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Rapid time decay of weak solutions to the generalized Hall-magneto-hydrodynamics equations

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Abstract: The rapid time decay for solutions to the generalized Hall-magneto-hydrodynamics equations was studied. By developing the classic Fourier splitting methods, the more rapid L^2 decay rate of the weak solutions as $(1+t)^{-\frac{7}{4}}$ was derived. The trick is mainly based on the even lower frequency effect of the nonlinear term.

Key words: generalized Hall-MHD equations; time decay; frequency estimation; incompressible

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广义霍尔磁流体力学方程弱解的快速衰减

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摘要：主要研究了广义霍尔磁流体力学方程解的快速衰减问题. 利用经典的 Fourier 变换方法，得到了弱解的 L^2 衰减 $(1+t)^{-\frac{7}{4}}$. 这一结论主要是基于非线性项的低频效率.

关键词：广义 Hall-MHD 方程；时间衰减；频数估计；不可压的

0 Introduction

The incompressible resistive viscous Hall-magneto-hydrodynamics system (so called Hall-MHD) is an important mathematical model in the fluid dynamics[1], governed by the following nonlinear partial differential equations:

$$\left.\begin{aligned}&\partial_t u + u\cdot\nabla u + \nabla\pi = \mu\Delta u + (\nabla\times B)\times B,\\ &\partial_t B - \nabla\times(u\times B) + \nabla\times((\nabla\times B)\times B) = \\ &\quad \nu\Delta B,\\ &\nabla\cdot u = \nabla\cdot B = 0\end{aligned}\right\} \tag{1}$$

Here $u(x,t)=(u_1(x,t),u_2(x,t),u_3(x,t))$, $B(x,t)=(B_1(x,t),B_2(x,t),B_3(x,t))$ are the

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unknown velocity field and unknown magnetic field.

Due to their importance in both mathematics and physics, great importance has been attached to the well-posedness and large time behavior for solutions to the above Hall-magneto-hydrodynamics fluid model[1-3]. In particular, Chae and Schonbek[4] recently investigated the L^2 decay rates of solution for the Hall-magneto-hydrodynamics fluid as follows:

$$\| u \|_{L^2} + \| B \|_{L^2} \leqslant C(1+t)^{-\frac{3}{4}},\ t > 0 \quad (2)$$

When the velocity $u=0$, it is easy to find that the Hall-magneto-hydrodynamics system (so-called generalized Hall-MHD equations) reduces the following system:

$$\left.\begin{aligned} &\partial_t B - \nu\Delta B + \nabla \times ((\nabla \times B) \times B) = 0, \\ &\nabla \cdot B = 0 \end{aligned}\right\} \quad (3)$$

and the associated initial data

$$B(x,0) = B_0 \quad (4)$$

As stated by Chae and Schonbek[3], the strongly nonlinear term of the generalized Hall-MHD equations (3)～(4)

$$\nabla \times ((\nabla \times B) \times B)$$

has the complicated structure and seems difficult to deal with in a satisfied way. However, we find that when we take the Fourier transformation of that nonlinear term, it follows that

$$| F[\nabla \times ((\nabla \times B) \times B)] | = \left| \int_{\mathbb{R}^3} \nabla \times ((\nabla \times B) \times B) \mathrm{e}^{-ix\xi} \mathrm{d}x \right| \leqslant C |\xi|^2 \int_{\mathbb{R}^3} |B|^2 \mathrm{d}x.$$

Compared with the other nonlinear term of the Hall-MHD equations (1) where the Fourier transformation implies

$$| F[(\nabla \times B) \times B] | = \left| \int_{\mathbb{R}^3} (\nabla \times B) \times B \mathrm{e}^{-ix\xi} \mathrm{d}x \right| \leqslant C |\xi| \int_{\mathbb{R}^3} |B|^2 \mathrm{d}x$$

or

$$| F[u \cdot \nabla u] | = \left| \int_{\mathbb{R}^3} u \cdot \nabla u \mathrm{e}^{-ix\xi} \mathrm{d}x \right| \leqslant C |\xi| \int_{\mathbb{R}^3} |u|^2 \mathrm{d}x.$$

Obviously, the strongly nonlinear term $\nabla \times ((\nabla \times B) \times B)$ of the generalized Hall-MHD equations (3)～(4) actually exhibits a much more lower frequency effect for the finite energy solution than the nonlinear terms in (1). This new observation allows us to investigate the more rapid time decay rate of the generalized Hall-MHD equations (3)～(4) than the decay rate $(1+t)^{-\frac{3}{4}}$ of the Hall-MHD equations (1) derived in Ref. [4].

On the other hand, when the strong nonlinear term $\nabla \times ((\nabla \times B) \times B)$ or magnetic field B of the Hall-MHD equations (1) vanishes, the system reduces the classic magneto-hydrodynamics equations:

$$\left.\begin{aligned} &\partial_t u + u \cdot \nabla u + \nabla \pi = \mu\Delta u + (\nabla \times B) \times B, \\ &\partial_t B - \nabla \times (u \times B) = \nu\Delta B, \\ &\nabla \cdot u = \nabla \cdot B = 0 \end{aligned}\right\} \quad (5)$$

and the classic Navier-Stokes equations:

$$\left.\begin{aligned} &\partial_t u + u \cdot \nabla u + \nabla \pi = \mu\Delta u, \\ &\nabla \cdot u = 0 \end{aligned}\right\} \quad (6)$$

The time decay issues of the weak solutions have been extensively investigated by many authors both for the classic magneto-hydrodynamics equations[9,15-16] and for the classic Navier-Stokes equations[7,10,11,18].

The study is focused on the rapid decay rate of the solution to the generalized Hall-MHD equations (3)～(4). More precisely, we will study the more rapid decay rate of solutions by developing the generalized Fourier splitting methods which was first used by Schonbek[14] (See also Refs. [5, 17]) who studied the decay rate of incompressible Navier-Stokes equations. One may also refer to some interesting asymptotic behaviors results of the relevant nonlinear mathematical models[6,8,13,19].

The remainder of this paper is organized as follows. In Section 1, we will give some preliminaries and state the main results together with some remarks. In Section 2, we prove some important lemmas which play an important role in the proof of the main results to be stated in

Section 3.

1 Preliminaries and main results

Here and in what follows, C stands for the abstractly positive constant. $L^p(\mathbb{R}^3)$ with $1\leqslant p\leqslant\infty$ denotes the usual Lebesgue space. $H^s(\mathbb{R}^3)$ with $s\in\mathbb{R}$ is the fractional Sobolev space with the norm

$$\| f \|_{H^s} = (\int_{\mathbb{R}^3} |\xi|^{2s} |\hat{f}|^2 d\xi)^{1/2}.$$

$F[f]$ or $\hat{f}$ is the Fourier transformation of f which is defined by

$$F[f(\xi)] = \hat{f}(\xi) = \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x) dx.$$

In order to state our main results, we need to give a definition of the weak solutions generalized Hall-MHD equations (3)～(4) (see also Ref. [3]).

Definition 1.1 $B(x,t)$ is called a weak solution of the generalized Hall-MHD equations (3)～(4) if the following conditions are valid:

① $u\in L^\infty(0, T; L^2(\mathbb{R}^3))\cap L^2(0, T; H^1(\mathbb{R}^3))$, $\forall T>0$;

② for any smooth test function $\phi\in C_0^\infty(\mathbb{R}^3\times[0,T))$

$$\int_{\mathbb{R}^3} B(t)\phi(t)dx + \nu\int_0^t\int_{\mathbb{R}^3} \nabla B\cdot\nabla\phi dx d\tau + \int_0^t\int_{\mathbb{R}^3} \nabla\times((\nabla\times B)\times B)\phi dx d\tau = \int_0^t\int_{\mathbb{R}^3} B\partial_t\phi dx d\tau + \int_{\mathbb{R}^3} B_0\phi(0)dx,\ 0<t<T;$$

③ $B(x,t)$ satisfies energy inequality

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |B|^2 dx + \nu\int_{\mathbb{R}^3} |\nabla B|^2 dx \leqslant 0 \quad (7)$$

The following is our main result.

Theorem 1.1 Suppose $B_0\in L^2(\mathbb{R}^3)$ and satisfies the following frequency growth condition:

$$\rho(r)\equiv\int_0^{2\pi} |\hat{B}_0(r\omega)|^2 d\omega = C r^{2\gamma-3} + o(r^{2\gamma-3}),\ \text{for } \gamma>2 \text{ as } r\to 0 \quad (8)$$

Then for the weak solution $B(x,t)$ of the generalized Hall-MHD equations (3)～(4), we have the more rapid time decay rate as

$$\| B(t) \|_{L^2} \leqslant C(1+t)^{-\frac{7}{4}},\ \forall t>0 \quad (9)$$

Remark 1.1 Compared with the time decay rate (2) of the ususal Hall-MHD equations (1), our time decay rate here is obviously more rapid. Our result is mainly based on the new observation that the strongly nonlinear term $\nabla\times((\nabla\times B)\times B)$ of the generalized Hall-MHD equations (3)～(4) actually exhibits a much more lower frequency effect for the finite energy solution. This sort of lower frequency effect allows us to derive the more rapid algebraic time decay rate by developing the classic Fourier splitting methods.

Remark 1.2 It should be mentioned that although the strongly nonlinear term $\nabla\times((\nabla\times B)\times B)$ of the generalized Hall-MHD equations (3)～(4) may allow us to derive the more rapid decay upper bounds as

$$\| B(t) \|_{L^2} \leqslant C(1+t)^{-\frac{7}{4}},\ \forall t>0.$$

we do not know whether or not the generalized Hall-MHD equations (3)～(4) exhibits the same time decay lower bounds. That is to say, the weak solution to the generalized Hall-MHD equations (3)～(4) decays as

$$\| B(t) \|_{L^2} \geqslant C(1+t)^{-\frac{7}{4}},\ \forall t>0 \quad (10)$$

It seems that the more new observation on the strongly nonlinear term $\nabla\times((\nabla\times B)\times B)$ is required and we will focus on the challenge issue in the future.

2 Some lemmas

In order to investigate the time decay issue of the generalized Hall-MHD equations (3)～(4), we fist recall some time decay results of heat equation.

Lemma 2.1[12, Proposition 3] Suppose $b(x,t)$ is a solution to the linear heat equation

$$\left.\begin{aligned} \partial_t b - \nu\Delta b = 0,\\ b(x,0) = b_0 \end{aligned}\right\} \quad (11)$$

with the initial data $b_0\in L^2(\mathbb{R}^3)$ and satisfies Eq. (8), namely

$$\rho(r)\equiv\int_0^{2\pi} |\hat{b}_0(r\omega)|^2 d\omega = Cr^{2\gamma-3} + o(r^{2\gamma-3}),\ \text{for } \gamma>2 \text{ as } r\to 0,$$

then

$$\| b(t) \|_{L^2} \leqslant C(1+t)^{-\gamma},\ \forall t>0 \quad (12)$$

The next two lemmas are to investigate the frequency estimate of the weak solution of the generalized Hall-MHD equations (3)～(4) which play a central role in the proof of the next section.

Lemma 2.2 $B(x,t)$ is the weak solution to the generalized Hall-MHD equations (3)～(4) with the same condition in Theorem 1.1, then we have

$$|\hat{B}|\leqslant e^{-\nu|\xi|^2 t}|\hat{B}_0|+C|\xi|^2\int_0^t \|B\|_{L^2}^2 ds \tag{13}$$

Proof Firstly, applying Fourier transformation to (3), one shows that

$$\hat{B}_t+\nu|\xi|^2\hat{B}=-F[\nabla\times((\nabla\times B)\times B)] \tag{14}$$

According to the definition of weak solution and applying the properties of Fourier transformation, we have for the right hand side of (14)

$$|F[\nabla\times((\nabla\times B)\times B)]|=$$
$$\left|\int_{\mathbb{R}^3}\nabla\times((\nabla\times B)\times B)\,e^{-ix\xi}dx\right|\leqslant$$
$$C|\xi|^2\int_{\mathbb{R}^3}|B|^2dx.$$

Hence by solving the ordinary differential equation (14), it follows that

$$\hat{B}=e^{-\nu|\xi|^2 t}\hat{B}_0-$$
$$\int_0^t e^{-(t-s)\nu|\xi|^2}F[\nabla\times((\nabla\times B)\times B)]ds\leqslant$$
$$|e^{-\nu|\xi|^2 t}\hat{B}_0|+\left|\int_0^t e^{-(t-s)\nu|\xi|^2}|\xi|^2\|B\|_{L^2}^2 ds\right|\leqslant$$
$$|e^{-\nu|\xi|^2 t}\hat{B}_0|+C|\xi|^2\int_0^t\|B\|_{L^2}^2 ds \tag{15}$$

□

Lemma 2.3 $B(x,t)$ is the weak solution to the generalized Hall-MHD equations (3)～(4), then we have

$$\sup_{0\leqslant t\leqslant\infty}\|B(t)\|_{L^2}\leqslant\|B_0\|_{L^2} \tag{16}$$

The proof of Lemma 2.3 can be derived directly through energy inequality (7) and here we omit it.

3 Rapid time decay

In this section, we are now in a position to derive the rapid time decay of the weak solution to the generalized Hall-MHD equations (3)～(4).

Firstly, from energy inequality (7), namely

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|B|^2dx+\nu\int_{\mathbb{R}^3}|\nabla B|^2\leqslant 0,$$

we apply Plancherel's theorem to give

$$\frac{d}{dt}\|\hat{B}\|_{L^2}^2+2\nu\|\widehat{\nabla B}\|_{L^2}^2\leqslant 0 \tag{17}$$

and Fourier transformation property implies

$$\frac{d}{dt}\int_{\mathbb{R}^3}|\hat{B}(\xi,t)|^2d\xi+2\nu\int_{\mathbb{R}^3}|\xi|^2|\hat{B}(\xi,t)|^2d\xi\leqslant 0 \tag{18}$$

Let

$$\theta(t)=\left\{\xi\in\mathbb{R}^3,|\xi|^2\leqslant\frac{5}{2\nu(1+t)}\right\},$$

then

$$2\nu\int_{\mathbb{R}^3}|\xi|^2|\hat{B}(\xi,t)|^2d\xi=$$
$$2\nu\int_{\theta(t)}|\xi|^2|\hat{B}(\xi,t)|^2d\xi+$$
$$2\nu\int_{\theta^c(t)}|\xi|^2|\hat{B}(\xi,t)|^2d\xi\geqslant$$
$$2\nu\int_{\theta^c(t)}|\xi|^2|\hat{B}|^2d\xi\geqslant$$
$$\frac{5}{1+t}\int_{\theta^c(t)}|\hat{B}|^2d\xi=$$
$$\frac{5}{1+t}\int_{\mathbb{R}^3}|\hat{B}|^2d\xi-\frac{5}{1+t}\int_{\theta(t)}|\hat{B}|^2d\xi \tag{19}$$

therefore

$$\frac{d}{dt}\int_{\mathbb{R}^3}|\hat{B}(\xi,t)|d\xi+\frac{5}{1+t}\int_{\mathbb{R}^3}|\hat{B}(\xi,t)|^2d\xi\leqslant$$
$$\frac{5}{1+t}\int_{\theta(t)}|\hat{B}(\xi,t)|^2d\xi \tag{20}$$

Now multiplying by $(1+t)^5$ to both side of (20) yields

$$(1+t)^5\frac{d}{dt}\|\hat{B}\|_{L^2}^2+5(1+t)^4\int_{\mathbb{R}^3}|\hat{B}(\xi,t)|^2d\xi\leqslant$$
$$5(1+t)^4\int_{\theta(t)}|\hat{B}(\xi,t)|^2d\xi$$

or

$$\frac{d}{dt}((1+t)^5\|\hat{B}\|_{L^2}^2)\leqslant$$
$$C(1+t)^4\int_{\theta(t)}|\hat{B}(\xi,t)|^2d\xi \tag{21}$$

Applying Lemma 2.2 and Lemma 2.3, the right hand side of (21) is bounded by

$$\int_{\theta(t)}|\hat{B}(\xi,t)|^2d\xi\leqslant$$
$$\int_{\theta(t)}|e^{-\nu|\xi|^2 t}\hat{B}_0+$$

$$C|\xi|^2\int_0^t \|B\|_{L^2}^2 \mathrm{d}s|^2\mathrm{d}\xi \leqslant$$

$$\int_{\theta(t)} |\mathrm{e}^{-\nu|\xi|^2 t}\hat{B}_0|^2\mathrm{d}\xi +$$

$$C\int_{\theta(t)} ||\xi|^2\int_0^t \|B\|_{L^2}^2 \mathrm{d}s|^2\mathrm{d}\xi \leqslant$$

$$\int_{\mathbb{R}^3} |\mathrm{e}^{-\nu|\xi|^2 t}\hat{B}_0|^2\mathrm{d}\xi +$$

$$C\int_{\theta(t)} ||\xi|^2\int_0^t \|B\|_{L^2}^2 \mathrm{d}s|^2\mathrm{d}\xi =$$

$$I+J \tag{22}$$

Applying Lemma 2.1 and Plancherel's theorem, for I,

$$I=\int_{\mathbb{R}^3} |\mathrm{e}^{-\nu|\xi|^2 t}\hat{B}_0|^2\mathrm{d}\xi = \int_{\mathbb{R}^3} |\mathrm{e}^{-\nu\Delta t}B_0|^2\mathrm{d}x \leqslant C(1+t)^{-2\gamma} \tag{23}$$

For J,

$$J=\int_{\theta(t)} ||\xi|^2\int_0^t \|B\|_{L^2}^2 \mathrm{d}s|^2\mathrm{d}\xi \leqslant \int_{\theta(t)} ||\xi|^2\int_0^t \|B_0\|_{L^2}^2 \mathrm{d}s|^2\mathrm{d}\xi \leqslant C\int_{\theta(t)} |\xi|^4 t^2\mathrm{d}\xi \leqslant C\int_{|w|=1}\mathrm{d}w\int_0^{\left(\frac{5}{2\nu(1+t)}\right)^{1/2}} \rho^6 t^2\mathrm{d}\rho \leqslant C(1+t)^{-\frac{3}{2}} \tag{24}$$

Plugging the estimates I and J into (22) gives, note that $\gamma>2$

$$\int_{\theta(t)} |\hat{B}(\xi,t)|^2\mathrm{d}\xi \leqslant C(1+t)^{-\frac{3}{2}} \tag{25}$$

Then inserting the above auxiliary decay rate (25) into (21) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}((1+t)^5\|\hat{B}\|_{L^2}^2) \leqslant C(1+t)^4(1+t)^{-\frac{3}{2}} \leqslant C(1+t)^{\frac{5}{2}} \tag{26}$$

Integrating in time

$$\|\hat{B}\|_{L^2} \leqslant C(1+t)^{-\frac{3}{4}} \tag{26}$$

Now repeating the argument in the derivation of (22), it follows that

$$\int_{\theta(t)} |\hat{B}(\xi,t)|^2\mathrm{d}\xi \leqslant$$

$$\int_{\theta(t)} |\mathrm{e}^{-\nu|\xi|^2 t}\hat{B}_0 + C|\xi|^2\int_0^t \|B\|_{L^2}^2 \mathrm{d}s|^2\mathrm{d}\xi \leqslant$$

$$C(1+t)^{-2\gamma} +$$

$$C\int_{\theta(t)} ||\xi|^2\int_0^t (1+s)^{-\frac{3}{2}}\mathrm{d}s|^2\mathrm{d}\xi \leqslant$$

$$C(1+t)^{-2\gamma} + C\int_{\theta(t)} |\xi|^4\mathrm{d}\xi \leqslant$$

$$C(1+t)^{-2\gamma} + C\int_{|w|=1}\mathrm{d}w\int_0^{\left(\frac{5}{2\nu(1+t)}\right)^{1/2}} \rho^6\mathrm{d}\rho \leqslant$$

$$C(1+t)^{-\frac{7}{2}} \tag{27}$$

Inserting the decay estimate (27) into the right hand side of (21)

$$\frac{\mathrm{d}}{\mathrm{d}t}((1+t)^5\|\hat{B}\|_{L^2}^2) \leqslant C(1+t)^{\frac{1}{2}},$$

and then integrating in time, we obtain the rapid decay rate

$$\|\hat{B}\|_{L^2} \leqslant C(1+t)^{-\frac{7}{4}}$$

or

$$\|B(t)\|_{L^2} \leqslant C(1+t)^{-\frac{7}{4}} \tag{28}$$

The proof of Theorem 1.1 is completed. □

References

[1] Sermange M, Teman R. Some mathematical questions related to the MHD equations[J]. Comm Pure Appl Math, 1983, 36: 635-664.

[2] Acheritogaray M, Degond P, Frouvelle A, et al. Kinetic formulation and global existence for the Hall-magneto-hydrodynamics system [J]. Kinet Relat Models, 2011, 4: 901-918.

[3] Chae D, Degond P, Liu J. Well-posedness for Hall-magnetohydrodynamics[J]. Annales de l'Institut Henri Poincare (C) Non Linear Analysis, 2014, 31: 555-565.

[4] Chae D, Schonbek M. On the temporal decay for the Hall-magnetohydrodynamic equations[J]. J Differential Equations, 2013, 255: 3 971-3 982.

[5] Dong B, Li Y. Large time behavior to the system of incompressible non-Newtonian fluids in R^2[J]. J Math Anal Appl, 2004, 298: 667-676.

[6] Dong B, Chen Z. Asymptotic profiles of solutions to the 2D viscous incompressible micropolar fluid equations[J]. Discrete Contin Dyn Syst, 2009, 23: 765-784.

[7] Dong B, Song J. Global regularity and asymptotic behavior of the modified Navier-Stokes equations with fractional dissipation [J]. Discrete and Continuous Dynamical Systems, 2012, 32: 57-79.

[8] Guo Y, Wang Y. Decay of dissipative equations and negative Sobolev spaces[J]. Comm Partial Differential Equations, 2012, 37: 2 165-2 208.

[9] Han P, He C. Decay properties of solutions to the incompressible magneto-hydrodynamics equations in a

half space[J]. Math Methods Appl Sci, 2012, 35: 1 472-1 488.

[10] He C, Xin Z. On the decay properties of solutions to the non-stationary Navier-Stokes equations in R^3[J]. Proc Roy Soc Edinburgh Sect A, 2001, 131: 597-619.

[11] Kajikiya R, Miyakawa T. On L^2 decay of weak solutions of Navier-Stokes equations in R^n[J]. Math Zeit, 1986, 192: 135-148.

[12] Oliver M, Titi E S. Remark on the rate of decay of higher order derivatives for solutions to the Navier-Stokes equations in R^n[J]. J Funct Anal, 2000, 172: 1-18.

[13] Qin X, Wang Y. Large-time behavior of solutions to the inflow problem of full compressible Navier-Stokes equations[J]. SIAM J Math Anal, 2011, 43: 341-366.

[14] Schonbek M E. L^2 decay for weak solutions of the Navier-Stokes equations[J]. Arch Rational Mech Anal, 1985, 88: 209-222.

[15] Schonbek M E, Schonbek T P, Süli E. Large-time behaviour of solutions to the magneto-hydrodynamics equations[J]. Math Ann, 1996, 304: 717-756.

[16] Agapito R, Schonbek M E. Non-uniform decay of MHD equations with and without magnetic diffusion [J]. Comm Partial Differential Equations, 2007, 32: 1 791-1 812.

[17] Wiegner M. Decay results for weak solutions of the Navier-Stokes equations in R^n[J]. J London Math Soc, 1987, 35: 303-313.

[18] Zhang L. New results of general n-dimensional incompressible Navier-Stokes equations [J]. J Differential Equations, 2008, 245: 3 470-3 502.

[19] Zhao C, Liang Y, Zhao M. Upper and lower bounds of time decay rate of solutions to a class of incompressible third grade fluid equations[J]. Nonlinear Anal Real World Appl, 2014, 15: 229-238.

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[4] Wu J, Ruan Q, Yang Y H. Gradient estimate for exponentially harmonic functions on complete Riemannian manifolds[J]. Manuscripta Mathematica, 2014, 143(3-4): 483-489.

[5] Kotschwar B, Ni L. Local gradient estimates of p-harmonic functions, $1/H$-flow, and an entropy formula [J]. Annales scientifiques de l'École Normale Supérieure, 2009, 42(1): 1-36.

[6] Li P, Yau S T. On the parabolic kernel of the Schrödinger operator[J]. Acta Mathematica, 1986, 156(1): 153-201.

[7] Wei G, Wylie W. Comparison geometry for the Bakry-Emery Ricci tensor [J]. Journal of Differential Geometry, 2009, 83(2): 337-405.

[8] Li P. Lecture notes on geometric analysis[R]. Seoul: Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, 1993.