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M_p-embedded subgroups and the structure of finite groups

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Abstract: A subgroup H of G is called $\mathcal{M}_p$-embedded in G, if there exists a p-nilpotent subgroup B of G such that $H_p \in Syl_p(B)$ and B is $\mathcal{M}_p$-supplemented in G. The structure of finite groups is investigated by means of $\mathcal{M}_p$-embedded property of primary subgroups.

Key words: composition factor; $\mathcal{M}_p$-embedded subgroup; primary subgroup

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M_p 嵌入子群与有限群的结构

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摘要:设 H 是 G 的子群,如果 G 存在 p 幂零子群 B 使得 $H_p \in Syl_p(B)$ 且 B 在 G 中是 $\mathcal{M}_p$ 可补充的,称 H 是 G 的 $\mathcal{M}_p$ 嵌入子群. 利用准素子群的 $\mathcal{M}_p$ 嵌入性质研究了有限群的结构.

关键词:合成因子;$\mathcal{M}_p$ 嵌入子群;准素子群

0 Introduction

Throughout this paper, all groups are finite. We shall use the standard notation employed in Refs. [1-3]. In particular, let G denote a group, $|G|$ denote the order of G and $\pi(G)$ denote the set of all prime divisors of $|G|$.

It is well known that many scholars have been studying the relationship between solvability and generalized supplementation of some primary subgroups. For example, in 1937, Hall[4] proved that G is solvable if and only if each Sylow subgroup of G is complemented in G. In 1982, Arad and Ward[5] showed that G is solvable if and only if every Sylow 2-subgroup and Sylow 3-subgroup of G are complemented in G. In 2000, Ballester-Bolinches, et al.[6] asserted that G is solvable if and only if each Sylow subgroup of G is

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c-supplemented in G. In 2009, Miao and Qian[7] stated that G is solvable if every Sylow subgroup of G is $\mathscr{M}$-supplemented in G. In 2014, Heliel[8] showed that G is solvable if and only if every Sylow subgroup of odd order of G is c-supplemented in G. In 2015, Zhang and Li[9] proved that G is solvable if and only if every Sylow subgroup of G satisfies Φ^*-property in G.

Note that every condition above has an essential characteristics: they picked at least two prime divisors of $|G|$ and considered primary subgroups corresponding to the prime divisors. Further, it is easy to see that the relationship between solvability and composition factors is very close. Naturally, we shall consider this question: If we only consider single prime divisors of $|G|$, how about the structure of G?

Recently, Zhang and Miao[10] introduced the concept of $\mathscr{M}_p$-embedded subgroups which is closely related to $\mathscr{M}_p$-supplementation. In this study, our purpose is to investigate the construction of finite groups and obtain some new characterization about composition factors of G by using the embedded property of primary subgroups corresponding to single prime divisors of $|G|$.

Definition 0.1[11] Let π be a set of primes. A subgroup H of a group G is called $\mathscr{M}_\pi$-supplemented in G, if there exists a subgroup B of G such that $G=HB$ and $H_1B<G$ for every maximal subgroup H_1 of H with $\pi(|H:H_1|)\subseteq\pi$. In particular, if $\pi=\{p\}$, then H is called $\mathscr{M}_p$-supplemented in G.

Definition 0.2[10, Definition 1.2] A subgroup H of G is called $\mathscr{M}_p$-embedded in G, if there exists a p-nilpotent subgroup B of G such that $H_p\in Syl_p(B)$ and B is $\mathscr{M}_p$-supplemented in G.

The following example indicates that an $\mathscr{M}_p$-embedded subgroup is not $\mathscr{M}_p$-supplemented in G.

Example 0.3 Consider the group $G=A_5$. Let $H=\{(1),(12345),(13524),(14253),(15432),(15)(24),(14)(23),(13)(45),(12)(35),(25)(34)\}$. Clearly, H is $\mathscr{M}_5$-embedded in G. But H is not $\mathscr{M}_5$-supplemented in G.

1 Preliminaries

For the sake of convenience, we first list here some known results which will be useful in the sequel.

Lemma 1.1[10, Lemma 2.1] Let G be a group. Then

① Let $N\trianglelefteq G$ and $N\leqslant H$. If H is $\mathscr{M}_p$-embedded in G, then H/N is $\mathscr{M}_p$-embedded in G/N.

② Let π be a set of primes. Let N be a normal π'-subgroup and H be a π-subgroup of G. If H is $\mathscr{M}_p$-embedded in G, then HN/N is $\mathscr{M}_p$-embedded in G/N.

Lemma 1.2[11, Lemma 4] Let H be a $\mathscr{M}_\pi$-supplemented subgroup in a group G and B be a $\mathscr{M}_\pi$-supplement to H. If H_1 is a maximal subgroup of H and $\pi(|H:H_1|)\subseteq\pi$, then

$$|G:H_1B|=|H:H_1|.$$

Lemma 1.3[12, Lemma 4] If P is a Sylow p-subgroup of a group G and $N\trianglelefteq G$ such that $P\cap N\leqslant\Phi(P)$, then N is p-nilpotent.

Lemma 1.4 Let G be a group and P a Sylow 5-subgroup of G. Suppose that P is $\mathscr{M}_5$-embedded in G. Then every composition factor of G satisfies one of the following conditions: ① U/W is cyclic of order 5, ② U/W is $5'$-group, ③ $U/W\cong A_5$.

Proof Perform mathematical induction for $|G|$. Since P is $\mathscr{M}_5$-embedded in G, there exists a 5-nilpotent subgroup B of G such that $P\in Syl_5(B)$ and B is $\mathscr{M}_5$-supplemented in G. Further, we may choose a subgroup K such that $G=BK$ and $B_iK=P_iB_{5'}K<G$ for every maximal subgroup P_i of P. By Lemma 1.2, $|G:B_iK|=5$ and $G/(B_iK)_G$ is isomorphic to a subgroup of the symmetric group S_5.

If G is simple, then G is isomorphic to a subgroup of the symmetric group S_5. Hence G holds by the structure of S_5.

If G is not simple, then we have

Case 1 there exists some i such that $(B_iK)_G=1$ and G holds by the structure of S_5.

Case 2 $(B_iK)_G\neq1$ for all i. Since $P\cap$

$(\bigcap(B_iK)_G)=P\bigcap(\bigcap(P_iB_{5'}K)_G)=\bigcap P_i=\Phi(P)$, $\bigcap(B_iK)_G$ is 5-nilpotent by Lemma 1.3. Set $T=\bigcap(B_iK)_G$. Clearly, $G/T\hookrightarrow\times S_5$. If $T=1$, then G holds. Next, we assume that $T\neq1$. If $T_{5'}=1$, then G/T holds, G also holds. If $T_{5'}\neq1$, then $G/T_{5'}$ holds, G also holds. □

Lemma 1.5[13, Theorem 5.4] If $(15,|G|)=1$, then G is solvable.

Lemma 1.6 Let G be a group and P a Sylow 7-subgroup of G. Suppose that P is $\mathscr{M}_7$-embedded in G. Then every composition factor of G satisfies one of the following conditions: ① U/W is cyclic of order 7, ② U/W is $7'$-group, ③ $U/W\cong A_7$, ④ $U/W\cong PSL(2,7)$.

Proof Perform mathematical induction for $|G|$. Since P is $\mathscr{M}_7$-embedded in G, there exists a 7-nilpotent subgroup B of G such that $P\in Syl_7(B)$ and B is $\mathscr{M}_7$-supplemented in G. Further, we may choose a subgroup K such that $G=BK$ and $B_iK=P_iB_{7'}K<G$ for every maximal subgroup P_i of P. By Lemma 1.2, $|G:B_iK|=7$ and $G/(B_iK)_G$ is isomorphic to a subgroup of the symmetric group S_7.

If G is simple, then G is isomorphic to a subgroup of the symmetric group S_7. Hence G holds by the structure of S_7.

If G is not simple, then we have

Case 1 there exists some i such that $(B_iK)_G=1$ and G holds by the structure of S_7.

Case 2 $(B_iK)_G\neq1$ for all i. Since $P\bigcap(\bigcap(B_iK)_G)=P\bigcap(\bigcap(P_iB_{7'}K)_G)=\bigcap P_i=\Phi(P)$, $\bigcap(B_iK)_G$ is 7-nilpotent by Lemma 1.3. Set $T=\bigcap(B_iK)_G$. Clearly, $G/T\hookrightarrow\times S_7$. If $T=1$, then G holds. Next, we assume that $T\neq1$. If $T_{7'}=1$, then G/T holds, G also holds. If $T_{7'}\neq1$, then $G/T_{7'}$ holds, G also holds. □

2 Main results

Theorem 2.1 Let G be a group and P a Sylow 5-subgroup of G. Suppose that P has a subgroup D such that $1<D\leqslant P$. If every subgroup H of P with $|H|=|D|$ is $\mathscr{M}_5$-embedded in G, then every composition factor of G satisfies one of the following conditions: ① U/W is cyclic of order 5, ② U/W is $5'$-group, and ③ $U/W\cong A_5$.

Proof Assume that the assertion is false and choose G to be a counterexample of minimal order. Further, by Lemma 1.4, $1<D<P$ and we may choose a subgroup H of P such that $|H|=|D|$. Since H is $\mathscr{M}_5$-embedded in G, there exists a 5-nilpotent subgroup B of G such that $H\in Syl_5(B)$ and B is $\mathscr{M}_5$-supplemented in G. Further, we may choose a subgroup K such that $G=BK$ and $B_iK=H_iB_{5'}K<G$ for every maximal subgroup H_i of H. By Lemma 1.2, $|G:B_iK|=5$ and $G/(B_iK)_G$ is isomorphic to a subgroup of the symmetric group S_5.

If G is simple, then G is isomorphic to a subgroup of the symmetric group S_5. Further, G holds by the structure of S_5, a contradiction. Hence G is not simple. Next, if there exists some i such that $(B_iK)_G=1$ and G holds by the structure of S_5, a contradiction.

Hence $(B_iK)_G\neq1$ for all i. Set $N_1=(B_iK)_G$ for some i. Clearly, $|(N_1)_5|\geqslant|D|$ where $(N_1)_5$ is a Sylow 5-subgroup of N_1. Then we may pick a subgroup E such that $|E|=|D|$ and $E\leqslant(N_1)_5$. Since E is $\mathscr{M}_5$-embedded in G, there exists a 5-nilpotent subgroup B^* of G such that $E\in Syl_5(B^*)$ and B^* is $\mathscr{M}_5$-supplemented in G. Further, we may choose a subgroup K^* such that $G=B^*K^*=N_1B_i^*K^*$ and $B_i^*K^*=E_iB_{5'}^*K^*<G$ for every maximal subgroup E_i of E. Hence $N_1(B_i^*K^*)_G/(B_i^*K^*)_G\cong N_1/N_1\bigcap(B_i^*K^*)_G$ is isomorphic to a subgroup of the symmetric group S_5.

If $N_1\bigcap(B_i^*K^*)_G=1$, then N_1 is isomorphic to a subgroup of the symmetric group S_5. Hence G holds by the structure of S_5. If $N_1\bigcap(B_i^*K^*)_G\neq1$, then we set $N_2=N_1\bigcap(B_i^*K^*)_G\trianglelefteq G$. With the similar discussion above, we have a series $N_s\trianglelefteq N_{s-1}\trianglelefteq\cdots\trianglelefteq N_1\trianglelefteq N_0=G$ such that N_i/N_{i+1} is isomorphic to a subgroup of the symmetric group S_5 and $|(N_s)_5|\leqslant|D|$ where $(N_s)_5$ is a Sylow 5-subgroup of N_s, $i=0,1,\cdots,s-1$.

Then we may choose a subgroup F such that $|F|=|D|$ and there exists a 5-nilpotent subgroup O of G such that $F\in Syl_5(O)$, O is $\mathscr{M}_5$-supplemented in G and $(N_s)_5\leqslant F$. We consider subgroup ON_s. Clearly, ON_s holds by the choice of G and N_s holds. Hence we also have a series $1=N_{s+t}\trianglelefteq\cdots\trianglelefteq N_s\trianglelefteq N_{s-1}\trianglelefteq\cdots\trianglelefteq N_1\trianglelefteq N_0=G$. By the structure of S_5, G holds, a contradiction.

The final contradiction completes the proof. □

By Lemma 1.5 and Theorem 2.1, we have the following corollary:

Corollary 2.2 Let G be a group. Then G is solvable provided G satisfies the following two conditions:

① If $3\in\pi(G)$, then every Sylow 3-subgroup P of G has a subgroup D_1 such that $1<D_1\leqslant P$ and every subgroup H_1 of P with $|H_1|=|D_1|$ is $\mathscr{M}_3$-embedded in G.

② If $5\in\pi(G)$, then every Sylow 5-subgroup Q of G has a subgroup D_2 such that $1<D_2\leqslant Q$ and every subgroup H_2 of Q with $|H_2|=|D_2|$ is $\mathscr{M}_5$-embedded in G.

Theorem 2.3 Let G be a group and P a Sylow 7-subgroup of G. Suppose that P has a subgroup D such that $1<D\leqslant P$. If every subgroup H of P with $|H|=|D|$ is $\mathscr{M}_7$-embedded in G, then every composition factor of G satisfies one of the following conditions: ① U/W is cyclic of order 7, ② U/W is $7'$-group, ③ $U/W\cong A_7$, ④ $U/W\cong PSL(2,7)$.

Proof Assume that the assertion is false and choose G to be a counterexample of minimal order. Further, by Lemma 1.6, $1<D<P$ and we may choose a subgroup H of P such that $|H|=|D|$. Since H is $\mathscr{M}_7$-embedded in G, there exists a 7-nilpotent subgroup B of G such that $H\in Syl_7(B)$ and B is $\mathscr{M}_7$-supplemented in G. Further, we may choose a subgroup K such that $G=BK$ and $B_iK=H_iB_{7'}K<G$ for every maximal subgroup H_i of H. By Lemma 1.2, $|G:B_iK|=7$ and $G/(B_iK)_G$ is isomorphic to a subgroup of the symmetric group S_7.

If G is simple, then G is isomorphic to a subgroup of the symmetric group S_7. Further, G holds by the structure of S_7, a contradiction. Hence G is not simple. Next, if there exists some i such that $(B_iK)_G=1$ and G holds by the structure of S_7, a contradiction.

Hence $(B_iK)_G\neq 1$ for all i. Set $N_1=(B_iK)_G$ for some i. Clearly, $|(N_1)_7|\geqslant|D|$ where $(N_1)_7$ is a Sylow 7-subgroup of N_1. Then we may pick a subgroup E such that $|E|=|D|$ and $E\leqslant(N_1)_7$. Since E is $\mathscr{M}_7$-embedded in G, there exists a 7-nilpotent subgroup B^* of G such that $E\in Syl_7(B^*)$ and B^* is $\mathscr{M}_7$-supplemented in G. Further, we may choose a subgroup K^* such that $G=B^*K^*=N_1B_i^*K^*$ and $B_i^*K^*=E_iB_{7'}^*K^*<G$ for every maximal subgroup E_i of E. Hence $N_1(B_i^*K^*)_G/(B_i^*K^*)_G\cong N_1/N_1\cap(B_i^*K^*)_G$ is isomorphic to a subgroup of the symmetric group S_7.

If $N_1\cap(B_i^*K^*)_G=1$, then N_1 is isomorphic to a subgroup of the symmetric group S_7. Hence G holds by the structure of S_7. If $N_1\cap(B_i^*K^*)_G\neq 1$, then we set $N_2=N_1\cap(B_i^*K^*)_G\trianglelefteq G$. With the similar discussion above, we have a series $N_s\trianglelefteq N_{s-1}\trianglelefteq\cdots\trianglelefteq N_1\trianglelefteq N_0=G$ such that N_i/N_{i+1} is isomorphic to a subgroup of the symmetric group S_7 and $|(N_s)_7|\leqslant|D|$ where $(N_s)_7$ is a Sylow 7-subgroup of N_s, $i=0,1,\cdots,s-1$.

Then we may choose a subgroup F such that $|F|=|D|$ and there exists a 7-nilpotent subgroup X of G such that $F\in Syl_7(X)$, X is $\mathscr{M}_7$-supplemented in G and $(N_s)_7\leqslant F$. We consider subgroup XN_s. Clearly, XN_s holds by the choice of G and N_s holds. Hence we also have a series $1=N_{s+t}\trianglelefteq\cdots\trianglelefteq N_s\trianglelefteq N_{s-1}\trianglelefteq\cdots\trianglelefteq N_1\trianglelefteq N_0=G$. By the structure of S_7, G holds, a contradiction.

The final contradiction completes the proof. □

References

[1] DOERK K, HAWKES T. Finite Solvable Groups [M]. Berlin/New York: de Gruyter, 1992.

[2] HUPPERT B. Endliche Gruppen Ⅰ [M]. Berlin/

Heidelberg/New York: Springer-Verlag, 1967.
[3] GUO W. The Theory of Classes of Groups[M]. Beijing/New York/Dordrecht/Boston/London: Science Press/Kluwer Academic Publishers, 2000.
[4] HALL P. A characteristic property of solvable groups [J]. J London Math Soc, 1937, 12: 198-200.
[5] ARAD Z, WARD M B. New criteria for the solvability of finite groups[J]. J Algebra, 1982, 77(1): 234-246.
[6] BALLESTER-BOLINCHES A, WANG Y, GUO X. C-supplemented subgroups of finite groups [J]. Glasgow Math J, 2000, 42(3): 383-389.
[7] MIAO L, QIAN G. A condition for the solvability of finite groups[J]. Siberian Math J, 2009, 50(4): 687-691.
[8] HELIEL A A. A note on c-supplemented subgroups of finite groups[J]. Comm Algebra, 2014, 42(4): 1 650-1 656.
[9] ZHANG L, LI B. Solvability of finite groups[J]. Journal of University of Science and Technology of China, 2015, 45(6): 443-448.
[10] ZHANG J, MIAO L. On M_p-embedded primary subgroups of finite groups[J]. Studia Scientiarum Mathematicarum Hungarica, 2016,53(4):429-439.
[11] MONAKHOV V S, SHNYPARKOV A V. On the p-supersolvability of a finite group with an M-supplemented Sylow p-subgroup[J]. Siberian Math J, 2009, 50(4): 681-686.
[12] TATE J. Nilpotent quotient groups[J]. Topology, 1964, 3: 109-111.
[13] CHEN Z. Inner Outer $\sum$-Group and Minimal Non-$\sum$-Group[M]. Chongqing: Southwest Normal University Press, 1988.

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Recall that $a=\mathrm{e}^{-t/2}$. From the above sequence of equalities we obtain

$$T_t f(x)=\int_{\mathbb{R}} f(\mathrm{e}^{-t/2}x+\sqrt{1-\mathrm{e}^{-t}}z)\gamma(\mathrm{d}z),$$

which is Mehler's formula we wanted to prove.

References

[1] GWYNNE E. Functional inequalities for Gaussian and Log-Concave probability measures [D]. Chicago: Northwestern University, 2013.
[2] HSU E P. Stochastic analysis and applications[R]. Chicago: Northwestern University, 2016.
[3] BOGACHEV V I. Gaussian Measures [M]. Providence, RI: American Mathematical Society, 1998.