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On S-c-propermutable subgroups of finite groups

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Abstract: A subgroup H of a group G is said to be S-c-propermutable in G if G has a subgroup B such that $G=N_G(H)B$ and for every Sylow subgroup A of B, there exists an element $x\in B$ such that $HA^x=A^xH$. Here, S-c-propermutable subgroups were used to study the structure of finite groups and some new criteria of supersoluble groups were obtained.

Key words: S-permutable subgroup; S-c-propermutable subgroup; Sylow subgroup; Hall subgroup

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有限群的 S-c 置换子群

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摘要:群 G 的一个子群被称为 S-c 置换的,如果 G 有一个子群 B 满足 $G=N_G(H)B$ 并且对于 B 的任意 Sylow 子群 A,存在元素 $x\in B$ 使得 $HA^x=A^xH$.利用 S-c 置换子群研究了有限群的结构,得到了超可解群的一些新的判别准则.

关键词:S 置换子群;S-c 置换子群;Sylow 子群;Hall 子群

0 Introduction

Throughout this paper, all groups considered are finite. G always denotes a group and p denotes a prime. let π denote a set of primes and $\pi(G)$ denote the set of all prime divisors of $|G|$. Let $|G|_p$ denote the order of Sylow p-subgroups of G. All unexplained notation and terminology are standard, as in Refs. [1-2].

A class of groups $\mathscr{F}$ is called a formation if it is closed under taking homomorphic images and subdirect products. A formation $\mathscr{F}$ is called

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saturated (resp. solubly saturated) if $G \in \mathscr{F}$ whenever $G/\Phi(G) \in \mathscr{F}$ (resp. $G/\Phi(N) \in \mathscr{F}$ for a soluble normal subgroup N of G). For a class of groups $\mathscr{F}$, a chief factor L/K of G is said to be $\mathscr{F}$-central in G if $L/K \rtimes G/C_G(L/K) \in \mathscr{F}$. A normal subgroup N of G is called $\mathscr{F}$-hypercentral in G if either $N=1$ or every chief factor of G below N is $\mathscr{F}$-central in G. Let $Z_{\mathscr{F}}(G)$ denote the $\mathscr{F}$-hypercentre of G, that is, the product of all $\mathscr{F}$-hypercentral normal subgroups of G. We say that a chief factor L/K of G is Frattini (resp. non-Frattini) if $L/K \leqslant \Phi(G/K)$ (resp. $L/K \nleqslant \Phi(G/K)$). The $\mathscr{F}$-residual of G, denoted by $G^{\mathscr{F}}$, is the smallest normal subgroup of G with quotient in $\mathscr{F}$. Moreover, the subgroup $F^*(G)$ of G is called generalized fitting subgroup of G which is the set of all elements x of which induce an inner automorphism on every chief factor of G (for details, see Ref. [3, Chapter X, Definition 13.9]. We use $\mathscr{U}$ and $\mathscr{N}$ to denote the formations of all supersoluble groups and nilpotent groups, respectively.

Recently, Ref. [4] gave the concept of S-propermutable subgroups: a subgroup H of G is said to be S-propermutable in G provided that there is a subgroup B of G such that $G=N_G(H)B$ and H permutes with all Sylow subgroups of B. In 2007, Ref. [5] proposed S-conditionally permutable subgroups: a subgroup H of G is said to be S-conditionally permutable in G if for every Sylow subgroup A of G there exists an element $x\in G$ such that $HA^x = A^xH$. Hence naturally, by S-conditionally permutable, we can weaken the definition of S-propermutable and so give the following new notion:

Definition 0.1 A subgroup H of a group G is said to be S-c-propermutable in G if G has a subgroup B such that $G=N_G(H)B$ and for every Sylow subgroup A of B, there exists an element $x\in B$ such that $HA^x=A^xH$.

It is easy to see that S-propermutable subgroups of G are S-c-propermutable in G. But the converse does not hold in general.

Example 0.2 Let $A=S_4$ be a symmetric group of degree 4 and $B=C_5$ be a cyclic group of order 5. Let $G=A \wr B$ be the regular wreath product of A by B. Let K be the base group of G. Clearly, $G=N_G(B)K$. It is easy to see that for any $p\in \pi(K)$, there exists some Sylow p-subgroup P of K such that $BP=PB$. Hence B is S-c-propermutable in G. But B is not S-propermutable in G. Clearly,

$$P_2 = K_4\langle(12)\rangle \times K_4\langle(13)\rangle \times K_4\langle(23)\rangle \times K_4\langle(24)\rangle \times K_4\langle(34)\rangle$$

is a Sylow 2-subgroup of K, where $K_4=\{(1),(12)(34),(13)(24),(14)(23)\}$. If $P_2B=BP_2$, then $P_2=K\cap P_2B \trianglelefteq P_2B$, and so $B\leqslant N_G(P_2)$. This is impossible. Hence B is not S-propermutable in G.

In the present paper, we derive some criteria for a finite group to be a supersoluble subgroup.

1 Preliminaries

Lemma 1.1 Let $H\leqslant G$ and $N \trianglelefteq G$. If H is S-c-propermutable in G, then HN/N is S-c-propermutable in G/N.

Proof By hypothesis, G has a subgroup B such that $G=N_G(H)B$ and H permutes with some Sylow p-subgroup of B, where p is any prime divisor of $|B|$. Clearly,

$$G/N=(N_G(H)N/N)(BN/N)= N_{G/N}(HN/N)(BN/N).$$

Let p be any prime dividing $|BN/N|$. Then there exists a Sylow p-subgroup B_p of B such that $HB_p=B_pH$. It follows that

$$(HN/N)(B_pN/N)=(B_pN/N)(HN/N)$$

and B_pN/N is a Sylow p-subgroup of BN/N. Hence HN/N is S-c-propermutable in G/N.

Lemma 1.2[6, Lemma 2.1] Let $\mathscr{F}$ be a non-empty saturated formation, $H\leqslant G$ and $N \trianglelefteq G$. Then $Z_{\mathscr{F}}(G)N/N\leqslant Z_{\mathscr{F}}(G/N)$.

Let P be a p-group. If P is not a non-abelian 2-group, then we use $\Omega(P)$ to denote the subgroup $\Omega_1(P)$. Otherwise, $\Omega(P)=\Omega_2(P)$.

Lemma 1.3[7, Theorem 2.8] Let $\mathscr{F}$ be a solubly saturated formation. Suppose that P is a normal p-

subgroup of G and C is a Thompson critical subgroup of P(see Ref. [8, Chapter 5]). If either $P/\Phi(P)\leqslant Z_{\mathcal{F}}(G/\Phi(P))$ or $\Omega(C)\leqslant Z_{\mathcal{F}}(G)$, then $P\leqslant Z_{\mathcal{F}}(G)$.

Lemma 1.4[7, Lemma 2.10] Let C be a Thompson critical subgroup of a nontrivial p-group of P.

(Ⅰ) If p is odd, then the exponent of $\Omega_1(C)$ is p.

(Ⅱ) If P is an abelian 2-group, then the exponent of $\Omega_1(C)$ is 2.

(Ⅲ) If $p=2$, then the exponent of $\Omega_2(C)$ is at most 4.

Lemma 1.5[9, Lemma 2.14] If the generalized fitting subgroup $F^*(G)$ of G is soluble, then $F^*(G)=F(G)$.

Lemma 1.6[10, Theorem B] Let $\mathcal{F}$ be any formation and E a normal subgroup of G. If $F^*(E)\leqslant Z_{\mathcal{F}}(G)$, then $E\leqslant Z_{\mathcal{F}}(G)$.

Lemma 1.7 Let $\mathcal{F}$ be a formation containing all supersoluble groups. Suppose that E is a normal subgroup of G such that $G/E\in\mathcal{F}$.

(Ⅰ) If $\mathcal{F}$ be a solubly saturated formation and $E\leqslant Z_{\mathcal{U}}(G)$, then $G\in\mathcal{F}$[11, Lemma 2.11].

(Ⅱ) If $\mathcal{F}$ be a saturated formation and E is cyclic, then $G\in\mathcal{F}$[12, Lemma 2.16].

Lemma 1.8[13, Theorem 2] Let G be a p-soluble group. Then for any $q\in\pi(G)$, G has Hall $\{p,q\}$-subgroup.

2 New characterizations of p-supersolvability of groups

Theorem 2.1 Let P be a normal p-subgroup of G. Suppose that P has a subgroup D such that $1<|D|<|P|$ and all subgroups H of P with order $|H|=|D|$ or all subgroups with order $|H|=2|D|$ (if P is a non-abelian 2-group, $|P:D|>2$ and $\exp(H)>2$) is S-c-propermutable in G, then $P\leqslant Z_{\mathcal{U}}(G)$.

Proof Suppose that the result is false and let (G,P) be a counterexample for which $|G|+|P|$ is minimal. Let N be a minimal normal subgroup of G contained in P. We now proceed via the following steps.

(Ⅰ) If $|N|<|D|$, then N is a unique minimal normal subgroup of G contained in P such that $P/N\leqslant Z_{\mathcal{U}}(G/N)$ and $|N|>p$.

Let H/N be a subgroup of P/N such that $|H/N|=|D|/|N|$ or $|H/N|=2|D|/|N|$ (if P/N is a non-abelian 2-group, $|P/N:D/N|>2$ and $\exp(H/N)>2$). Then it is easy to see that H is a subgroup of P such that $|H|=|D|$ or $|H|=2|D|$ (if P is a non-abelian 2-group, $|P:D|>2$ and $\exp(H)>2$). Thus by hypothesis, H is S-c-propermutable in G. Then by Lemma 1.1, H/N is S-c-propermutable in G/N. The choice of G implies that $P/N\leqslant Z_{\mathcal{U}}(G/N)$. If $|N|=p$, then $P\leqslant Z_{\mathcal{U}}(G)$, a contradiction. Thus $|N|>p$. Let R be another minimal normal subgroup of G contained in P such that $N\neq R$. Since $NR/N\leqslant Z_{\mathcal{U}}(G/N)$ and NR/N is a minimal normal subgroup of G/N, we have $|R|=|NR/N|=p$. It implies that $|R|\leqslant|N|<|D|$. A similar discussion to the above can deduce that $P/R\leqslant Z_{\mathcal{U}}(G/R)$, and so $P\leqslant Z_{\mathcal{U}}(G)$, a contradiction. Hence N is a unique minimal normal subgroup of G contained in P.

(Ⅱ) $|N|=|D|$.

If $|N|>|D|$, suppose that N_1 is a subgroup of N such that $|N_1|=|D|$ and N_1 is normal in some Sylow p-subgroup G_p of G. By hypothesis, G has a subgroup B such that $G=N_G(N_1)B$ and for any $q\in\pi(B)$ with $p\neq q$, there exists a Sylow q-subgroup Q of B such that $N_1Q=QN_1$. Since $N_1=N\cap N_1Q\trianglelefteq N_1Q$, we have that $Q\leqslant N_G(N_1)$. Let B_p be a Sylow p-subgroup of B. Then there exists an element $x\in G$ such that $B_p=B\cap G_p^x$. Clearly, $G_p^x\leqslant N_G(N_1^b)$ for $N_1\trianglelefteq G_p$, where $b\in B$, and so $B_p\leqslant N_G(N_1^b)$. Since $Q^b\leqslant N_G(N_1^b)$, we have that $B\leqslant N_G(N_1)$. Therefore, $N_1\trianglelefteq G$. It implies that $N_1=1$ or $N_1=N$, which is impossible because $1<|D|<|N|$.

Now assume that $|N|<|D|$. By (1), $P/N\leqslant Z_{\mathcal{U}}(G/N)$. If $N\leqslant\Phi(P)$, then by Lemma 1.2, $P/\Phi(P)\leqslant Z_{\mathcal{U}}(G/\Phi(P))$, and so $P\leqslant Z_{\mathcal{U}}(G)$ by Lemma 1.3, a contradiction. Thus assume that $N\nleqslant\Phi(P)$. It follows from (1) that $\Phi(P)=1$. Let

U be a complement subgroup of N in P. Let N_1 be a maximal subgroup of N such that N_1 is normal in some Sylow p-subgroup G_p of G. Since

$$|D|<|P|=|U||N_1|p,$$

we have $|U|\geqslant|D|/|N_1|$. Hence we take a subgroup V of order $|D|/|N_1|$ of U. Let $T=N_1V$, then $|T|=|N_1V|=|D|$. By hypothesis, T is S-c-propermutable in G. Therefore there exists a subgroup B of G such that $G=N_G(T)B$ and for any $q\in\pi(B)$ with $p\neq q$, there exists a Sylow q-subgroup B_q of B such that $TB_q=B_qT$. Since $N\cap T=N\cap TB_q\trianglelefteq TB_q$, $B_q\leqslant N_G(N\cap T)$. Let B_p be a Sylow p-subgroup of B. Then there exists an element $x\in G$ such that $B_p=B\cap G_p^x$. Note that $N\cap T=N_1$. Thus $G_p^x\leqslant N_G(N\cap T^b)$ for $G_p\leqslant N_G(N_1)$, where $b\in B$, and so $B_p^{b^{-1}}\leqslant N_G(N\cap T)$. It implies that $B\leqslant N_G(N\cap T)$. Obviously, $N_G(T)\leqslant N_G(N\cap T)$. Hence $G\leqslant N_G(N\cap T)=N_G(N_1)$. Then $N_1\trianglelefteq G$ and so $|N|=p$, which contradicts (Ⅰ). Therefore, $|N|=|D|$.

(Ⅲ) $|D|=p$.

Let G_p be a Sylow p-subgroup of G and N/M a chief factor of G_p. Then $|N/M|=p$. Suppose that L/N is a normal subgroup of order p of G_p/N contained in P/N. Then $|L/M|=p^2$. First, we consider that L/M is an elementary abelian p-group and assume that $L/M=N/M\times T/M$, where $|N/M|=|T/M|=p$. Then $|T|=|N|=|D|$, and so by hypothesis, T is S-c-propermutable in G. Clearly, $M=T\cap N$. By applying a same argument as in (Ⅱ), we can derive that M is normal in G, and so $|N|=p$. Now assume that L/M is a cyclic group of order p^2. Then there exists an element $a\in L\backslash M$ such that $L=M\langle a\rangle$. It follows that $N=M(N\cap\langle a\rangle)$. It is easy to see that $a\notin N$ and $|N\cap\langle a\rangle|=p$. It implies that $M\cap\langle a\rangle=1$ and so $|\langle a\rangle|=p^2$. Let $\mho_1(L)=\langle l^p\mid l\in L\rangle$. Clearly, $\mho_1(L)\leqslant\Phi(L)<N$. Let N_1 be a maximal subgroup of N such that $\mho_1(L)\leqslant N_1$ and $N_1\trianglelefteq G_p$. Note that $N_1\neq M$ for $M\cap\langle a\rangle=1$. Then $\langle a^p\rangle\leqslant N_1$ and $|N_1\langle a\rangle|=|N|=|D|$. By hypothesis, $N_1\langle a\rangle$ is S-c-propermutable in G. Since $N_1=N\cap N_1\langle a\rangle$, then by using a similar discussion as in (Ⅱ), we have that $N_1\trianglelefteq G$, and so $|N|=p$.

(Ⅳ) Final contradiction.

By (Ⅲ) and the hypothesis of the theorem, we know that every cyclic subgroup of P of order prime or order 4 (when P is a non-abelian 2-group) is S-c-propermutable in G. First, we claim that G has a unique normal subgroup R such that P/R is a chief factor of G, $R\leqslant Z_{\mathcal{U}}(G)$ and $|P/R|>p$.

Let P/R be a chief factor of G. Clearly, (G, R) satisfies the hypothesis of the theorem. The choice of (G, P) implies that $R\leqslant Z_{\mathcal{U}}(G)$. If $|P/R|=p$, then $P/R\leqslant Z_{\mathcal{U}}(G/R)$ and so $P\leqslant Z_{\mathcal{U}}(G)$, a contradiction. Hence $|P/R|>p$. Assume that P/L is a chief factor of G with $P/R\neq P/L$. By a same discussion as above, we have that $L\leqslant Z_{\mathcal{U}}(G)$. It follows from Lemma 1.2 that $P/R=RL/R\leqslant RZ_{\mathcal{U}}(G)/R\leqslant Z_{\mathcal{U}}(G/R)$, which can derive a same contradiction as above. Therefore, G has a unique normal subgroup R such that P/R is a chief factor of G, $R\leqslant Z_{\mathcal{U}}(G)$ and $|P/R|>p$.

Let C be a Thompson critical subgroup of P. If $\Omega(C)<P$, then $\Omega(C)\leqslant R\leqslant Z_{\mathcal{U}}(G)$, and so $P\leqslant Z_{\mathcal{U}}(G)$ by Lemma 1.3, a contradiction. Hence $P=C=\Omega(C)$. Then by Lemma 1.4, the exponent of P is p or 4 (when P is a non-abelian 2-group).

Since $P/R\cap Z(G_p/R)>1$, we put $L/R\leqslant P/R\cap Z(G_p/R)$ with $|L/R|=p$, where G_p is a Sylow p-subgroup of G. Let $x\in L\backslash R$ and $H=\langle x\rangle$. Then $L=HR$ and $|H|=p$ or 4 (when P is a non-abelian 2-group). By hypothesis, H is S-c-propermutable in G. It follows from Lemma 1.1 that $HR/R=L/R$ is S-c-propermutable in G/R. Thus G/R has a subgroup B/R such that $G/R=N_{G/R}(L/R)(B/R)$ and for any $q\in\pi(B/R)$ with $p\neq q$, there exists a Sylow q-subgroup Q/R of B/R such that

$$(L/R)(Q/R)=(Q/R)(L/R).$$

By a similar discussion to (Ⅱ), we have that $L/R\trianglelefteq G/R$. It implies that $|L/R|=|P/R|=p$. The final contradiction completes that proof of the theorem. □

Corollary 2.2 Let $\mathscr{F}$ be a solubly saturated formation containing the class of all supersoluble groups. If G has a normal subgroup E such that $G/E\in\mathscr{F}$ and $F^*(E)$ is soluble. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1<|D|<|P|$ and all subgroups H of P with order $|H|=|D|$ or with order $|H|=2|D|$ (if P is a non-abelian 2-group, $|P:D|>2$ and $\exp(H)>2$) is S-c-propermutable in G, then $G\in\mathscr{F}$.

Proof Since $F^*(E)$ is soluble, by Lemma 1.5, $F^*(E)=F(E)$. Let P be a Sylow p-subgroup of $F(E)$ for arbitrary $p\in\pi(F(E))$. Then $P\trianglelefteq G$. If P is non-cyclic, then P satisfies the hypothesis of Theorem 2.1, thus $P\leqslant Z_{\mathscr{U}}(G)$. Assume that P is cyclic. Let L/K be any chief factor of G below P. Then $|L/K|=p$, and so L/K is $\mathscr{F}$-central in G. Hence $P\leqslant Z_{\mathscr{U}}(G)$. It implies that $F^*(E)\leqslant Z_{\mathscr{U}}(G)$. By Lemma 1.6, $E\leqslant Z_{\mathscr{U}}(G)$. Consequently, $G\in\mathscr{F}$ by Lemma 1.7. □

Theorem 2.3 Let G be a soluble group. If every maximal subgroup of every Sylow subgroup of G is S-c-propermutable in G, then $G=D\rtimes C$ is supersoluble, where $D=G^{\mathscr{N}}$ is a nilpotent Hall subgroup of G of odd order whose maximal subgroups are normal in G.

Proof Suppose that the result is false and let G be a counterexample with $|G|$ minimal.

(Ⅰ) Let N be any normal subgroup of G. Then $G/N=D/N\rtimes C/N$ is supersoluble, where $D/N=(G/N)^{\mathscr{N}}$ is a nilpotent Hall subgroup of G/N of odd order whose maximal subgroups are normal in G/N.

Let P/N be a Sylow p-subgroup of G/N and M/N be a maximal subgroup of P/N, where p is an arbitrary prime divisor of $|G/N|$. Then there exists a Sylow subgroup G_p of G such that $P=G_pN$ and $M\cap G_p$ is a maximal subgroup of G_p. Clearly, $M/N=(M\cap G_p)N/N$. By Lemma 1.1, M/N is S-c-propermutable in G/N. This shows that the hypothesis holds for G/N. Hence $G/N=DN/N\rtimes C/N$ is supersoluble, where $DN/N=(G/N)^{\mathscr{N}}$ is a nilpotent Hall subgroup of G/N of odd order whose maximal subgroups are normal in G/N.

(Ⅱ) G is supersoluble.

Let N be a minimal normal subgroup of G. Since G is soluble, N is an abelian p-group, where $p\in\pi(G)$. Let P be a Sylow p-subgroup of G such that $N\leqslant P$. If $N\leqslant\Phi(G)$, since the class of all supersoluble groups is a saturated formation, G is supersoluble by (Ⅰ). Hence assume that $N\nleqslant\Phi(G)$, then $N\nleqslant\Phi(P)$. Therefore there exists a maximal subgroup P_1 of P such that $P=NP_1$. By hypothesis, G has a subgroup B such that $G=N_G(P_1)B$ and for any $q\in\pi(B)$ with $p\neq q$, there exists a Sylow q-subgroup of B such that $P_1Q=QP_1$. Since $N\cap P_1Q=N\cap P_1\trianglelefteq P_1Q$, we have $Q\leqslant N_G(N\cap P_1)$. Let B_p be a Sylow p-subgroup of B. Then there exists an element $x\in G$ such that $B_p=B\cap P^x$. Clearly, $P\leqslant N_G(N\cap P_1)$. Thus $P^x\leqslant N_G(N\cap P_1^b)$, where $b\in B$, and so

$$B_p\leqslant N_G(N\cap P_1^b).$$

It is easy to see that $Q^b\leqslant N_G(N\cap P_1^b)$. Hence $B\leqslant N_G(N\cap P_1)$. Obviously, $N_G(P_1)\leqslant N_G(N\cap P_1)$, and so $N\cap P_1\trianglelefteq G$. It implies that $N\leqslant P_1$ or $N\cap P_1=1$. If $N\leqslant P_1$, then $P=P_1$, a contradiction. If $N\cap P_1=1$, then $|N|=p$, by Lemma 1.7, G is supersoluble.

(Ⅲ) D is a nilpotent Hall subgroup of G.

By (Ⅱ), we have that G' is nilpotent. Thus $D=G^{\mathscr{N}}$ is nilpotent. We show that D is a Hall subgroup of G. Assume that D is not a Hall subgroup of G. Then $G\neq D\neq 1$. Let P be a Sylow p-subgroup of D such that $1<P<G_p$ for some Sylow p-subgroup of G, where p is a prime divisor of $|D|$. We now proceed via the following steps.

(ⅰ) If N is a minimal normal subgroup of G contained in D, then $N=O_p(D)=P$ is a Sylow p-subgroup of D.

Let N be a minimal normal subgroup of G contained in D. Since D is nilpotent, then N is a subgroup of prime power order. Suppose that N is an abelian q-group, where q is a prime divisor of $|D|$ with $p\neq q$. By (Ⅰ), $D/N=(G/N)^{\mathscr{N}}$ is a Hall subgroup of G/N. Since PN/N is a Sylow p-subgroup of D/N, PN/N is a Sylow p-subgroup of G/N. It follows that P is a Sylow p-subgroup of G, which is a contradiction. Hence $p=q$ and $N=$

$O_p(D)=P$.

(ⅱ) $O_{p'}(G)=1$, $G_p \trianglelefteq G$ and $D=P=N$.

Assume that $O_{p'}(G)\neq 1$. Let R be a minimal normal subgroup of G such that $R\leqslant O_{p'}(G)$. If $R\leqslant D$, then by (Ⅰ), $D/R=(G/R)^{\mathcal{N}}$ is a Hall subgroup of G/R. Since NR/R is a Sylow p-subgroup of D/R, NR/R is a Sylow p-subgroup of G/R, and thereby N is a Sylow p-subgroup of G, a contradiction. Thus $R\cap D=1$. By (Ⅰ) again, $DR/R=(G/R)^{\mathcal{N}}$ is a Hall subgroup of G/R. It follows that D is a Hall subgroup of G, which is impossible. Therefore $O_{p'}(G)=1$, and so $O_{p'}(D)=1$. By (Ⅱ), $G_p \trianglelefteq G$. Hence

$$D=P=N\leqslant F(G)=G_p.$$

(ⅲ) $\Phi(G_p)=1$ and every subgroup of G_p is normal in G.

Assume that $\Phi(G_p)\neq 1$. Let L be a minimal normal subgroup of G contained in $\Phi(G_p)$. Assume that $L\neq D$. It follows from (Ⅰ) that $DL/L=(G/L)^{\mathcal{N}}$ is a Hall subgroup of G/L. Then $DL=G_p$ and so $D=G_p$, a contradiction. Thus $L=D$. This implies that $G=D$ is nilpotent, which is impossible. Hence $\Phi(G_p)=1$. Let P_1 be any maximal subgroup of G_p. By hypothesis, G has a subgroup B such that $G=N_G(P_1)B$ and for any $q\in\pi(B)$ with $p\neq q$, there exists a Sylow q-subgroup Q of B such that $P_1Q=QP_1$. Then $P_1=P_1Q\cap G_p \trianglelefteq P_1Q$, and so $Q\leqslant N_G(P_1)$. Clearly, $B\cap G_p\leqslant N_G(P_1)$. Hence $B\leqslant N_G(P_1)$ and so $P_1 \trianglelefteq G$. This deduces that every subgroup of G_p is normal in G for $\Phi(G_p)=1$.

(ⅳ) Final contradiction of (Ⅲ).

Since $\Phi(G_p)=1$, there exists a maximal subgroup V of G_p such that $D\nleqslant V$. By (ⅲ), $V \trianglelefteq G$. Let L be a minimal normal subgroup of G such that $L\leqslant V$. By (Ⅱ), $DL/L=(G/L)^{\mathcal{N}}$ is a Hall subgroup of G/L. Then $G_p=DL$ and $|G_p|=p^2$. Let $D=\langle a\rangle$ and $\langle a_2\rangle$. Then $G_p=\langle a\rangle\times\langle a_2\rangle$. Write $a_1=aa_2$. It is easy to see that $\langle a_1\rangle\cap\langle a_2\rangle=1$. Thus $G_p=\langle a_1\rangle\times\langle a_2\rangle$. Since $\langle a_1\rangle=D\langle a_1\rangle/D\leqslant Z(G/D)$, it can derive $\langle a_1\rangle\leqslant Z(G)$. Similarly, we have that $\langle a_2\rangle\leqslant Z(G)$, and so $G_p\leqslant Z(G)$, and so G is nilpotent for $D\leqslant G_p$, a contradiction. Therefore, D is a Hall subgroup of G.

(Ⅳ) Final contradiction.

Suppose that p is any prime divisor of $|D|$ and P is a Sylow p-subgroup of D. By (Ⅲ), P is a Sylow p-subgroup of G and $P \trianglelefteq G$. Let P_1 be a maximal subgroup of P. A similar argument to step (ⅲ) of the proof of (Ⅲ), we have that $P_1 \trianglelefteq G$. It follows that every maximal subgroup of D is normal in G. If $2\mid |D|$, then D has a maximal subgroup M such that $|D/M|=2$. It implies that $G=C_G(D/M)$ and $D/M\leqslant Z(G/M)$. Therefore, G/M is nilpotent. It follows that $D\leqslant M$, a contradiction. Hence $|D|$ is odd. Consequently, by Schur-Zassenhaus Theorem, $G=D\rtimes C$ is supersoluble, where $D=G^{\mathcal{N}}$ is a nilpotent Hall subgroup of G of odd order whose maximal subgroups are normal in G. This final contradiction completes the proof of theorem. □

Theorem 2.4 Let G be a p-soluble group. Then G is p-supersoluble if and only if for any non-Frattini p-chief factor H/K of G, there exists a maximal subgroup P_1 of some Sylow p-subgroup of G being S-c-propermutable in G and $H/K\leqslant P_1K/K$.

Proof First we prove the sufficient part of the theorem. Suppose that the result is false and let G be a counterexample with $|G|$ minimal. Let N be a minimal normal subgroup of G. We claim that G/N satisfies the hypothesis of the theorem. Suppose that $(H/N)/(K/N)$ is a non-Frattini p-chief factor G/N, then $(H/N)/(K/N)\nleqslant\Phi((G/N)/(K/N))$, and so H/K is a p-chief factor of G and $H/K\nleqslant\Phi(G/K)$. By hypothesis, there exists a maximal subgroup P_1 of some Sylow p-subgroup P of G being S-c-propermutable in G and $H/K\leqslant P_1K/K$. If N is a p'-subgroup, then P_1N/N is a maximal subgroup of PN/N. Assume that N is a p-subgroup and $N\nleqslant P_1$, then $P=NP_1$. Since Sylow p-subgroup of G covers all p-chief factors of G, $H\leqslant PK=P_1K$, which contradicts $H/K\nleqslant P_1K/K$. Hence $N\leqslant P_1$, and so P_1/N is a maximal subgroup of P/N. This follows from Lemma 1.1 that P_1N/N is S-c-propermutable in

G/N. Clearly, $(H/N)/(K/N) \not\leqslant (P_1N/N)/(K/N)$. The choice of G implies that G/N is p-supersoluble. If $N \leqslant O_{p'}(G)$, then G is p-supersoluble, a contradiction. Thus N is an abelian p-subgroup. Clearly, N is a non-Frattini chief factor of G. By hypothesis, there exists a maximal subgroup G_1 of some Sylow p-subgroup G_p of G being S-c-propermutable in G and $N \not\leqslant G_1$. Then $G_p = NG_1$. Hence G has a subgroup B such that $G = N_G(G_1)B$ and for any $q \in \pi(B)$ with $p \neq q$, there exists a Sylow q-subgroup of B such that $G_1Q = QG_1$. A similar argument to Step (Ⅱ) of the proof of Theorem 2.3, we obtain that $N \cap G_1 \trianglelefteq G$. This implies that $N \cap G_1 = 1$ or $N \leqslant G_1$, which are all impossible. Consequently, the sufficiency holds.

Suppose that G is p-supersoluble and H/K is a non-Frattini p-chief factor of G. Then $|H/K| = p$ and there exists a maximal subgroup of G such that $H \not\leqslant M$ but $K \leqslant M$. Clearly, $|G:M| = p$. Let P be a Sylow p-subgroup of M, then P is a maximal subgroup of some Sylow p-subgroup G_p of G and $H/K \not\leqslant PK/K$. Obviously, $G = HM$. Since H/K is a p-chief factor of G, $H/K \leqslant G_pK/K$, and so $G/K = (G_pK/K)(M/K) = (N_G(P)K/K)(M/K)$. It follows that $G = N_G(P)M$. Let q be any prime divisor of $|M|$ with $p \neq q$. Since M is p-soluble, there exists a Sylow q-subgroup Q of M such that $PQ = QP$ by Lemma 1.8. It implies that P is S-c-propermutable in G. Hence the necessary part holds. □

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