embedding properties is Π-property of subgroups, which was introduced by Li in Ref. [4]:

Definition 0.1[4, Definition 1.1] Let H be a subgroup of G. We call that H satisfies Π-property

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in G if for any G-chief factor L/K, $|G/K: N_{G/K}(HK/K\cap L/K)|$ is a $\pi(HK/K\cap L/K)$-number, where $\pi(HK/K\cap L/K)$ denotes the set of all prime divisors of $|HK/K\cap L/K|$.

The Π-property of subgroups covers many known embedding properties of subgroups and has been widely studied in many publications, see, for example, Refs. [5-7].

It is also well known that subgroups with prime power indices play an important role in the research of the structure of groups. For example, G is solvable if the index of every maximal subgroup of G is a prime or a square of a prime; G is nilpotent if and only if every maximal subgroup of G is normal in G with prime index; G is supersolvable if and only if every maximal subgroup of G has prime index. Also, keep in mind that a subgroup H of G is said to be quasinormal or permutable[8] if $HK=KH$ for any subgroup K of G.

The following introduces that weakly Π-embedded subgroup is closely related to the above notions.

Definition 0.2 A subgroup H is weakly Π-embedded in G if there exists a subgroup pair (T,S), where T is a quasinormal subgroup of G containing H_G and $S/H_G\leqslant H/H_G$ satisfies Π-property in G/H_G, such that $|G:HT|$ is a power of a prime and $(H\cap T)/H_G\leqslant S/H_G$.

As we know, a class $\mathscr{F}$ of groups is called a formation if either $\mathscr{F}=\emptyset$ or $1\in\mathscr{F}$ and for any group G, every homomorphic image of $G/G^{\mathscr{F}}$ belongs to $\mathscr{F}$, where $G^{\mathscr{F}}=\bigcap\{N\mid N\trianglelefteq G,\ G/N\in\mathscr{F}\}$. A formation $\mathscr{F}$ is saturated if $G\in\mathscr{F}$ whenever $G/\Phi(G)\in\mathscr{F}$. A normal subgroup N of G is said to be $\mathscr{F}$-hypercentrally embedded in G if for every G-chief factor H/K below N, $(H/K)\rtimes(G/C_G(H/K))\in\mathscr{F}$. The product of all normal $\mathscr{F}$-hypercentrally embedded subgroups is called the $\mathscr{F}$-hypercentre of G and denoted by $Z_{\mathscr{F}}(G)$. We use $\mathscr{U}$ and $\mathscr{N}$ to denote the saturated formations of supersolvable groups and nilpotent groups, respectively. Then $Z_{\mathscr{U}}(G)$ is the product of all normal subgroups N of G such that every G-chief factor below N has prime order. Also, we use $Z_\infty(G)$ to denote the $\mathscr{N}$-hypercentre of G. Moreover, the generalized fitting subgroup $F^*(G)$ of G is the maximal quasinilpotent subgroup of G (for details, see Ref. [9, Chap. X, Section 13]).

In this paper, we investigate the influence of weakly Π-embedded subgroups on the structure of finite groups. Our main results are as follows.

Theorem 0.1 Let X and E be normal subgroups of G such that $X\leqslant E$. Suppose that for every prime divisor p of $|X|$ and every non-cyclic Sylow p-subgroup P of X, all maximal subgroups of P are weakly Π-embedded in G. Then $E\leqslant Z_{\mathscr{U}}(G)$ when $X=E$ or $F^*(E)$.

Theorem 0.2 Let E be a normal subgroup of G. Suppose that for every prime divisor p of $|E|$ and every non-cyclic Sylow p-subgroup P of E, every cyclic subgroup of P with order p or 4 (if P is a non-abelian 2-group) is weakly Π-embedded in G. Then $E\leqslant Z_{\mathscr{U}}(G)$.

The following results follow directly from Theorems 0.1 and 0.2.

Corollary 0.1 Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and $X\leqslant E$ normal subgroups of G such that $G/E\in\mathscr{F}$. Suppose that for every prime divisor p of $|X|$ and every non-cyclic Sylow p-subgroup P of X, every maximal subgroup of P is weakly Π-embedded in G. Then $G\in\mathscr{F}$ when $X=E$ or $F^*(E)$.

Corollary 0.2 Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and E a normal subgroup of G such that $G/E\in\mathscr{F}$. Suppose that for every prime divisor p of $|E|$ and every non-cyclic Sylow p-subgroup P of E, every cyclic subgroup of P with prime order or 4 (if P is a non-abelian 2-group) is weakly Π-embedded in G. Then $G\in\mathscr{F}$.

1 Preliminaries

Lemma 1.1 Assume that H is a quasinormal subgroup of G, $E\leqslant G$ and $N\trianglelefteq G$.

① $H^G/H_G\leqslant Z_\infty(G/H_G)$[10, Theorem]. Particularly, H is subnormal in G.

② $H\cap E$ is a quasinormal subgroup of E[11, Lemma 1.2.14(4)].

③ HN/N is a quasinormal subgroup of G/N[11, Lemma 1.2.7(2)].

④ Suppose that E is subnormal in G such that $|G:E|$ is a power of p, for a prime divisor p of $|G|$. Then $O^p(G)\leqslant E$[11, Lemma 1.1.11].

Lemma 1.2 Let $H\leqslant G$ and $N\trianglelefteq G$.

① If H satisfies Π-property in G, then HN/N satisfies Π-property in G/N[4, Proposition 2.1(1)].

② Assume that H is weakly Π-embedded in G and N satisfies either $N\leqslant H$ or $(|H|,|N|)=1$. Then HN/N is weakly Π-embedded in G/N.

Proof ② Let (T, S) be a pair such that H is weakly Π-embedded in G. By Lemma 1.1③, TN/N is quasinormal in G/N. And $|G/N:HTN/N|=|G:HT|/|HTN:HT|$ is a power of a prime. If $N\leqslant H$, then $H\cap TN=(H\cap T)N$ by the modular law. Assume that $(|H|,|N|)=1$. Since

$$(|HN\cap T:H\cap T|,|HN\cap T:N\cap T|)=(|N\cap HT|,|H\cap NT|)=1,$$

we have $HN\cap T=(H\cap T)(N\cap T)$ by Ref. [1, Chap. A, Lemma 1.6(b)]. Hence $HN\cap TN=(HN\cap T)N=(H\cap T)N$. Generally speaking,

$$(HN\cap TN)(HN)_G/(HN)_G=(H\cap T)(HN)_G/(HN)_G\leqslant S(HN)_G/(HN)_G,$$

where

$$S(HN)_G/(HN)_G\cong (S/H_G)((HN)_G/H_G)/((HN)_G/H_G)$$

satisfies Π-property in $G/(HN)_G$ by ①.

Lemma 1.3 Let $\mathscr{F}$ be a saturated formation and F the canonical local satellite of $\mathscr{F}$ (see Ref. [1, Chap. Ⅳ, Theorem 3.7]). Let E be a normal p-subgroup of G. Then $E\leqslant Z_{\mathscr{F}}(G)$ if and only if one of the following holds:

① $G/C_G(E)\in F(p)$[12, Lemma 2.14]. In particular, $E\leqslant Z_\infty(G)$ if and only if $[O^p(G), E]=1$.

② $E/\Phi(E)\leqslant Z_{\mathscr{F}}(G/\Phi(E))$[13, Lemma 2.8].

Lemma 1.4[12, Theorem A(ii)] Let $\mathscr{F}$ be any formation and E a normal subgroup of G. If $F^*(E)$ is $\mathscr{F}$-hypercentral in G, then E is also $\mathscr{F}$-hypercentral in G.

Lemma 1.5[14, Lemma 2.4] Let P be a p-group and α a p'-automorphism of P.

① If $[\alpha, \Omega_2(P)]=1$, then $\alpha=1$.

② If $[\alpha, \Omega_1(P)]=1$ and either p is odd or P is abelian, then $\alpha=1$.

2 Proof of Theorem 0.1

The following propositions are the main steps in the proof of Theorem 0.1.

Proposition 2.1 Assume that P is a normal p-subgroup of G. If every maximal subgroup of P is weakly Π-embedded in G, then $P\leqslant Z_{\mathscr{U}}(G)$.

Proof Suppose that the assertion is false and consider a counterexample G of minimal order. Let G_p be a Sylow p-subgroup of G.

① P is not a minimal normal subgroup of G.

Assume that P is a minimal normal subgroup of G. Let P_1 be a non-trivial maximal subgroup of P such that $P_1\trianglelefteq G_p$. Clearly, $(P_1)_G=1$. Let (T,S) be a pair such that P_1 is weakly Π-embedded in G. If $P\leqslant T^G$ and $P\cap T_G=1$, then $PT_G/T_G\leqslant Z_\infty(G/T_G)$ by Lemma 1.1 ① and consequently, $P\leqslant Z_\infty(G)\leqslant Z_{\mathscr{U}}(G)$ by the G-isomorphism $P\cong PT_G/T_G$. This contradiction shows that either $P\leqslant T$ or $P\cap T^G=1$. In the former case, P_1 satisfies Π-property in G, so $|G:N_G(P_1\cap P)|=|G:N_G(P_1)|$ is a power of p. Moreover, $P_1\trianglelefteq G$ by the choice of P_1, which is absurd. In the latter case, $P\cap T=1$. Suppose that $|G:P_1T|$ is a power of p. Then $|G:T|=|G:P_1T|\cdot|P_1:P_1\cap T|$ is also a power of p and so $O^p(G)\leqslant T$ by Lemma 1.1①④. We have $P\cap O^p(G)=1$ and $[O^p(G),P]=1$. Hence $P\leqslant Z_\infty(G)$ by Lemma 1.3①. This contradiction shows that $|G:P_1T|=q^\alpha$, where $q(\neq p)$ is a prime and $\alpha\geqslant 0$ an integer. Obviously, $P\leqslant P_1T$ and then $P=P_1(P\cap T)=P_1$, a contradiction. Thus ① holds.

② The minimal normal subgroup of G contained in P is unique, denoted with N. Moreover, $P/N\leqslant Z_{\mathscr{U}}(G/N)$ and $|N|>p$.

By ①, $N<P$. We have $P/N\leqslant Z_{\mathscr{U}}(G/N)$ by Lemma 1.2② and the choice of G. So $|N|>p$. Assume that G has another minimal normal subgroup L contained in P. Analogously, $P/L\leqslant$

$Z_{\mathcal{U}}(G/L)$. However, the G-isomorphism $N \cong NL/L$ implies that $|N| = p$, a contradiction.

③ $\Phi(P) \neq 1$, which gives the final contradiction.

Assume that $\Phi(P)=1$. Then $P=N\times B$ where B is a complement of N in P. Let N_1 be a maximal subgroup of N such that $N_1 \trianglelefteq G_p$. Then $K=N_1B$ is a maximal subgroup of P such that $K_G=1$ and $K\cap N=N_1$. Let (T, S) be a pair such that K is weakly Π-embedded in G. If $N\leqslant T^G$ and $N\cap T_G=1$, then $NT_G/T_G \leqslant Z_\infty(G/T_G)$ by Lemma 1.1① and so $|N|=p$ by the G-isomorphism $N\cong NT_G/T_G$, which contradicts ②. Hence $N\leqslant T$ or $P\cap T^G=1$. In the latter case, $P\cap T=1$, which would arrive at a contradiction similarly to ①. We should, therefore, assume that $N\leqslant T$. Then $N_1 = K\cap N\leqslant K\cap T\leqslant S$ and so $N_1=K\cap N=S\cap N$. Since S satisfies Π-property in G,

$$|G:N_G(S\cap N)|=|G:N_G(N_1)|$$

is a power of p. Therefore, $N_1 \trianglelefteq G$ by the choice of N_1. This contradiction shows that $\Phi(P)\neq 1$. Hence $N\leqslant \Phi(P)$ and $P/\Phi(P)\leqslant Z_{\mathcal{U}}(G/\Phi(P))$ by ② and Ref.[15, Lemma 2.2]. Consequently, $P\leqslant Z_{\mathcal{U}}(G)$ by Lemma 1.3②. This completes the proof.

Proposition 2.2 Assume that E is a normal subgroup of G and P a non-cyclic Sylow p-subgroup of E, for a prime divisor p of $|E|$ with $(|E|, p-1)=1$. If every maximal subgroup of P is weakly Π-embedded in G, then E is p-nilpotent.

Proof Suppose that the result is false and let G be a counterexample of minimal order. Let G_p be a Sylow p-subgroup of G containing P.

① $O_{p'}(E)=1$ (It follows directly from Lemma 1.2② and the choice of G).

② $O_p(E)>1$.

Assume that $O_p(E)=1$ and N is a minimal normal subgroup of G contained in E. Let M/N be any maximal subgroup of PN/N. Then $M=P_1N$ where $P_1=P\cap M$ is a maximal subgroup of P. Assume that (T, S) is a pair such that P_1 is weakly Π-embedded in G. Obviously, TN/N is quasinormal in G/N by Lemma 1.1③, and $|G/N:MT/N|=|G:P_1T|/|MT:P_1T|$ is a power of a prime. Since $P_1\cap N=P\cap N$ is a Sylow p-subgroup of N and $|P_1T\cap N:T\cap N|=|P_1\cap NT:P_1\cap T|$ is a power of p, we have

$$P_1T\cap N=(P_1\cap N)(T\cap N)$$

by Ref.[1, Chap. A, Lemma 1.6(b)], and

$$M\cap TN=(P_1\cap T)N$$

by Ref.[1, Chap. A, Lemma 1.2]. Hence

$$(M\cap TN)M_G/M_G=(P_1\cap T)M_G/M_G\leqslant SM_G/M_G,$$

where SM_G/M_G satisfies Π-property in G/M_G by Lemma 1.2①. Generally speaking, G/N satisfies the hypothesis for G. Therefore E/N is p-nilpotent and N is the unique minimal normal subgroup of G contained in E. Since S satisfies Π-property in G, $|G:N_G(S\cap N)|$ is a power of p. If $S\cap N>1$, then $N\leqslant (S\cap N)^G=(S\cap N)^{G_p}\leqslant G_p$ and so N is a p-group, a contradiction. So $S\cap N=1$.

Assume that $N\leqslant T^G$ and $N\cap T_G=1$. We have $NT_G/T_G\leqslant Z_\infty(G/T_G)$ and N is central in G by Lemma 1.1① and the G-isomorphism $N\cong NT_G/T_G$, a contradiction. Hence either $N\leqslant T$ or $E\cap T^G=1$ by the uniqueness of N. In the former case, $P_1\cap N\leqslant P_1\cap T\leqslant S$, so $P_1\cap N=S\cap N=1$ and then $N\leqslant O_{p'}(E)$, which contradicts ①. In the latter case, $E\cap T=1$. Assume that $|G:P_1T|$ is a power of p. Then $|G:T|=|G:P_1T|\cdot|P_1:P_1\cap T|$ is also a power of p. Thus $O^p(G)\leqslant T$ by Lemma 1.1①④ and so $N\cap O^p(G)=1$. Consequently N has order p by the G-isomorphism $N\cong NO^p(G)/O^p(G)$, a contradiction. Therefore $|G:P_1T|$ is a power of a prime $q(\neq p)$. On the other hand, since P is a Sylow p-subgroup of E, $G=N_G(P)E$ by the Frattini argument. Then there exists $g=ke\in G$, where $k\in N_G(P)$ and $e\in E$, such that $P^g\leqslant {G_p}^g\leqslant P_1T$, that is, $P^e\leqslant P_1T$. Consequently $P^e\leqslant P_1T\cap E=P_1(T\cap E)=P_1$, a contradiction. So we have ②.

③ Final contradiction.

Let N be a minimal normal subgroup of G contained in $O_p(E)$. Then by Lemma 1.2② and the choice of G, N is the unique minimal normal subgroup of G contained in $O_p(E)$ and E/N is p-

nilpotent. Moreover, $|N|>p$ and $N\nleqslant\Phi(G)$. Then $G=N\rtimes D$ for some maximal subgroup D of G and $E=N\rtimes M$ where $M=E\cap D$. Denote $M_p=P\cap M$ and $D_p=G_p\cap D$. Then M_p is a Sylow p-subgroup of M, D_p is a Sylow p-subgroup of D containing M_p and $P=NM_p$, $G_p=ND_p$. Let N_1 be a maximal subgroup of N such that $N_1\trianglelefteq G_p$. Then $P_1=N_1M_p$ is a maximal subgroup of P with $(P_1)_G=1$, and $W=N_1D_p$ is a maximal subgroup of G_p. Let (T,S) be a pair such that P_1 is weakly Π-embedded in G. Since S satisfies Π-property in G, $|G:N_G(S\cap N)|$ is a power of p. If $S\cap N>1$, then $N\leqslant(S\cap N)^G=(S\cap N)^{G_p}\leqslant W$ and so $N=N_1(N\cap D_p)=N_1$, a contradiction. Thus $S\cap N=1$.

Similarly as in ②, $N\leqslant T^G$ and $N\cap T_G=1$ would imply that N has order p. Therefore we should assume that either $N\leqslant T$ or $E\cap T^G=1$. Assume that $N\leqslant T$. In the same manner as ②, we have $N_1=P_1\cap N=S\cap N=1$ and so $|N|=p$, a contradiction. So we assume that $E\cap T^G=1$. Particularly, $P\cap T=1$. However this will also obtain a contradiction just like ②.

Proof of Theorem 0.1 We prove by induction.

Firstly, assume that $X=E$. Let p be the smallest prime divisor of $|E|$. By Burnside Theorem and Proposition 2.2, E is p-nilpotent. Let $E_{p'}$ be the normal Hall p'-subgroup of E. Clearly, $(G, E_{p'})$ and $(G/E_{p'}, E/E_{p'})$ satisfy the hypothesis for (G, E). So $E_{p'}\leqslant Z_{\mathscr{U}}(G)$ by induction and $E/E_{p'}\leqslant Z_{\mathscr{U}}(G/E_{p'})$ by Proposition 2.1. Consequently, $E\leqslant Z_{\mathscr{U}}(G)$. Secondly, if $X=F^*(E)$, then $F^*(E)\leqslant Z_{\mathscr{U}}(G)$. Therefore, $E\leqslant Z_{\mathscr{U}}(G)$ by Lemma 1.4.

3 Proof of Theorem 0.2

The following propositions are useful in the proof of Theorem 0.2, which also have independent meanings.

Proposition 3.1 Assume that P is a normal p-subgroup of G. If every cyclic subgroup of P with order p or 4 (if P is a non-abelian 2-group) is weakly Π-embedded in G, then $P\leqslant Z_{\mathscr{U}}(G)$.

Proof Suppose that the assertion is false and consider a counterexample (G, P) for which $|G|+|P|$ is minimal. We denote $\Omega=\Omega_1(P)$ when $p>2$ or P is abelian. Otherwise, $\Omega=\Omega_2(P)$.

① G has a normal subgroup R such that P/R is a non-cyclic G-chief factor. Moreover, $R\leqslant Z_{\mathscr{U}}(G)$ and $V\leqslant R$ for any normal subgroup V of G satisfying $V<P$.

Obviously, ① holds when P is a minimal normal subgroup in G. Now assume that $R<P$ such that P/R is a G-chief factor. Since (G, R) satisfies the hypothesis, $R\leqslant Z_{\mathscr{U}}(G)$ and P/R is non-cyclic by the choice of (G, P). Let V be any normal subgroup of G satisfying $V<P$. Similarly, $V\leqslant Z_{\mathscr{U}}(G)$. If V covers P/R, then $P=VR\leqslant Z_{\mathscr{U}}(G)$, a contradiction. Thus V avoids P/R, that is, $V\leqslant R$.

② $\Omega=P$.

If $\Omega<P$, then $G/C_G(\Omega)\in F(p)$ and $C_G(\Omega)/C_G(P)\in\mathscr{N}_p$ by ①, Lemmas 1.3① and 1.5, where F is the canonical local satellite of $\mathscr{U}$ and $\mathscr{N}_p$ the class of p-groups. Consequently, $G/C_G(P)\in\mathscr{N}_pF(p)=F(p)$ and so $P\leqslant Z_{\mathscr{U}}(G)$ by Lemma 1.3① again. This contradiction shows that ② holds.

② Final contradiction.

Let $H/R\leqslant P/R\cap Z(G_p/R)$ be a cyclic subgroup, where G_p is a Sylow p-subgroup of G. Take $x\in H\backslash R$. Then $H=LR$, where $L=\langle x\rangle$ is cyclic of order p or 4 by ②. By the hypothesis, there exists a pair (T, S) such that L is weakly Π-embedded in G.

By ①, this can be separated into three cases: (a) $P\cap T_G\leqslant P\cap T^G\leqslant R$; (b) $P\leqslant T$; (c) $P\cap T_G\leqslant R$ and $P\leqslant T^G$. In case (a), $P\cap T\leqslant R$ and $P/R\cap TR/R=1$. If $|G:LT|$ is a power of p, then

$|G/R:TR/R|=|G:LT|\cdot|L:L\cap T|/|RT:T|$

is also a power of p and so $O^p(G/R)\leqslant TR/R$ by Lemma 1.1①④. Then $P/R\cap O^p(G/R)=1$ and $[P/R, O^p(G/R)]=1$. By Lemma 1.3①, $P/R\leqslant Z_\infty(G/R)$, a contradiction. Then we have that $|G:LT|$ is a power of a prime $q(\neq p)$. Hence $P\leqslant LT$ and $P=L(P\cap T)=H$. Then $P/R=H/R$ is cyclic, a contradiction. If case (b) holds, then $P\leqslant$

T and L/L_G satisfies Π-property in G/L_G. By ① and Lemma 1.2①, $H/R \cong (L/L_G)(R/L_G)/(R/L_G)$ satisfies Π-property in G/R, so

$|G/R : N_{G/R}(H/R \cap P/R)| = |G/R : N_{G/R}(H/R)|$

is a power of p. By the choice of H/R, H/R is normal in G/R. Therefore $P/R = H/R$ is cyclic, which contradicts ①. Now assume that case (c) is true. By Lemma 1.1①, $PT_G/T_G \leqslant Z_\infty(G/T_G)$ and then $PT_G/RT_G \leqslant Z_\infty(G/RT_G)$ (see Ref. [15, Lemma 2.2]). Therefore, by the G-isomorphism

$$PT_G/RT_G \cong P/R(P \cap T_G) = P/R,$$

P/R is cyclic. This contradiction completes the proof.

Proposition 3.2 Let E be a normal subgroup of G and P a non-cyclic Sylow p-subgroup of E for a prime divisor p of $|E|$ with $(|E|, p-1)=1$. Suppose that every cyclic subgroup of P with order p or 4 (if P is a non-abelian 2-group) is weakly Π-embedded in G. Then E is p-nilpotent.

Proof Suppose that the result is false and let (G,E) be a counterexample such that $|G|+|E|$ is minimal. Then $|P|>p$.

① $O_{p'}(E)=1$ (It follows directly from Lemma 1.2② and the choice of (G, E)).

② $O_p(E) \leqslant Z_\infty(E)$.

By Proposition 3.1, $O_p(E) \leqslant Z_{\mathcal{U}}(G) \cap E \leqslant Z_{\mathcal{U}}(E)$ (see Ref. [15, Lemma 2.2]). Therefore, $O_p(E) \leqslant Z_\infty(E)$.

③ $E=P$, which gives final contradiction.

Suppose that $O_p(E)<E$. Let (H, K) be a pair for which $|H|+|K|$ is minimal such that H/K is a G-chief factor below E, $K \leqslant O_p(E)$ and $H \nleqslant O_p(E)$. Note that $K \leqslant Z_\infty(E) \cap H \leqslant Z_\infty(H)$ by ②. If H/K is a p'-group, then H is p-nilpotent. Thus the Hall p'-subgroup $H_{p'}$ of H is normal in G, which contradicts ①. So H/K is non-abelian. By the Feit-Thompson's theorem, $p=2$. Moreover $O_2(E) \leqslant Z_\infty(G)$ by Proposition 3.1.

Let A be a subgroup of G such that $K<A<H$ and A is a minimal non-2-nilpotent group. Hence by Ref. [16, Theorems 3.4.7 and 3.4.11], $A = A_2 \rtimes A_q$, where A_2 is the normal Sylow 2-subgroup of A and A_q a cyclic Sylow q-subgroup of A with $q \neq 2$. Moreover, the following conclusions hold: (i) $A_2/\Phi(A_2)$ is a non-cyclic A-chief factor; (ii) $A_2 = A^{\mathcal{N}}$; (iii) the exponent of A_2 is 2 or 4 (when A_2 is non-abelian). Note that $K \leqslant Z_\infty(H) \cap A \leqslant Z_\infty(A)$. A is 2-nilpotent if $K = A_2$, which is impossible. So $K \leqslant \Phi(A_2)$ by (i). Take $x \in A_2 \backslash \Phi(A_2)$. Then $L = \langle x \rangle$ is cyclic of order 2 or 4 by (iii). Moreover, $L \nleqslant K$ and $L_G \leqslant K$. In fact, if $L_G \nleqslant K$, then $H = L_G K$. Consequently, H is a p-group, a contradiction. Assume that (T, S) is a pair such that L is weakly Π-embedded in G. By Lemma 1.1②, $T_0 = T \cap A$ is quasinormal in A. By (i), we should break the proof into three cases, which are: (a) $A_2 \cap (T_0)_A \leqslant A_2 \cap (T_0)^A \leqslant \Phi(A_2)$, (b) $A_2 \leqslant (T_0)_A$ and (c) $A_2 \cap (T_0)_A \leqslant \Phi(A_2)$ and $A_2 \leqslant (T_0)^A$. First assume that case (a) holds. If $|G:LT|$ is a power of p, then $|G:T| = |G:LT| \cdot |L:L \cap T|$ is also a power of p and so $O^p(G) \leqslant T$ by Lemmas 1.1① and ④. We have that $A_2 = A^{\mathcal{N}} \leqslant G^{\mathcal{N}} \leqslant O^p(G) \leqslant T$ and then $A_2 = A_2 \cap T = A_2 \cap T_0 \leqslant \Phi(A_2)$, a contradiction. Hence $|G:LT|$ is a power of a prime $r(\neq p)$. Moreover, $|A:LT_0| = |ALT:LT|$ and then $(p, |A:LT_0|)=1$. Then $A_2 \leqslant LT_0$ and so $A_2 = L(A_2 \cap T_0) = L\Phi(A_2) = L$. Consequently, $A_2/\Phi(A_2)$ is cyclic, which contradicts (i). Second, suppose that case (b) holds, that is, $A_2 \leqslant T$. Then L/L_G satisfies Π-property in G/L_G. Hence

$$1 < LK/K \cong (L/L_G)(K/L_G)/(K/L_G)$$

satisfies Π-property in G/K by Lemma 1.2①. We have that $|G/K : N_{G/K}(LK/K \cap H/K)| = |G/K : N_{G/K}(LK/K)|$ is a power of 2 and $H/K \leqslant (LK/K)^{G/K} = (LK/K)^{G_2K/K} \leqslant G_2K/K$ where G_2 is a Sylow 2-subgroup of G containing A_2, that is, H/K is a 2-group, a contradiction. Last, if case (c) holds, then $A_2(T_0)_A/(T_0)_A \leqslant Z_\infty(A/(T_0)_A)$ by Lemma 1.1①. Moreover,

$$A_2(T_0)_A/\Phi(A_2)(T_0)_A \leqslant Z_\infty(A/\Phi(A_2)(T_0)_A).$$

Note that

$$A_2(T_0)_A/\Phi(A_2)(T_0)_A \cong A_2/\Phi(A_2)(A_2 \cap (T_0)_A) = A_2/\Phi(A_2)$$

by (i). So $A_2/\Phi(A_2)$ is cyclic, which contradicts

(ⅰ). Finally, $E=P$, which completes the proof.

Proof of Theorem 0.2 We prove by induction on $|G|$. Let p be the smallest prime divisor of $|E|$. By Burnside Theorem and Proposition 3.2, E is p-nilpotent. Assume that $E_{p'}$ is the normal Hall p'-subgroup of E. Clearly $(G, E_{p'})$ and $(G/E_{p'}, E/E_{p'})$ satisfy the hypothesis for G. Hence $E_{p'} \leqslant Z_{\mathcal{U}}(G)$ by induction, and $E/E_{p'} \leqslant Z_{\mathcal{U}}(G/E_{p'})$ by Proposition 3.1. Consequently, $E \leqslant Z_{\mathcal{U}}(G)$.

4 Some applications

Recall that, a subgroup H of G is said to be c-normal[17] in G if there exists a normal subgroup N of G such that $G=HN$ and $H\cap N\leqslant H_G$. It is easy to see that c-normal subgroups and the subgroups satisfying Π-property in G are all weakly Π-embedded in G. The following examples show that the converse is not true in general. Note that, a subgroup H of G is said to be Π-normal[4] in G if there exists a subnormal subgroup T of G such that $G=HN$ and $H\cap N\leqslant I\leqslant H$, where I satisfies Π-property in G. Example 4.2 also shows that weakly Π-embedded subgroups are different from Π-normal subgroups.

Example 4.1 Assume that $G=S_4$ is the symmetric group of degree 4. Let $H=Z_3$ be a cyclic subgroup of G of order 3. Since K_4 is a normal subgroup of G such that $|G:HK_4|=2$ and $H\cap K_4=1$, H is weakly Π-embedded in G. However, H is not c-normal in G. In fact, if H is c-normal in G, then G has a normal subgroup of order 8, which is impossible.

Example 4.2 Let $L_1=\langle a, b \mid a^5=b^5=1, ab=ba\rangle$ and $L_2=\langle a', b'\rangle$ be a copy of L_1. Assume that α is an automorphism of L_1 of order 3 satisfying $a^\alpha=b$, $b^\alpha=a^{-1}b^{-1}$. Put $G=(L_1\times L_2)\rtimes\langle\alpha\rangle$ and $H=\langle a\rangle\times\langle a'\rangle$. According to Ref. [18, Example 1.3], G has a normal subgroup $T=\langle aa'b, a^{-1}b'\rangle$ such that $|G:HT|=|G:L_1L_2|=3$ and $H\cap T=1$. It follows that H is weakly Π-embedded in G. However, $|G:N_G(H\cap L_1)|=|G:N_G(\langle a\rangle)|=3$ is not a 5-number, that is, H does not satisfy Π-property in G.

Note that, any normal subgroup N of G containing α satisfies that

$$(a, 1)(1, \alpha^{-1})(a^{-1}, 1)(1, \alpha)=(ab^{-1}, 1)\in N,$$
$$(b, 1)(1, \alpha^{-1})(b^{-1}, 1)(1, \alpha)=(ab^2, 1)\in N.$$

Consequently, a and b belong to N. Assume that there exists a subnormal subgroup T of G such that $G=HT$. Then $\alpha\in T$, and so $a, b\in T$. Therefore, $H\cap T=\langle a\rangle$ or H. However, neither $\langle a\rangle$ nor H satisfies Π-property in G. This shows that H is not a Π-normal subgroup of G.

Therefore, some related known results are corollaries of our theorems, for example, Theorem C in Ref. [19], Theorems 3.1 and 3.4 in Ref. [20], Theorems 1 and 3 in Ref. [21], Theorems 4.1 and 4.2 in Ref. [17], Theorems 3.1 and 3.2 in Ref. [22] and so on.

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