第45卷第12期 中国科学技术大学学报 Vol. 45, No. 12
2015年12月 JOURNAL OF UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA Dec. 2015

Article ID: 0253-2778(2015)12-0972-04

New upper and lower bound for the signless Laplacian spectral radius

ZHAO Hongting[1,2], ZHANG Hailiang[2]

(1. *School of Mathematical Sciences, University of Science and Technology of China, Hefei* 230026, *China*;
2. *Department of Mathematics, Taizhou University, Linhai* 317000, *China*)

Abstract: Let D be the degree diagonal matrix of G, A be the adjacency matrix of G, $Q=D+A$ be the signless Laplacian matrix of G. Let $\xi(G)$ be the signless Laplacian spectral radius of G. Here the degree of graph was extended to k-degree, and average degree to k-average degree of a graph. A new upper and a new lower bound for the signless spectral radius of a graph G was obtained. Comparisons were made of the result with several classical results on the $\xi(G)$.

Key words: graph; Laplacian spectral radius; signless Laplacian spectral radius; k-degree; average k-degree

CLC number: O157.1 **Document code**: A doi: 10.3969/j.issn.0253-2778.2015.12.002

2010 Mathematics Subject Classification: 05C50

Citation: Zhao Hongting, Zhang Hailiang. New upper and lower bound for the signless Laplacian spectral radius [J]. Journal of University of Science and Technology of China, 2015,45(12):972-975,988.

图的无符号拉普拉斯谱半径的一个新上下界

赵宏挺[1,2],张海良[2]

(1.中国科学技术大学数学科学学院,安徽合肥 230026;2.台州学院,浙江临海 317000)

摘要:D 为图的 G 度序列对角矩阵,A 为图的邻接矩阵. $Q=D+A$ 为图的无符号拉普拉斯矩阵. Q 的最大特征值 $\xi(G)$ 称为图 G 的无符号拉普拉斯谱半径.这里将图的 2 度,平均 2 度等概念推广到 k 度与平均 k 度,得到了图的关于无符号拉普拉斯谱半径的一个新的上、下界.最后举例与图的几个已知经典的界进行了比较.

关键词:简单图;拉普拉斯谱半径;无符号拉普拉斯谱;k 度;平均 k 度

0 Introduction

Let G be a simple graph, V(G) and E(G) be the vertex set and edge set of G, respectively. A is the adjacency matrix of G. For $v_i \in V(G)$, $N(v_i)=\{v_j \mid v_iv_j \in E(G)\}$ denoted the neighbors of v_i, and d_i be the degree of a vertex v_i. Let $D=\mathrm{Diag}(d_1, d_2, \cdots, d_n)$ be the diagonal degree matrix of G. $L=D-A$ and $Q=D+A$ be the Laplacian matrix and signless Laplacian matrix, respectively.

Received: 2014-11-03; **Revised**: 2015-06-01

Foundation item: Supported by the National Science Foundation of Zhejiang (Y6110054).

Biography: ZHAO Hongting (corresponding author), male born in 1979, master/lecturer. Research field: combinatorial analysis. E-mail: zhtbear@mail.ustc.edu.cn

Let $\rho(G)$, $\mu(G)$ and $\xi(G)$ be the largest eigenvalues of matrices A, L and Q. In graph theory, we call these eigenvalues the adjacency spectral radius, the Laplacian spectral radius and the signless spectral radius. The 2-degree is defined to be the sum of degrees of the vertices adjacent to v_i denoted by $d_i^{(2)} = \sum_{v_j \in N(v_i)} d_j$, and the average 2-degree of v_i is defined to $\overline{d}_i^{(2)} = d_i^{(2)}/d_i$. Recursively, we can define the k-degree of v_i by $d_i^{(k)} = \sum_{v_j \in N(v_i)} d_j^{(k-1)}$ and the average k-degree of v_i denoted by $\overline{d}_i^{(k)} = d_i^{(k)}/d_i^{(k-1)}$.

Up to now, many upper and lower bounds have been given for three spectral radius, see Refs. [4-15]. In this paper, we give an upper bound and a lower bound for the signless Laplacian spectral radius of simple graphs with respect to k-degree and average k-degree of G.

By using the similar transformation to the matrices, we obtained

$$\min\left\{\frac{d_i + d_j + \sqrt{(d_i - d_j)^2 + 4\overline{d}_i^{(k)}\,\overline{d}_j^{(k)}}}{2}\right\} \leqslant \xi(G) \leqslant \max\left\{\frac{d_i + d_j + \sqrt{(d_i - d_j)^2 + 4\overline{d}_i^{(k)}\,\overline{d}_j^{(k)}}}{2},\ v_iv_j \in E(G)\right\}$$

for the signless Laplacian matrix of G. Moreover, equality holds for a particular value of k if and only if G is a bipartite graph and $\xi(G) = d_i + \overline{d}_i^{(k)}$ for all $v_i \in V(G)$. At the end of this paper we give an example and compare it with some of the bounds which were given by other results.

1 Basic lemmas

In this section we give some basic results of nonnegative irreducible matrix and some classical bounds of Laplacian and signless Laplacian matrices of graph G.

Lemma 1.1[2] The maximal eigenvalue of an irreducible nonnegative matrix is a simple positive real number and the associate eigenvector is strictly positive.

Lemma 1.1 is the best known and most important part of Perron-Frobenius theory in nonnegative matrix theory.

Lemma 1.2[2] If a nonnegative matrix has row sum $r_1, r_2, \cdots, r_n$, let $r = \min r_i$, $R = \max r_i$, $i = 1, 2, \cdots, n$, ρ be the maximal eigenvalue, then

$$r \leqslant \rho \leqslant R.$$

If A is irreducible, then each equality holds if and only if $r_1 = r_2 = \cdots = r_n$.

Lemma 1.3[14] For a connected simple graph G,

$$\mu(G) \leqslant \xi(G),$$

equality holds if and only if G is a bipartite graph.

The following Lemmas 1.4 and 1.5 are lower bounds and upper bounds for the signless Laplacian matrix of graph.

Lemma 1.4[3] Let G be a graph on n vertices, with vertex degrees $d_1, d_2, \cdots, d_n$. Then

① $\min d_i \leqslant \xi(G) \leqslant \max d_i$, equality hold on the either side if and only if G is a regular graph;

② $\min(d_i + d_j) \leqslant \xi(G) \leqslant \max(d_i + d_j)$, $v_iv_j \in E(G)$. For a connected graph, equality holds in either side if and only if G is regular or semi-regular.

Let $D = \mathrm{Diag}(d_1, d_2, \cdots, d_n)$, apply similar transformation $D^{-1}QD$ to the matrix Q and by Lammas 1.1 and 1.2, we can easily have Lemma 1.5.

Lemma 1.5 Let G be a simple connected graph. Then

$$\xi(G) \leqslant \max\left\{\frac{d_u + \sqrt{d_u^2 + 8d_u\overline{d}_u^{(2)}}}{2},\ u \in V(G)\right\}.$$

Moreover, the equality holds if and only if G is a regular graph.

In Ref. [16], Das gives a bound for the $\xi(G)$ when he studies the Laplacian spectral radius of connected graphs. We cite it here as Lemma 1.6.

Lemma 1.6[16] For a simple connected graph G,

$$\xi(G) \leqslant \max\left\{\frac{d_u + d_v + \sqrt{(d_u - d_v)^2 + 4\overline{d}_u^{(2)}\,\overline{d}_v^{(2)}}}{2}:\right.$$

$uv \in E(G)\Big\}$,

with the inequality holding if and only if G is a bipartite regular graph or a bipartite semi-regular graph.

2 Main result

In this section, we give another lower bound and upper bound for the signless Laplacian spectral radius of G. The inner product of two vectors:

$$\langle Qx, x\rangle = \sum_{v_iv_j \in E(G)} (x_i + x_j)^2.$$

Theorem 2.1 Let G be a connected simple graph. For any positive integer k, we have

$$\min\left\{\frac{d_i + d_j + \sqrt{(d_i - d_j)^2 + 4\bar{d}_i^{(k)}\bar{d}_j^{(k)}}}{2}\right\} \leqslant \xi(G) \leqslant \max\left\{\frac{d_i + d_j + \sqrt{(d_i - d_j)^2 + 4\bar{d}_i^{(k)}\bar{d}_j^{(k)}}}{2}, v_iv_j \in E(G)\right\}$$

Moreover, equality holds for a particular value of k if and only if G is a bipartite graph and $\xi(G) = d_i + \bar{d}_i^{(k)}$ for all $v_i \in V(G)$.

Proof Let Q be the signless Laplacian matrix of G, $\xi(G)$ be the spectral radius of Q, $D = \mathrm{Diag}(d_1^{(k-1)}, d_2^{(k-1)}, \cdots, d_n^{(k-1)})$. Let $Q' = D^{-1}QD$. By Lemma 1.1 there exists a nonnegative eigenvector $X = (x_1, x_2, \cdots, x_n)^t$ associated with $\xi(G)$. Since $D^{-1}QD$ and Q have the same spectral radius, we have

$$Q'X = D^{-1}QDX = \xi(G)X \tag{1}$$

Assume that the $x_1 = 1$ is the largest coordinate and $x_p = \max\{x_k : v_k \in N(v_1)\}$. Then

$$\xi(G)x_1 = d_1x_1 + \sum_{v_j \in N(v_1)} \frac{d_j^{(k-1)}}{d_1^{(k-1)}} x_j \leqslant d_1x_1 + \bar{d}_1^{(k)}x_p \tag{2}$$

Similarly, we have

$$\xi(G)x_p = d_px_p + \sum_{v_i \in N(v_p)} \frac{d_i^{(k-1)}}{d_p^{(k-1)}} x_i \leqslant d_px_p + \bar{d}_p^{(k)}x_1 \tag{3}$$

From (2) and (3), we have

$$(\xi(G) - d_1)(\xi(G) - d_p) \leqslant \bar{d}_1^{(k)}\bar{d}_p^{(k)} \tag{4}$$

By solving inequality (5) we have

$$\xi(G) \leqslant \frac{d_1 + d_p + \sqrt{(d_1 - d_p)^2 + 4\bar{d}_1^{(k)}\bar{d}_p^{(k)}}}{2} \tag{5}$$

Similarly, we can give a lower bound for the signless Laplacian spectral radius. Without loss of generality, we assume that

$$x_1 = 1 = \min\{x_i : v_i \in V(G)\},$$
$$x_q = \min\{x_j : v_j \in N(v_1)\}.$$

By the first row of equation (2), we get

$$\xi(G)x_1 = d_1x_1 + \sum_{v_j \in N(v_1)} \frac{d_j^{(k-1)}}{d_1^{(k-1)}} x_j \geqslant d_1x_1 + \bar{d}_1^{(k)}x_q \tag{6}$$

From the qth equation of (2), we get

$$\xi(G)x_q = d_qx_q + \sum_{v_i \in N(v_q)} \frac{d_q^{(k-1)}}{d_q^{(k-1)}} x_i \geqslant d_qx_q + \bar{d}_q^{(k)}x_1 \tag{7}$$

From (6) and (7), we have

$$(\xi(G) - d_1)(\xi(G) - d_q) \geqslant \bar{d}_1^{(k)}\bar{d}_q^{(k)} \tag{8}$$

By solving this inequality, we get

$$\xi(G) \geqslant \frac{d_1 + d_q + \sqrt{(d_1 - d_q)^2 + 4\bar{d}_1^{(k)}\bar{d}_q^{(k)}}}{2} \tag{9}$$

Now, suppose that the other side of the inequality holds, without losing generality, let

$$\xi(G) = \max\left\{\frac{d_u + d_v + \sqrt{(d_u - d_v)^2 + 4\bar{d}_u^{(k)}\bar{d}_v^{(k)}}}{2} : uv \in E(G)\right\} \tag{10}$$

where $x_u = 1 = \max\{x_i : v_i \in V(G)\}$, $x_p = \max\{x_p : v_p \in N(u)\}$. Then all inequalities in the above argument must be equalities. In particular, inequality (2) must be an equality. Then $x_j = x_p$ for all $x_j \in N(u)$. Form inequality (3), we have $x_i = x_u = 1$ for all $v_i \in N(v_p)$.

Let $V_1 = \{v_l : x_l = 1\}$, $V_2 = \{v_l : x_l = x_p\}$ and $x_u \neq x_p$. So $N(u) \in V_2$ and $N(v) \in V_1$. Further for any $v_i \in N(N(u))$ there exists a vertex $v_j \in N(u)$ such that $v_iv_j \in E(G)$, $uv_j \in E(G)$. Therefore $x_i = 1$, hence $v_i \in V_1$, that is $N(N(u)) \subseteq V_1$. By a similar argument, we can show that $N(N(v)) \subseteq V_2$. Continuing this procedure, it is easy to see, since G is connected, that $G = V_1 \cup V_2$, and all

Tab. 1　The k-average degree of graph

$d_i=\overline{d}_i^{(1)}=d_i^{(1)}$	$N(v_i)$	$d_i^{(2)}$	$\overline{d}_i^{(2)}$	$d_i^{(3)}$	$\overline{d}_i^{(3)}$	$d_i^{(4)}$	$\overline{d}_i^{(4)}$	…
$d(v_1)=3$	v_2, v_6, v_7	8	2.667	24	3	68	2.833	…
$d(v_2)=3$	v_1, v_3, v_6	9	3	26	2.889	76	2.923	…
$d(v_3)=3$	v_2, v_4, v_5	9	3	26	2.889	76	2.923	…
$d(v_4)=3$	v_3, v_5, v_7	8	2.667	24	3	68	2.883	…
$d(v_5)=3$	v_3, v_4, v_6	9	3	26	2.889	76	2.923	…
$d(v_6)=3$	v_1, v_2, v_5	9	3	26	2.889	76	2.923	…
$d(v_7)=2$	v_1, v_4	6	3	16	2.667	48	3	…

subgraphs induced by V_1 and V_2 are empty graphs. Hence G is a bipartite graph.

Now, we prove that all the coordinates of the Perron vector are equal. Suppose that $x_p<1$, $v_p\in V_2$. Let $v_i\in V_1$ with the minimal degree and $v_j\in N(v_i)$. If $d(v_i)+\overline{d}(v_i)^{(k)}\leqslant d(v_j)+\overline{d}(v_j)^{(k)}$ possible for $v_iv_j\in E(G)$, let $x_j<1$. Then $\xi(G)=d(v_i)+\overline{d}_i^{(k)}x_p$ for $v_i\in V_1$, and $\xi(G)=d(v_j)+\overline{d}(v_j)^{(k)}\frac{1}{x_p}$ for $v_j\in V_2$. Since $\xi(G)$ is a unique largest eigenvalue of $Q(G)$, then

$$d_i+\overline{d}_i^{(k)}=d_j+\overline{d}_j^{(k)}\frac{1}{x_p}.$$

For $x_p<1$, we get

$$d_i+\overline{d}_i^{(k)}<d_j+\overline{d}_j^{(k)}.$$

That is a contradiction. Therefore $x_j=1$ for $v_j\in V_2$. By the above argument, we have all that the coordinates of the Perron vector of Q' are equal. So DX is an eigenvector associated with eigenvalues $\xi(G)D$ of Q. That is

$$Q(d_1^{(k-1)}x_1,d_2^{(k-1)}x_2,\cdots,d_n^{(k-1)}x_n)^t=\xi(G)(d_1^{(k-1)}x_1,d_2^{(k-1)}x_2,\cdots,d_n^{(k-1)}x_n)^t,$$

$$Q(d_1^{(k-1)},d_2^{(k-1)},\cdots,d_n^{(k-1)})^t=\xi(G)(d_1^{(k-1)},d_2^{(k-1)},\cdots,d_n^{(k-1)})^t,$$

$$d_id_i^{(k-1)}+\sum_{j\sim i}a_{ji}d_j^{(k-1)}=d_i+\overline{d}_i^{(k)}=\xi(G).$$

Hence

$$\xi(G)=d_i+\overline{d}_i^{(k)},$$

Conversely, it is easy to verify that all coordinates of the Perron vector are equal. Then $\xi(G)=d_i+\overline{d}_i^{(k)}$. □

In the end of this paper, we give an example to our bounds. Let G be the graph in Fig. 1. Tab. 1 gives the k-average degree of vertices of graph in Fig. 1.

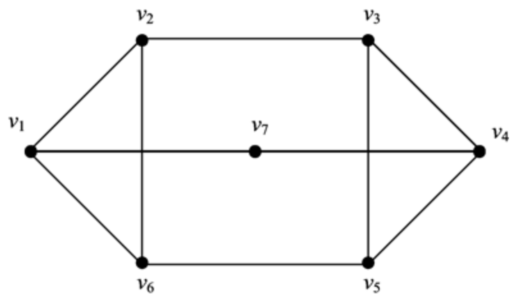

Fig. 1　Graph to the example

From our example in Fig. 1, $\xi(G)=5.855\ 8$. Tab. 2 gives four lower and upper bounds according to k. From Tab. 2, we can see when k increase the signless spectral radius $\xi(G)$ is close to the real value.

Tab. 2　The change of spectral radius with respect to k

$\xi(G)$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
max	6	6	5.944	5.923	5.917 9	5.909
min	5	5.372 5	5.372 5	5.457 9	5.429 5	5.453

In Tab. 3, we give the upper and the lower bounds according to Theorem 2.1 and make a comparison with the other bounds.

Tab. 3　The compare of different spectral radius

$\xi(G)$	Lemma 1.5	Lemma 1.6	Lemma 1.7	Theorem 2.1
max	6	6	6	5.909
min	5	—	—	5.453

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