Measuring the leading order hadronic contribution to the muon $g - 2$ in the space-like region

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Abstract：Recently a novel approach to determining the leading hadronic corrections to the muon $g-2$ has been proposed. It consists in a measurement of the effective electromagnetic coupling in the space-like region extracted from Bhabha scattering data. The new method may become feasible at flavor factories，leading to an alternative determination，possibly competitive with the accuracy of the present evaluations based on the dispersive approach via time-like data.

Key words：$e^+ e^-$ collisions；vacuum polarization；anomalous magnetic moment

在类空区域测量强子对缪子 $g - 2$ 因子的领头阶贡献

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摘要：最近提出了一种确定缪子领头阶强子修正项的新方法,基于从 Bhabha 散射数据提取的类空区域有效电磁耦合的测量. 新方法对于味工厂是可行的,可作为一种可选方案,其精度与基于类时数据的色散方法所做的评估是有竞争性的.

关键词：正负电子对撞；真空极化；反常磁矩

0　Introduction

In a recent paper[1]，we explored the possibility to evaluate the leading-order（LO）hadronic contribution to the muon anomalous magnetic moment a_μ^{HLO}，measuring the effective running electromagnetic coupling in the space-like region.

The discrepancy between experiment and the Standard Model（SM）prediction of the muon anomalous magnetic moment a_μ has kept the hadronic corrections under close scrutiny for several years[2-9]：the hadronic uncertainty dominates that of the SM value and is comparable with the experimental one. When the new results from the $g - 2$ experiments at Fermilab and J-PARC will reach the unprecedented precision of 0. 14

parts per million (or better)[10-12], the uncertainty of the hadronic corrections will become the main limitation.

An intense research program is under way to improve the evaluation of the leading order (LO) hadronic contribution to a_μ, due to the hadronic vacuum polarization correction to the one-loop diagram[13-15], as well as the next-to-leading order (NLO) hadronic one. The latter is further divided into the $O(\alpha^3)$ contribution of diagrams containing hadronic vacuum polarization insertions[16], and the leading hadronic light-by-light term, also of $O(\alpha^3)$[3,17-23]. Even the next-to-next-to leading order (NNLO) hadronic contributions have been studied: insertions of hadronic vacuum polarizations were computed in Ref. [24], while hadronic light-by-light corrections have been estimated in Ref. [25].

The evaluation of the hadronic LO contribution a_μ^{HLO} involves long-distance QCD for whose perturbation theory cannot be employed. However, using analyticity and unitarity, it was shown long ago that this term can be computed via a dispersion integral using the cross section for low-energy hadronic $e^+ e^-$ annihilation[26-28]. At low energy this cross-section is highly fluctuating due to resonances and particle production threshold effects.

An alternative determination of a_μ^{HLO} can be obtained measuring the effective electromagnetic coupling in the space-like region extracted from Bhabha ($e^+ e^- \rightarrow e^+ e^-$) scattering data, as detailed in Ref. [1]. A method to determine the running of the electromagnetic coupling in small-angle Bhabha scattering was proposed in Ref. [29] and applied to LEP data in Ref. [30]. As vacuum polarization in the space-like region is a smooth function of the squared momentum transfer, the accuracy of its determination is only limited by the statistics and by the control of the systematics of the experiment. Also, as at flavor factories the Bhabha cross section is strongly enhanced in the forward region, the space-like determination of a_μ^{HLO} may not be limited by statistics and, although challenging, may become competitive with standard results obtained with the dispersive

approach via time-like data.

1   Theoretical framework

The leading-order hadronic contribution to the muon $g-2$ is given by the well-known formula[9,26-28]

$$a_\mu^{\mathrm{HLO}} = \frac{\alpha}{\pi^2} \int_0^\infty \frac{\mathrm{d}s}{s} \, K(s) \, \mathrm{Im} \, \Pi_{\mathrm{had}}(s + \mathrm{i}\varepsilon) \qquad (1)$$

where $\Pi_{\mathrm{had}}(s)$ is the hadronic part of the photon vacuum polarization, $\varepsilon > 0$,

$$K(s) = \int_0^1 \mathrm{d}x \, \frac{x^2(1-x)}{x^2 + (1-x)(s/m_\mu^2)} \qquad (2)$$

is a positive kernel function and m_μ is the muon mass. As the total cross section for hadron production in low-energy $e^+ e^-$ annihilations is related to the imaginary part of $\Pi_{\mathrm{had}}(s)$ via the optical theorem, the dispersion integral in Eq. (1) is computed integrating experimental time-like ($s > 0$) data up to a certain value of s[3,31-32]. The high-energy tail of the integral is calculated using perturbative QCD[33].

Alternatively, if we exchange the x and s integrations in Eq. (1) we obtain[34]

$$a_\mu^{\mathrm{HLO}} = \frac{\alpha}{\pi} \int_0^1 \mathrm{d}x (x-1) \bar{\Pi}_{\mathrm{had}}[t(x)] \qquad (3)$$

where $\bar{\Pi}_{\mathrm{had}}(t) = \Pi_{\mathrm{had}}(t) - \Pi_{\mathrm{had}}(0)$ and

$$t(x) = \frac{x^2 m_\mu^2}{x - 1} < 0 \qquad (4)$$

is a space-like squared four-momentum. If we invert Eq. (4), we get $x = (1-\beta) t/(2m_\mu^2)$, with $\beta = (1 - 4m_\mu^2/t)^{1/2}$, and from Eq. (3) we obtain

$$a_\mu^{\mathrm{HLO}} = \frac{\alpha}{\pi} \int_{-\infty}^0 \bar{\Pi}_{\mathrm{had}}(t) \left(\frac{\beta-1}{\beta+1} \right)^2 \frac{\mathrm{d}t}{t\beta} \qquad (5)$$

Eq. (5) has been used for lattice QCD calculations of a_μ^{HLO}[35-38]; while the results are not yet competitive with those obtained with the dispersive approach via time-like data, their errors are expected to decrease significantly in the next few years[39].

The effective fine-structure constant at squared momentum transfer q^2 can be defined by

$$\alpha(q^2) = \frac{\alpha}{1 - \Delta\alpha(q^2)} \qquad (6)$$

where $\Delta\alpha(q^2) = -\mathrm{Re} \, \bar{\Pi}(q^2)$. The purely leptonic part, $\Delta\alpha_{\mathrm{lep}}(q^2)$, can be calculated order-by-order in perturbation theory — it is known up to three

loops in QED[40] (and up to four loops in specific q^2 limits[41-43]). As Im $\bar{\Pi}(q^2) = 0$ for negative q^2, Eq. (3) can be rewritten in the form[44-45]

$$a_\mu^{\text{HLO}} = \frac{\alpha}{\pi} \int_0^1 \mathrm{d}x (1-x) \Delta\alpha_{\text{had}}[t(x)] \qquad (7)$$

Eq. (7) can be evaluated by measuring the effective electromagnetic coupling in the space-like region (see also Ref. [15]), for instance from Bhabha scattering data.

A few considerations about Eq. (7) are in order here: in Fig. 1 (a) the integrand $(1-x)\Delta\alpha_{\text{had}}[t(x)]$ is plotted, using the output of the routine hadr5n12[46-47] for $\Delta\alpha_{\text{had}}(t)$. The range $x \in (0,1)$ corresponds to $t \in (-\infty, 0)$. The peak of the integrand occurs at $x_{\text{peak}} \simeq 0.914$ where $t_{\text{peak}} \simeq -0.108$ GeV^2 and $\Delta\alpha_{\text{had}}(t_{\text{peak}}) \simeq 7.86 \times 10^{-4}$ (see Fig. 1(b)). Such relatively low t values can be explored at e^+e^- colliders with $\sqrt{s}$ around or below 10 GeV (the so called "flavor factories").

Depending on s and θ, the integrand of Eq. (7) can be measured in the range $x \in [x_{\text{min}}, x_{\text{max}}]$, as shown in Fig. 2 (a). Note that to span low x intervals, larger θ ranges are needed as the collider energy decreases. In this respect, $\sqrt{s} \sim 3$ GeV
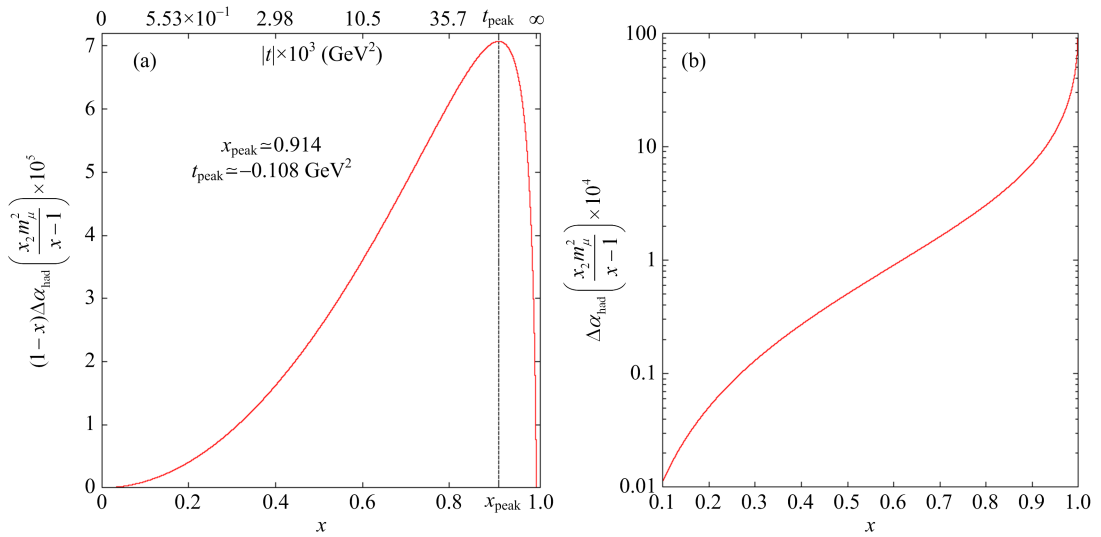
Fig. 1 (a) The integrand $(1-x)\Delta\alpha_{\text{had}}[t(x)] \times 10^5$ as a function of x and t. (b) $\Delta\alpha_{\text{had}}[t(x)] \times 10^4$
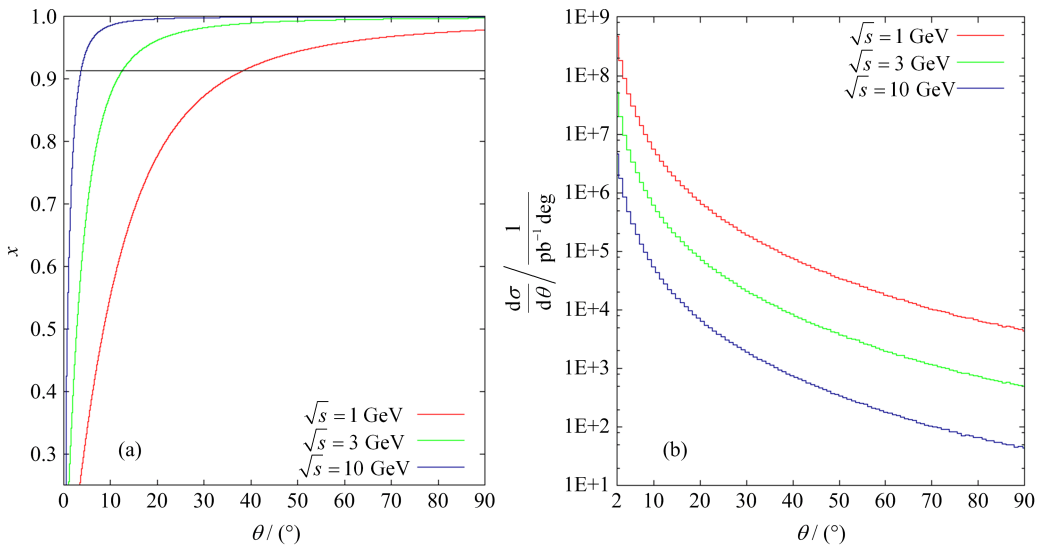
Fig. 2 (a) Ranges of x values as a function of the electron scattering angle θ for three different center-of-mass energies. The horizontal line corresponds to $x = x_{\text{peak}} \simeq 0.914$. (b) Bhabha differential cross section obtained with BabaYaga[50] as a function of θ for the same three values of $\sqrt{s}$ in the angular range $2° < \theta < 90°$

appears to be very convenient, as an x interval $[0.30, 0.98]$ can be measured varying θ between $\sim 2°$ and $28°$. It is also worth remarking that data collected at flavor factories, such as DAΦNE (Frascati), VEPP-2000 (Novosibirsk), BEPC-II (Beijing), PEP-II (SLAC) and SuperKEKB (Tsukuba), and possibly at a future high-energy $e^+ e^-$ collider, like FCC-ee (TLEP)[48] or ILC[49], can help to cover different and complementary x regions. Furthermore, given the smoothness of the integrand, values outside the measured x interval may be interpolated with some theoretical input.

2 $\Delta\alpha_{\mathrm{had}}(t)$ from Bhabha scattering data

The hadronic contribution to the running of α in the space-like region, $\Delta\alpha_{\mathrm{had}}(t)$, can be extracted comparing Bhabha scattering data to Monte Carlo (MC) predictions. The LO Bhabha cross section receives contributions from t- and s-channel photon exchange amplitudes. At NLO in QED, it is customary to distinguish corrections with an additional virtual photon or the emission of a real photon (photonic NLO) from those originated by the insertion of the vacuum polarization corrections into the LO photon propagator (VP). The latter goes formally beyond NLO when the Dyson resumed photon propagator is employed, which simply amounts to rescaling the α coupling in the LO s- and t-diagrams by the factor $1/(1-\Delta\alpha(q^2))$ (see Eq. (6)). In MC codes, e. g. in BabaYaga[50], VP corrections are also applied to photonic NLO diagrams, in order to account for a large part of the effect due to VP insertions in the NLO contributions. Beyond NLO accuracy, MC generators consistently include also the exponentiation of (leading-log) QED corrections to provide a more realistic simulation of the process and to improve the theoretical accuracy. We refer the reader to Ref. [51] for an overview of the status of the most recent MC generators employed at flavor factories. We stress that, given the inclusive nature of the measurements, any

contribution to vacuum polarization which is not explicitly subtracted by the MC generator will be part of the extracted $\Delta\alpha(q^2)$. This could be the case, for example, of the contribution of hadronic states including photons (which, although of higher order, are conventionally included in a_μ^{HLO}), and that of W bosons or top quark pairs.

The analytic dependence of the MC Bhabha predictions on $\alpha(t)$ (and, in turn, on $\Delta\alpha_{\mathrm{had}}(t)$) is not trivial, and a numerical procedure has to be devised to extract it from the data. In formulae, we have to find a function $\alpha(t)$ such that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}\bigg|_{\mathrm{data}} = \frac{\mathrm{d}\sigma}{\mathrm{d}t}(\alpha(t), \alpha(s))\,|_{\mathrm{MC}} \qquad (8)$$

where we explicitly kept apart the dependence on the time-like VP $\alpha(s)$ because we are only interested in $\alpha(t)$. Being the Bhabha cross section in the forward region dominated by the t-channel exchange diagram, we checked that the present $\alpha(s)$ uncertainty induces in this region a relative error on the θ distribution of less than $\sim 1 \times 10^{-4}$ (which is part of the systematic error).

We propose to perform the numerical extraction of $\Delta\alpha_{\mathrm{had}}(t)$ from the Bhabha distribution of the t Mandelstam variable. The idea is to let $\alpha(t)$ vary in the MC sample around a reference value and choose, bin by bin in the t distribution, the value that minimizes the difference with data. The procedure is detailed in Ref. [1] and here we only remark that the algorithm does not assume any simple dependence of the cross section on $\alpha(t)$, which can in fact be general, mixing s, t channels and higher order radiative corrections, relevant (or not) in different t domains.

In order to check our procedure, we performed a "pseudo-experiment", generating pseudo-data using the parameterization $\Delta\alpha_{\mathrm{had}}^l(t)$ of Refs. [32,52-53] and checking if it can be recovered by inserting in the MC the (independent) parameterization $\Delta\alpha_{\mathrm{had}}^{II}(t)$ of Ref. [46-47] by means of our algorithm. For this exercise, we used the generator BabaYaga in its most complete setup.

In Fig. 3, $\Delta\alpha_{\mathrm{had}}^{\mathrm{extr}}$ is the result extracted with our

algorithm：the figure shows that the method is capable of recovering the underlying function $\Delta\alpha_{\text{had}}(t)$ inserted into the "data". As the difference between $\Delta\alpha_{\text{had}}^{I}$ and $\Delta\alpha_{\text{had}}^{\text{extr}}$ is hardly visible on an absolute scale，in Fig. 3 all the functions have been divided by $\Delta\alpha_{\text{had}}^{II}$ to display better the comparison between $\Delta\alpha_{\text{had}}^{I}$ and $\Delta\alpha_{\text{had}}^{\text{extr}}$.
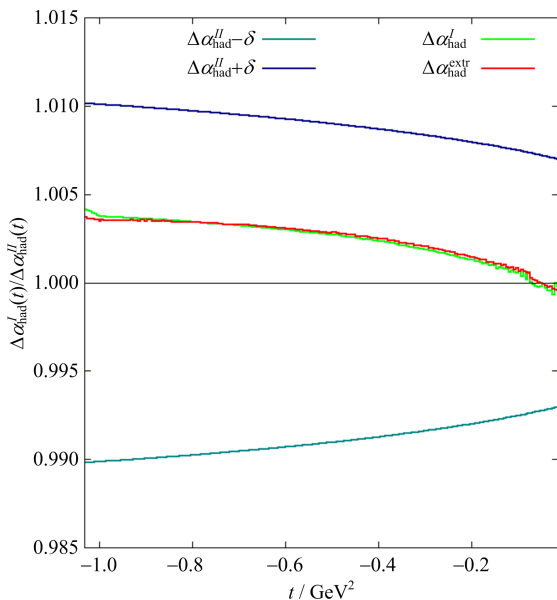
The functions $\Delta\alpha_{\text{had}}^{II}(t) \pm \delta(t)$ are shown to display the range spanned by the MC samples. All functions have been divided by $\Delta\alpha_{\text{had}}^{II}(t)$. The tiny difference between $\Delta\alpha_{\text{had}}^{I}$ and $\Delta\alpha_{\text{had}}^{\text{extr}}$ is due to the binning discretization

Fig. 3　The extracted function $\Delta\alpha_{\text{had}}^{\text{extr}}(t)$ compared to the function $\Delta\alpha_{\text{had}}^{I}(t)$ used in the pseudo-data（see text）.

In order to assess the achievable accuracy on $\Delta\alpha_{\text{had}}(t)$ with the proposed method，we remark that the LO contribution to the cross section is quadratic in $\alpha(t)$，thus we have

$$\frac{1}{2}\frac{\delta\sigma}{\sigma} \simeq \frac{\delta\alpha}{\alpha} \simeq \delta\Delta\alpha_{\text{had}} \qquad (9)$$

which relates the absolute error on $\Delta\alpha_{\text{had}}$ with the relative error on the Bhabha cross section. From the theoretical point of view，the present accuracy of the MC predictions[51] is at the level of about $0.5‰$，which implies that the precision that our method can，at best，set on $\Delta\alpha_{\text{had}}(t)$ is $\delta\Delta\alpha_{\text{had}}(t) \simeq 2\times10^{-4}$. Any further improvement requires the inclusion of the NNLO QED corrections into the MC codes，which is at present not available

（although not out of reach）[51].

From the experimental point of view，a measurement of a_{μ}^{HLO} from space-like data competitive with the current time-like evaluations would require an $O(1\%)$ accuracy. Statistical considerations show that a 3% fractional accuracy on the a_{μ}^{HLO} integral can be obtained by sampling the integrand in ~ 10 points around the x peak with a fractional accuracy of 10%. Given the value of $O(10^{-3})$ for $\Delta\alpha_{\text{had}}$ at $x = x_{\text{peak}}$，this implies that the cross section must be known with relative accuracy of $\sim 2\times10^{-4}$. Such a statistical accuracy，although challenging，can be obtained at flavor factories，as shown in Fig. 2（b）：with an integrated luminosity of $O(1)$，$O(10)$，$O(100)$ fb^{-1} at $\sqrt{s} = 1$，3 and 10 GeV，respectively，the angular region of interest can be covered with a 0.01% statistical accuracy per degree，which must be matched by a similar systematic error.

A fraction of the latter comes from the knowledge of the machine luminosity，which is normalized by calculating a theoretical cross section in principle not depending on $\Delta\alpha_{\text{had}}$. We devise two possible options for the normalization process：

① using the $e^{+}e^{-} \rightarrow \gamma\gamma$ process，which has no dependence on $\Delta\alpha_{\text{had}}$，at least up to NNLO order；

② using the Bhabha process at $t \sim 1\times10^{-3}$ GeV^{2}，where the dependence on $\Delta\alpha_{\text{had}}$ is of $O(10^{-5})$ and can be safely neglected.

Both processes have advantages and disadvantages；a dedicated study of the optimal choice will be considered in a future detailed study.

3　Conclusion

We discussed a novel approach to determining the leading hadronic correction to the muon $g-2$ by measuring the running of $\alpha(t)$ in the space-like region from Bhabha scattering data. Although challenging，we argue that this alternative determination may become feasible with a dedicated experimental and theoretical effort using data collected at present flavor factories and possibly also at a future high-energy $e^{+}e^{-}$ collider.

The proposed determination can become competitive with the accuracy of the present results obtained with the standard dispersive approach via time-like data.

References

[1] CARLONI CALAME C M, PASSERA M, TRENTADUE L, et al. A new approach to evaluate the leading hadronic corrections to the muon $g-2$[J]. Phys Lett B, 2015, 746: 325-329.

[2] BENNETT G W, BOUSQUET B, BROWN H N, et al (Muon $g-2$ Collaboration). Final report of the E821 muon anomalous magnetic moment measurement at BNL[J]. Phys Rev D, 2006, 73: 072003.

[3] JEGERLEHNER F, NYFFELER A. The muon $g-2$ [J]. Phys Rep, 2009, 477: 1-110.

[4] BLUM T, DENIG A, LOGASHENKO I, et al. The muon ($g-2$) theory value: Present and future [DB/OL]. arXiv:1311.2198 [hep-ph].

[5] MELNIKOV K, VAINSHTEIN A. Theory of the Muon Anomalous Magnetic Moment[M]. New York: Springer, 2006.

[6] DAVIER M, MARCIANO W J. The theoretical prediction for the muon anomalous magnetic moment [J]. Ann Rev Nucl Part Sci, 2004, 54: 115-140.

[7] PASSERA M. The standard model prediction of the muon anomalous magnetic moment [J]. J Phys G, 2005, 31: R75-R94.

[8] KNECHT M. The anomalous magnetic moment of the muon: A theoretical introduction [J]. Lect Notes Phys, 2004, 629: 37-84.

[9] JEGERLEHNER F. The Anomalous Magnetic Moment of the Muon[M]. New York: Springer, 2008.

[10] GRANGE J, GUARINO V, WINTER P, et al (Muon $g-2$ Collaboration). Muon ($g-2$) technical design report[DB/OL]. arXiv: 1501.06858[physics. ins-det].

[11] VENANZONI G (for the Muon $g-2$ Collaboration). The new muon $g-2$ experiment at Fermilab[DB/OL]. arXiv:1411.2555 [physics. ins-det].

[12] SAITO N, J-PARC $g-2$/EDM Collaboration. A novel precision measurement of muon $g-2$ and EDM at J-PARC[J]. AIP Conf Proc, 2012, 1467: 45-46.

[13] VENANZONI G. Hadronic contribution to the anomalous magnetic moment of the muon[J]. Nuovo Cim C, 2014, 37(2): 165-171.

[14] VENANZONI G. Latest on $g-2$ from experiment[J]. Frascati Phys Ser, 2012, 54: 52-67.

[15] FEDOTOVICH G V (on behalf of the CMD-2 Collaboration). Prospects of improving accuracy of the hadronic cross section measurements to the 10^{-3} level at the VEPP-2000 $e^+ e^-$ collider: Experimental and theoretical problems [J]. Nucl Phys (Proc Suppl), 2008,181-182: 146-150.

[16] KRAUSE B. Higher-order hadronic contributions to the anomalous magnetic moment of leptons[J]. Phys Lett B, 1997, 390: 392-400.

[17] KNECHT M, NYFFELER A. Hadronic light-by-light corrections to the muon $g-2$: The pion-pole contribution[J]. Phys Rev D, 2002, 65: 073034.

[18] MELNIKOV K, VAINSHTEIN A. Hadronic light-by-light scattering contribution to the muon anomalous magnetic moment reexamined[J]. Phys Rev D, 2004, 70: 113006.

[19] PRADES J, DE RAFAEL E, VAINSHTEIN A. The hadronic light-by-light scattering contribution to the muon and electron anomalous magnetic moments [DB/OL]. arXiv:0901.0306 [hep-ph].

[20] COLANGELO G, HOFERICHTER M, PROCURA M, et al. Dispersive approach to hadronic light-by-light scattering[J]. JHEP, 2014, 9: 91.

[21] COLANGELO G, HOFERICHTER M, KUBIS B, et al. Towards a data-driven analysis of hadronic light-by-light scattering[J]. Phys Lett B, 2014, 738: 6-12.

[22] PAUK V, VANDERHAEGHEN M. Anomalous magnetic moment of the muon in a dispersive approach [J]. Phys Rev D, 2014, 90:113012.

[23] BLUM T, CHOWDHURY S, HAYAKAWA M, et al. Hadronic light-by-light scattering contribution to the muon anomalous magnetic moment from lattice QCD[J]. Phys Rev Lett, 2015, 114: 012001.

[24] KURZ A, LIU T, MARQUARD P, et al. Hadronic contribution to the muon anomalous magnetic moment to next-to-next-to-leading order [J]. Phys Lett B, 2014, 734: 144-147.

[25] COLANGELO G, HOFERICHTER M, NYFFELER A, et al. Remarks on higher-order hadronic corrections to the muon $g-2$[J]. Phys Lett B, 2014, 735: 90-91.

[26] BOUCHIAT C, MICHELL. La résonance dans la diffusion méson π-méson π et le moment magnétique anormal du méson μ[J]. J Phys Radium, 1961, 22: 121.

[27] DURAND L. Pionic contributions to the magnetic

moment of the muon[J]. Phys Rev, 1962, 128: 441-448(Erratum-ibid 1963, 129: 2835).

[28] GOURDIN M, DE RAFAEL E. Hadronic contributions to the muon g-factor[J]. Nucl Phys B, 1969, 10: 667-674.

[29] ARBUZOV A B, HAIDT D, MATTEUZZI C, et al. The running of the electromagnetic coupling α in small angle Bhabha scattering[J]. Eur Phys J C, 2004, 34: 267-275.

[30] ABBIENDI G, et al (OPAL Collaboration). Measurement of the running of the QED coupling in small-angle Bhabha scattering at LEP[J]. Eur Phys J C, 2006, 45: 1-21.

[31] DAVIER M, HOECKER A, MALAESCU B, et al. Reevaluation of the hadronic contributions to the muon $g-2$ and to $\alpha(M_Z^2)$[J]. Eur Phys J C, 2011, 71: 1515 (Erratum-ibid, 2012, 72: 1874).

[32] HAGIWARA K, LIAO R, MARTIN A D, et al. $(g-2)_\mu$ and $\alpha(M_Z^2)$ re-evaluated using new precise data [J]. J Phys G, 2011, 38: 085003.

[33] HARLANDER R V, STEINHAUSER M. rhad: A program for the evaluation of the hadronic R-ratio in the perturbative regime of QCD[J]. Comput Phys Commun, 2003, 153: 244-274.

[34] LAUTRUP B E, PETERMAN A, DE RAFAEL E. Recent developments in the comparison between theory and experiments in quantum electrodynamics[J]. Phys Rep, 1972, 3: 193-259.

[35] AUBIN C, BLUM T. Calculating the hadronic vacuum polarization and leading hadronic contribution to the muon anomalous magnetic moment with improved staggered quarks[J]. Phys Rev D, 2007, 75: 114502.

[36] BOYLE P, DEL DEBBIO L, KERRANE E, et al. Lattice determination of the hadronic contribution to the muon $g-2$ using dynamical domain wall fermions [J]. Phys Rev D, 2012, 85: 074504.

[37] FENG X, JANSEN K, PETSCHLIES M, et al. Two-flavor QCD correction to lepton magnetic moments at leading order in the electromagnetic coupling[J]. Phys Rev Lett, 2011, 107: 081802.

[38] DELLA MORTE M, JAGER B, JUTTNER A, et al. Towards a precise lattice determination of the leading hadronic contribution to $(g-2)_\mu$[J]. JHEP, 2012, 1203: 055.

[39] BLUM T, HAYAKAWA M, IZUBUCHI T. Hadronic corrections to the muon anomalous magnetic moment from lattice QCD[D]. PoS, 2012: PoS(Lattice 2012)022.

[40] STEINHAUSER M. Leptonic contribution to the effective electromagnetic coupling constant up to three loops[J]. Phys Lett B, 1998, 429: 158-161.

[41] BAIKOV P A, CHETYRKIN K G, KUHN J H, et al. The relation between the QED charge renormalized in $\overline{MS}$ and on-shell schemes at four loops, the QED on-shell β-function at five loops and asymptotic contributions to the muon anomaly at five and six loops [J]. Nucl Phys B, 2013, 867: 182-202.

[42] STURM C. Leptonic contributions to the effective electromagnetic coupling at four-loop order in QED[J]. Nucl Phys B, 2013, 874: 698-719.

[43] BAIKOV P A, MAIER A, MARQUARD P. The QED vacuum polarization function at four loops and the anomalous magnetic moment at five loops[J]. Nucl Phys B, 2013, 877: 647-661.

[44] JEGERLEHNER F. Hadronic Contributions to the photon vacuum polarization and their role in precision physics [C]// Proceedings of Fifty Years of Electroweak Physics: A Symposium In Honour of Professor Alberto Sirlin's 70th Birthday. New York: New York University, 2000.

[45] JEGERLEHNER F. Hadronic contributions to the photon vacuum polarization and their role in precision physics[J]. J Phys G, 2003, 29: 101-110.

[46] EIDELMAN S, JEGERLEHNER F. Hadronic contributions to $(g-2)$ of the leptons and to the effective fine structure constant $\alpha(M_Z^2)$[J]. Z Phys C, 1995, 67: 585-601.

[47] JEGERLEHNER F. The running fine structure constant $\alpha(E)$ via the Adler function[J]. Nucl Phys Proc Suppl, 2008, 181-182: 135-140.

[48] BICER M, YILDIZ H D, YILDIZ I, et al (TLEP Design Study Working Group Collaboration). First look at the physics case of TLEP[DB/OL]. arXiv: 1308.6176 [hep-ex].

[49] DJOUADI A, LYKKEN J, MÖNIG K, et al (ILC Collaboration). International linear collider reference design report volume 2: Physics at the ILC[DB/OL]. arXiv: 0709.1893 [hep-ph].

[50] BALOSSINI G, CARLONI CALAME C M, MONTAGNA G, et al. Matching perturbative and parton shower corrections to Bhabha process at flavour factories[J]. Nucl Phys B, 2006, 758: 227-253.

[51] ACTIS S, ARBUZOV A, BALOSSINI G, et al., Quest for precision in hadronic cross sections at low energy: Monte Carlo tools vs. experimental data[J]. Eur Phys J C, 2010, 66: 585-686.

[52] HAGIWARA K, MARTIN A D, NOMURA D, et al. Improved predictions for $g-2$ of the muon and $\alpha_{QED}(M_Z^2)$[J]. Phys Lett B, 2007, 649: 173-179.

[53] HAGIWARA K, MARTIN A D, NOMURA D, et al. Predictions for $g-2$ of the muon and $\alpha_{QED}(M_Z^2)$[J]. Phys Rev D, 2004, 69: 093003.