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A direct proof of Mehler's formula for the Ornstein-Uhlenbeck semigroup

ZHANG Qihao

(*School of Mathematical Sciences, University of Science and Technology of China, Hefei* 230026, *China*)

Abstract: A direct proof of Mehler's formula for the Ornstein-Uhlenbeck semigroup was given using an integral representation of the Hermite polynomials.

Key words: Hermite polynomials; Ornstein-Uhlenbeck semigroup; Mehler's formula; Gaussian measure

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关于 Ornstein-Uhlenbeck 半群下 Mehler 方程的一种证明

张启浩

(中国科学技术大学数学科学学院,安徽合肥 230026)

摘要:利用 Hermite 多项式,给出了一个关于 Ornstein-Uhlenbeck 半群下 Mehler 方程的直接证明.

关键词:Mehler 方程;Ornstein-Uhlenbeck 半群;Hermite 多项式;高斯测度

0 Introduction

Let γ be the standard Gaussian measure on $\mathbb{R}$. For a nonnegative integer n, we define the nth Hermite polynomials H_n as

$$H_n(x)=\frac{(-1)^n}{\sqrt{n!}}\mathrm{e}^{x^2/2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{e}^{-x^2/2}\qquad(1)$$

The Hermite polynomials on $\mathbb{R}$ are obtained by orthogonalizing the sequence of the powers of x in $L^2(\mathbb{R},\gamma)$ with respect to the standard Gaussian measure by the well known Gram-Schmidt procedure. They can be introduced in several other alternative ways; for example, we have the integral representation

$$H_n(x)=\frac{1}{\sqrt{n!}}\int_{\mathbb{R}}(x+\mathrm{i}u)^n\gamma(\mathrm{d}u)\qquad(2)$$

The generating function for the Hermite polynomials is given by

$$\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}H_n(x)=\mathrm{e}^{\lambda x-\lambda^2/2}$$

We will prove these formulas in the next section.

The Hermite polynomials are related to the heat equation for the Ornstein-Uhlenbeck operator. If $D=\mathrm{d}/\mathrm{d}x$ is the usual differentiation

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Biography: ZHANG Qihao, male, born in 1991, master. Research field: Probability measure. E-mail: zqhzqh@mail.ustc.edu.cn

operator on $\mathbb{R}$, then its dual D^* with respect to the standard Gaussian measure γ is given by

$$D^* = -\frac{\mathrm{d}}{\mathrm{d}x} + x.$$

The Ornstein-Uhlenbeck operator is defined by

$$L = -D^* D = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - x\frac{\mathrm{d}}{\mathrm{d}x} \tag{3}$$

The solution to the initial value problem

$$\frac{\partial u}{\partial t} = \frac{1}{2}Lu,\ u(0,\cdot) = f \tag{4}$$

is given by $u(t,x)=T_t f(x)$, where $\{T_t,\ t\geqslant 0\}$ is called the Ornstein-Uhlenbeck semigroup. It can be shown that each T_t is given by an integral kernel, i. e.,

$$T_t f(x) = \int_{\mathbb{R}} T(t,x,y) f(y)\gamma(\mathrm{d}y) \tag{5}$$

We will prove that in terms of the Hermite polynomials,

$$T(t,x,y) = \sum_{n=0}^{\infty} \mathrm{e}^{-nt/2} H_n(x) H_n(y) \tag{6}$$

The function $T(t, x, y)$ is called the Mehler kernel.

Mehler's formula refers to the explicit identification of the kernel $T(t, x, y)$ in the following manner:

$$T_t f(x) = \int_{\mathbb{R}} f(\mathrm{e}^{-t/2}x + \sqrt{1-\mathrm{e}^{-t}}z)\gamma(\mathrm{d}z) \tag{7}$$

There are many ways to prove Mehler's formula. For a conventional approach using partial differential equations, see Ref. [1]. There is also another approach using stochastic analysis in Ref. [2]. In this article, we will give a simple proof of Mehler's formula based on the integral representation (2) of the Hermite polynomials.

1 Integral representation and generating function of the Hermite polynomials

We first prove the integral representation (2) of the Hermite polynomials. Let γ be the standard Gaussian measure on $\mathbb{R}$.

Theorem 1.1

$$H_n(x) = \frac{1}{\sqrt{n!}}\int_{\mathbb{R}} (x+\mathrm{i}u)^n \gamma(\mathrm{d}u).$$

Proof First we calculate the Fourier transform of the standard Gaussian measure γ.

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx}\gamma(\mathrm{d}t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\left[-\frac{t^2-2\mathrm{i}tx}{2}\right]\mathrm{d}t = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\left[-\frac{(t-\mathrm{i}x)^2}{2} - \frac{x^2}{2}\right]\mathrm{d}t = \mathrm{e}^{-x^2/2}\cdot\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\left[-\frac{(t-\mathrm{i}x)^2}{2}\right]\mathrm{d}t = \mathrm{e}^{-x^2/2}.$$

In Definition (1) of the Hermite polynomial H_n we replace $\mathrm{e}^{-x^2/2}$ by $\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx}\gamma(\mathrm{d}t)$. We have

$$H_n(x) = \mathrm{e}^{x^2/2}\frac{1}{\sqrt{2\pi}}\frac{(-1)^n}{\sqrt{n!}}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx-t^2/2}\mathrm{d}t = \mathrm{e}^{x^2/2}\frac{1}{\sqrt{2\pi}}\frac{(-1)^n}{\sqrt{n!}}\int_{-\infty}^{\infty}(\mathrm{i}t)^n \mathrm{e}^{\mathrm{i}tx-t^2/2}\mathrm{d}t = \frac{1}{\sqrt{2\pi}}\frac{(-1)^n}{\sqrt{n!}}\int_{-\infty}^{\infty}(\mathrm{i}t)^n \mathrm{e}^{-(\mathrm{i}x-t)^2/2}\mathrm{d}t.$$

Making the change of variable $u = \mathrm{i}x - t$, we have

$$H_n(x) = \frac{1}{\sqrt{2\pi}}\frac{(-1)^n}{\sqrt{n!}}\int_{-\infty}^{\infty}(-x-\mathrm{i}u)^n \mathrm{e}^{-u^2/2}\mathrm{d}u = \frac{1}{\sqrt{n!}}\int_{-\infty}^{\infty}(x+\mathrm{i}u)^n\gamma(\mathrm{d}u).$$

The generating function for the Hermite polynomials is given as follows.

Theorem 1.2

$$\sum_{k=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}H_n(x) = \mathrm{e}^{\lambda x-\lambda^2/2} \tag{8}$$

Proof Using the integral representation (2) we have

$$\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}H_n(x) = \int_{\mathbb{R}}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}(x+\mathrm{i}y)^n\gamma(\mathrm{d}y) = \int_{\mathbb{R}}\mathrm{e}^{\lambda(x+\mathrm{i}y)}\gamma(\mathrm{d}y) = \mathrm{e}^{\lambda x}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}y\lambda}\gamma(\mathrm{d}y).$$

We replace $\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}y\lambda}\gamma(\mathrm{d}y)$ by $\mathrm{e}^{-\lambda^2/2}$ which has been proved before, and obtain

$$\mathrm{e}^{\lambda x}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}y\lambda}\gamma(\mathrm{d}y) = \mathrm{e}^{\lambda x-\lambda^2/2},$$

that is,

$$\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}H_n(x) = \mathrm{e}^{\lambda x-\lambda^2/2}. \qquad \square$$

2 Further properties of the Hermite polynomials

We will need the following properties of the Hermite polynomials H_n, which will be used later. See Ref. [3] for more thorough discussion of the Hermite polynomials.

Theorem 2.1 The Hermite polynomials H_n have the following properties:

① $\{H_n\}$ is an orthonormal basis for the Hilbert space $L^2(\mathbb{R},\gamma)$.

② $LH_n = -nH_n$, where the Ornstein-Uhlenbeck operator L is defined by Eq. (3).

Proof ① According to the generating function (8),

$$e^{tx-t^2/2} = \sum_{n=0}^{\infty} \frac{t^n}{\sqrt{n!}} H_n(x),$$

$$e^{sx-s^2/2} = \sum_{n=0}^{\infty} \frac{s^n}{\sqrt{n!}} H_n(x).$$

Integrating the function $e^{tx-t^2/2}e^{sx-s^2/2}$ in x with respect to the standard Gaussian measure γ, we have

$$\int_{\mathbb{R}} e^{tx-t^2/2} e^{sx-s^2/2} \gamma(dx) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left[-\frac{1}{2}(x-(t+s))^2 + ts\right] dx = e^{ts} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left[-\frac{1}{2}(x-(t+s))^2\right] dx = e^{ts} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-x^2/2} dx = e^{ts},$$

and e^{ts} can be written as $\sum_{n=0}^{\infty} \frac{(ts)^n}{n!}$. On the other hand,

$$\int_{\mathbb{R}} e^{tx-t^2/2} e^{sx-s^2/2} \gamma(dx) = \int_{\mathbb{R}} \sum_{n=0}^{\infty} \frac{t^n}{\sqrt{n!}} H_n(x) \sum_{k=0}^{\infty} \frac{s^k}{\sqrt{k!}} H_k(x) \gamma(dx) = \sum_{k,n\geqslant 0} \frac{t^n s^k}{\sqrt{n!k!}} (H_n, H_k)_{L^2(\mathbb{R},\gamma)},$$

that is

$$\sum_{n=0}^{\infty} \frac{(ts)^n}{n!} = \sum_{k,n\geqslant 0} \frac{t^n s^k}{\sqrt{n!k!}} (H_n, H_k)_{L^2(\mathbb{R},\gamma)}.$$

Comparing the coefficients of the double power series in s and t, we see that H_n are mutually orthogonal and have the unit norm. Note that H_n is a polynomial of degree n. Hence the linear span of $H_0, \cdots, H_n$ coincides with the space of polynomials of degree at most n.

We now show that the orthogonal system $\{H_n\}$ is complete in the Hilbert space $L^2(\mathbb{R},\gamma)$. Suppose that $f \in L^2(\mathbb{R},\gamma)$ is orthogonal to all H_n, then f is also orthogonal to all polynomials. Let

$$F(z) = \int_{\mathbb{R}} f(x) e^{zx-x^2/2} dx.$$

Note that the integral converges absolutely because by the Cauchy-Schwarz inequality,

$$\left[\int_{\mathbb{R}} |f(x)| e^{|z|\cdot|x|-x^2/2} dx\right]^2 \leqslant \int_{\mathbb{R}} |f(x)|^2 e^{-x^2/2} dx \cdot \int_{\mathbb{R}} e^{2|z|\cdot|x|-x^2/2} dx$$

and both integrals on the right side converges, the first by the hypothesis $f \in L^2(\mathbb{R},\gamma)$. Now we can expand the exponential function and write

$$F(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\mathbb{R}} f(x) x^n e^{-x^2/2} dx = \sqrt{2\pi} \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\mathbb{R}} f(x) x^n \gamma(dx) = 0.$$

Letting $z = it$, we have

$$F(it) = \int_{\mathbb{R}} f(x) e^{-x^2/2} e^{itx} dx = 0.$$

Using the uniqueness of Fourier transform we can conclude that $f=0$ almost everywhere. Thus every function orthogonal to all Hermite polynomials is zero. This shows that the Hermite polynomials are complete.

② Let $w(x,\lambda) = e^{\lambda x-\lambda^2/2}$. We have

$$\frac{\partial w}{\partial x} = \lambda e^{\lambda x-\lambda^2/2} = \lambda w,$$

that is

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} H'_n(x) = \frac{\partial w}{\partial x} = \lambda w = \sum_{n=0}^{\infty} \frac{\lambda^{n+1}}{\sqrt{n!}} H_n(x).$$

Moreover, using $H'_0 = 0$ we have

$$\sum_{n=1}^{\infty} \frac{\lambda^n}{\sqrt{n!}} H'_n(x) = \sum_{n=1}^{\infty} \frac{\lambda^n}{\sqrt{(n-1)!}} H_{n-1}(x),$$

that is to say

$$H'_n(x) = \sqrt{n} H_{n-1}(x).$$

On the other hand, from Definition (1) of H_n we have

$$H'_n(x) = \frac{(-1)^n}{\sqrt{n!}}\left[x e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2} + e^{x^2/2}\frac{d^{n+1}}{dx^{n+1}}e^{-x^2/2}\right] = xH_n(x) - \sqrt{n+1}H_{n+1}(x).$$

We now have

$$\frac{d^2}{dx^2}H_n = \frac{d}{dx}\sqrt{n}H_{n-1} = x\sqrt{n}H_{n-1} - nH_n$$

and

$$x\frac{d}{dx}H_n = x\sqrt{n}H_{n-1}.$$

Finally,

$$LH_n = \left(\frac{d^2}{dx^2} - x\frac{d}{dx}\right)H_n = -nH_n. \qquad \square$$

We now show the Mehler kernel for the Ornstein-Uhlenbeck semigroup (5) is given by (6). This boils down to show that for any $f \in L^2(\mathbb{R},\gamma)$, the solution of the heat equation (4) is given by

$$u(t,x) = \sum_{n=0}^{\infty} f_n e^{-nt/2} H_n(x) \tag{9}$$

where

$$f_n = \int_{\mathbb{R}} H_n(y) f(y) \gamma(dy).$$

Using the property $LH_n = -nH_n$ in Theorem 2.1 we can verify easily that each term $e^{-nt/2}H_n(x)$ is a solution of the heat equation, hence by linearity, the series also satisfies the heat equation. For the initial condition, we have $u(0, x) = \sum_{n=0}^{\infty} f_n H_n(x)$. By the completeness of the Hermite polynomials, this is nothing but the orthogonal expansion of the square integrable function f in terms of the Hermite polynomials. This shows that $u(t,x)$ defined in (9) is indeed the unique solution of the intial value problem (4). Now for any $f \in L^2(\mathbb{R},\gamma)$ we have

$$T_t f(x) = \int_{\mathbb{R}} T(t,x,y) f(y)\gamma(dy) = \int_{\mathbb{R}}\left[\sum_{n=0}^{\infty} e^{-nt/2} H_n(x) H_n(y)\right] f(y)\gamma(dy).$$

It follows that the Mehler kernel is given by

$$T(t,x,y) = \sum_{n=0}^{\infty} e^{-nt/2} H_n(x) H_n(y) \tag{10}$$

3 Proof of Mehler's formula

Now, we are ready to prove the Mehler's formula (7). Write $a = e^{-t/2}$ for simplicity. Using the integral representation of the Hermite polynomials (2) and the expansion (10) we have

$$T(t,x,y) = \int_{\mathbb{R}}\left[\sum_{n=0}^{\infty}\frac{a^n(x+iu)^n}{\sqrt{n!}}H_n(y)\right]\gamma(du).$$

The infinite series can be replaced by the generating function (8) of the Hermite polynomials and we have

$$T(t,x,y) = \int_{\mathbb{R}}\exp\left[ya(x+iu) - \frac{a^2(x+iu)^2}{2}\right]\gamma(du) = e^{y^2/2}\int_{\mathbb{R}}\exp\left[\frac{a^2}{2}\left(u + i\frac{y-ax}{a}\right)^2\right]\gamma(du).$$

For simplicity let $b = (y-ax)/a$. We have

$$T(t,x,y) = e^{y^2/2}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\exp\left[\frac{a^2}{2}(u+ib)^2 - \frac{u^2}{2}\right]du = e^{y^2/2}\frac{1}{\sqrt{2\pi}}\cdot\int_{\mathbb{R}}\exp\left[-\frac{1-a^2}{2}\left(u^2 + \frac{a^2b^2}{1-a^2} - i\frac{2a^2bu}{1-a^2}\right)\right]du = \frac{1}{\sqrt{1-a^2}}\exp\left[\frac{y^2}{2} - \frac{a^2b^2}{2(1-a^2)}\right].$$

Now using the representation (5) and $b = (y-ax)/a$, we have

$$T_t f(x) = \int_{\mathbb{R}} T(t,x,y) f(y)\gamma(dy) = \frac{1}{\sqrt{1-a^2}}\int_{\mathbb{R}} f(y)\exp\left[\frac{y^2}{2} - \frac{a^2b^2}{2(1-a^2)}\right]\gamma(dy) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{1-a^2}}\int_{\mathbb{R}} f(y)\exp\left[-\frac{(y-ax)^2}{2(1-a^2)}\right]dy.$$

Making the substitution $y = ax + \sqrt{1-a^2}z$ we have

$$T_t f(x) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{1-a^2}}\cdot\int_{\mathbb{R}} f(ax+\sqrt{1-a^2}z)e^{-z^2/2}d(ax+\sqrt{1-a^2}z) = \int_{\mathbb{R}} f(ax+\sqrt{1-a^2}z)\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz = \int_{\mathbb{R}} f(ax+\sqrt{1-a^2}z)\gamma(dz).$$

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