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Solubility of finite groups

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Abstract: Let H be a p-subgroup of G. Then: ① H satisfies Φ^*-property in G if H is a Sylow subgroup of some subnormal subgroup of G and for any non-solubly-Frattini chief factor L/K of G, $|G:N_G(K(H\cap L))|$ is a power of p; ② H is called Φ^*-embedded in G if there exists a subnormal subgroup T of G such that HT is S-quasinormal in G and $H\cap T\leqslant S$, where $S\leqslant H$ satisfies Φ^*-property in G. Here Φ^*-embedded subgroups were used to study the structure of finite groups and, in particular, some new characterizations for a group G to be soluble are obtained.

Key words: p-subgroups; Φ^*-property; Φ^*-embedded; Sylow subgroup

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有限群的可解性

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摘要:设 H 是有限群 G 的一个 p 子群. ①H 在 G 中满足 Φ^* 性质,如果对 G 的任一非可解 Frattini 主因子 L/K,$|G:N_G(K(H\cap L))|$ 是 p 的方幂;②H 称为在 G 中 Φ^* 嵌入的,如果存在 G 的次正规子群 T 使得 HT 是 G 的 S 拟正规子群且 $H\cap T\leqslant S$,其中,$S\leqslant H$ 在 G 中满足 Φ^* 性质. 这里主要利用 Φ^* 嵌入子群进一步研究有限群的结构,特别地,得到了群 G 可解的一些新判别准则.

关键词:p 子群;Φ^* 性质;Φ^* 嵌入;Sylow 子群

0 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. All unexplained notations and terminologies are standard. The reader is referred to Refs. [1-3].

A primary subgroup is a subgroup of prime power order. The primary subgroups play an important role in the study of finite groups. For example, it is well-known that a group G is

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nilpotent if and only if every Sylow subgroup of G is normal in G. Hall[4] proved that G is soluble if and only if every Sylow subgroup of G is complemented in G. Srinivasan[5] proved that G is supersoluble if all maximal subgroups of every Sylow subgroup are S-quasinormal in G.

In recent years, in order to investigate the structure of finite groups, a number of embedding properties of primary subgroups have been introduced by many authors. Recall that a p-subgroup H of G is said to be n-embedded[6] in G if for some normal subgroup T of G and some S-quasinormal subgroup S of G contained in H, HT is normal in G and $H\cap T\leqslant S$. A p-subgroup H of G is called S-embedded[7] in G if for some normal subgroup T of G and some S-quasinormal subgroup S of G contained in H, HT is S-quasinormal in G and $H\cap T\leqslant S$.

In this paper, we study the structure of finite groups by the following new defined embedding property of primary subgroups.

Definition 0.1 Let H be a p-subgroup of G. We say that:

① H satisfies Φ^*-property in G if H is a Sylow subgroup of some subnormal subgroup of G and for any non-solubly-Frattini chief factor L/K of G, $|G:N_G((H\cap L)K)|$ is a power of p (cf. Ref. [8, Definition 1.19(ⅰ)]).

② H is called Φ^*-embedded in G if there exists a subnormal subgroup T of G such that HT is S-quasinormal in G and $H\cap T\leqslant S$, where $S\leqslant H$ satisfies Φ^*-property in G.

Recall that a G-chief factor L/K is called non-solubly-Frattini if $L/K\nleqslant\Phi(R(G/K))$, where $R(G/K)$ is the largest soluble normal subgroup of G/K. Let H be a p-subgroup of G. If H is a normal subgroup, CAP-subgroup or S-quasinormal subgroup (that is, H permutes with all Sylow subgroups of G), then H satisfies Φ^*-property in G. For example, let H be an S-quasinormal p-subgroup of G. Then for any chief factor L/K of G, $(H\cap L)K/K$ is an S-quasinormal p-subgroup of G/K. Hence

$$O^p(G/K)\leqslant N_{G/K}((H\cap L)K/K)^{[9\text{-}10]}.$$

Consequently

$$|G:N_G((H\cap L)K)|=|G/K:N_{G/K}((H\cap L)K/K)|$$

is a power of p, which shows that H satisfies Φ^*-property in G. But the following example shows that the converse is not true.

Example 0.1 Let $G=S_4\times Z_3$ and $H=\{1,(123),(132)\}\times Z_3$. It is easy to check that H satisfies Φ^*-property in G. But H is not S-quasinormal in G since $H\nleqslant O_3(G)=Z_3$.

Recall also that a subgroup H of G is said to be c-normal[11] in G if there exists a normal subgroup T of G such that $G=HT$ and $H\cap T\leqslant H_G$. A subgroup H of G is called weakly s-permutable[12] in G if G has a subnormal subgroup T such that $G=HT$ and $H\cap T\leqslant H_{sG}$, where H_{sG} denotes the subgroup of H generated by all subgroups of H which are S-quasinormal in G. The following example shows the set of Φ^*-embedded subgroups of a group G is wider than the set of c-normal subgroups of G and the set of weakly s-permutable subgroups of G.

Example 0.2 Let $G=S_4\times S_3$ and $H=\{1,(123),(132)\}\times A_3$. It is easy to show that $\{1,(123),(132)\}\times 1$, $1\times A_3$ and H, all satisfy Φ^*-property in G. But $\{1,(123),(132)\}\times 1$ and H are not S-quasinormal since $O_3(G)=1\times A_3$. Clearly, H is Φ^*-embedded in G. However, H is neither c-normal nor weakly s-permutable in G.

In this paper, we will study the structure of finite groups by means of some Φ^*-embedded subgroups. In particular, some new characterizations of solubility of finite groups are obtained.

1 Preliminaries

We cite the following lemmas which will be useful in our proofs.

Lemma 1.1 Let G be a group and $A\leqslant K\leqslant G$, $B\leqslant G$.

① Suppose that A is normal in G. Then K/A is subnormal in G/A if and only if K is subnormal in G.

② If A and B are subnormal in G, then $A\cap B$ and $\langle A, B\rangle$ are subnormal in G.

③ If A is a subnormal subgroup of G, then $Soc(G)\leqslant N_G(A)$.

④ If A is subnormal in G, then $A\cap B$ is subnormal in B.

⑤ If A is subnormal in G and A is a π-subgroup of G, then $A\leqslant O_\pi(G)$. Moreover, if A is a Hall subgroup of G, then A is normal in G.

Proof For ①~④, one can refer to Ref. [1, A, (14.1)~(14.4)]. For ⑤, see Ref. [13]. □

Lemma 1.2[10] Let G be a group and A, $B\leqslant G$.

① If A and B are S-quasinormal in G, then $\langle A, B\rangle$ is S-quasinormal in G.

② If A is S-quasinormal in G, then A is a subnormal subgroup of G and A/A_G is nilpotent.

③ Suppose that B is normal in G. If A is S-quasinormal in G, then AB/B is S-quasinormal in G/B.

Lemma 1.3[8, Lemma 3.2] Let N be a normal subgroup of G and H a p-subgroup of G. If H satisfies Φ^*-property in G, then HN/N satisfies Φ^*-property in G/N.

Lemma 1.4 Let N be a normal subgroup and H a p-subgroup of G. If H is Φ^*-embedded in G and either $N\leqslant H$ or $(|H|,|N|)=1$, then HN/N is Φ^*-embedded in G/N.

Proof Suppose that H is Φ^*-embedded in G. Let T be a subnormal subgroup of G such that HT is S-quasinormal in G and $H\cap T\leqslant S$, where $S\leqslant H$ satisfies Φ^*-property in G. Then TN/N is subnormal in G/N by Lemma 1.1①, and HTN/N is S-quasinormal in G/N by Lemma 1.2③. If $N\leqslant H$, then $H\cap TN=(H\cap T)N$ by the Dedekind identity. Now we assume that $(|H|, |N|)=1$. Then $(|HN\cap T: H\cap T|, |HN\cap T: N\cap T|)=(|N\cap HT|, |H\cap TN|)=1$. It follows that

$$HN\cap T=(H\cap T)(N\cap T)$$

by Ref. [1, A, (1.6)(b)], and

$$HN\cap TN=(HN\cap T)N=(H\cap T)N.$$

Generally speaking, $(HN/N)\cap(TN/N)=(H\cap T)N/N\leqslant SN/N$, where $SN/N\leqslant HN/N$ satisfies Φ^*-property in G/N by Lemma 1.3. This shows that HN/N is Φ^*-embedded in G/N. □

Lemma 1.5[14, Lemma 2.12] Let p be a prime divisor of $|G|$ with

$$(|G|, (p-1)(p^2-1)\cdots(p^n-1))=1.$$

If $H\lhd G$ with $p^{n+1}\nmid |H|$ and G/H is p-nilpotent, then G is p-nilpotent. In particular, if $p^{n+1}\nmid |G|$, then G is p-nilpotent.

2 New characterizations of solubility of groups

Theorem 2.1 G is soluble if and only if every Sylow subgroup of G satisfies Φ^*-property in G.

Proof To prove the necessity, we suppose that the assertion is false and let G be a counterexample of minimal order. We proceed via the following steps:

① G is not a non-abelian simple group.

Assume that G is a non-abelian simple group. Then the unique chief factor of G is $G/1$ and obviously, $G/1$ is non-solubly- Frattini since $R(G)=1$. Let P be any Sylow subgroup of G. By the hypothesis, $|G: N_G(P\cap G)|=|G: N_G(P)|$ is a power of p. It follows that $G=N_G(P)$ and P is normal in G. Thus G is nilpotent. The contradiction shows that ① holds.

② If N is a non-identity normal subgroup of G, then G/N is soluble.

By Sylow's theorem, for any prime $p\in\pi(G/N)$, every Sylow p-subgroup of G/N must be of the form PN/N, where P is a Sylow p-subgroup of G. By Lemma 1.3, PN/N satisfies Φ^*-property in G/N. It follows that G/N satisfies the hypothesis for G. Consequently G/N is soluble by the choice of G.

③ $O_p(G)=1$, for any $p\in\pi(G)$. G has a unique minimal normal subgroup, N say, and $\Phi(G)=1$.

It follows directly from ②, $O_p(G)=1$ for any prime $p\in\pi(G)$. By ① and ②, G has a non-trivial minimal normal subgroup N and then G/N is soluble. Let L be another minimal normal subgroup of G distinct from N. Then $N\cap L=1$ and G/L is soluble by ②. It follows that $G\cong G/(N\cap L)$ is soluble. Now assume that $N\leqslant \Phi(G)$. Then G is soluble since the class of soluble groups is a saturated formation[2, Chapter 3]. Hence we have ③.

④ $N=1$, which gives the final contradiction.

Let $P\cap N$ be an arbitrary Sylow p-subgroup of N, where P is a Sylow p-subgroup of G. If $P\cap N\neq 1$, then $|G:N_G(P\cap N)|$ is a power of p, since P satisfies Φ^*-property in G and N is non-solubly-Frattini by ③. Therefore

$$N\leqslant (P\cap N)^G=(P\cap N)^{N_G(P\cap N)P}\leqslant O_p(G)=1.$$

The contradiction shows that $P\cap N=1$ for any prime $p\in\pi(G)$. Consequently $N=1$. The final contradiction completes the necessity.

Now let P be an arbitrary Sylow p-subgroup of G for any prime $p\in\pi(G)$. If G is soluble, then for any non-solubly-Frattini chief factor (which is equal to non-Frattini chief factor) H/K, H/K is an elementary abelian q-group for some prime $q\in\pi(G)$. By Sylow's theorem, $(P\cap H)K/K$ is a Sylow p-subgroup of H/K. If $q\neq p$, then

$$(P\cap H)K/K=1,$$

$$|G:N_G((P\cap H)K)|=|G:N_G(K)|=1.$$

Assume that $q=p$. Then

$$(P\cap H)K/K=H/K,$$

$$|G:N_G((P\cap H)K)|=|G:N_G(H)|=1.$$

This shows that P satisfies Φ^*-property in G. □

Theorem 2.2 G is soluble if every Sylow 2-subgroup of G is Φ^*-embedded in G.

Proof Suppose that the assertion is false and let G be a counterexample of minimal order. Then $|G|$ is even by Feit-Thompson's odd order theorem (that is, groups of odd order are soluble). We proceed via the following steps:

① G is not a non-abelian simple group.

Suppose that ① is false. Let P be a Sylow 2-subgroup of G. By the hypothesis, G has a subnormal subgroup T such that PT is S-quasinormal in G and $P\cap T\leqslant S$, where $S\leqslant P$ satisfies Φ^*-property in G. Since G is a non-abelian simple group, $T=1$ or G. If $T=1$, then $P=PT$ is S-quasinormal in G and so $P\leqslant O_p(G)=1$ by Lemma 1.2 ② and Lemma 1.1 ⑤. This contradiction implies that $T=G$, and so $P=P\cap G$ satisfies Φ^*-property in G. We have that P is normal in G similarly as the proof of ① in Theorem 2.1. Then G is soluble since G/P is soluble by Feit-Thompson's theorem. The contradiction shows that ① holds.

② If $N>1$ is a normal subgroup of G, then G/N satisfies the hypothesis. Consequently G/N is soluble.

Clearly, any Sylow 2-subgroup of G/N must be of the form PN/N for some Sylow 2-subgroup P of G. By the hypothesis, G has a subnormal subgroup T such that PT is S-quasinormal in G and $P\cap T\leqslant S$, where $S\leqslant P$ satisfies Φ^*-property in G. Then TN/N is subnormal in G/N by Lemma 1.1①, and $(PN/N)(TN/N)=(PT)N/N$ is S-quasinormal in G/N by Lemma 1.2 ③. Since $|PT\cap N: T\cap N|=|P\cap NT: P\cap T|$ is a power of p and $P\cap N$ is a Sylow p-subgroup of $PT\cap N$, we have $PT\cap N=(P\cap N)(T\cap N)$ by Ref. [1, A, (1.6)(b)]. It follows that $PN\cap TN=(P\cap T)N$ by Ref. [1, A, (1.2)]. Hence

$$(PN/N)\cap(TN/N)=(P\cap T)N/N\leqslant SN/N,$$

where $SN/N\leqslant PN/N$ satisfies Φ^*-property in G/N by Lemma 1.3. Therefore G/N satisfies the hypothesis and so G/N is soluble by choice of G.

③ $O_p(G)=1$, for any $p\in\pi(G)$ and particularly, $O_2(G)=1$ (It follows directly from ②).

④ $\Phi(G)=1$ and G has the unique minimal normal subgroup, N say.

By ①, G has a non-trivial normal subgroup N. Then ④ holds similarly, whose proof is similar to that of ② in Theorem 2.1.

⑤ N is soluble.

If N is of odd order, then N is soluble by Feit-Thompson's theorem. We may, therefore, assume that $|N|$ is even. By Sylow's theorem, any Sylow 2-subgroup of N must be of the form $P\cap N$ for some Sylow 2-subgroup P of G. By the hypothesis, G has a subnormal subgroup T such that PT is S-quasinormal in G and $P\cap T\leqslant S$, where $S\leqslant P$ satisfies Φ^*-property in G. If $S\cap N\neq 1$, then $|G:N_G(S\cap N)|$ is a power of 2 since $N\not\leqslant \Phi(G)=1$ by ④. It follows that

$$N=(S\cap N)^G=(S\cap N)^{N_G(S\cap N)P}\leqslant O_2(G)=1$$

by ③. The contradiction shows $S\cap N=1$. If $(PT)_G=1$, then $PT=1$. In fact, if $PT>1$, then P is subnormal in G by Lemma 1.2② and so P is normal in G by Lemma 1.1⑤, which contradicts ③. Hence $(PT)_G\neq 1$ and so $N\leqslant PT$ by the uniqueness of N. Then

$$N=N\cap PT=(N\cap P)(N\cap T),$$

whose proof is similar to that of ②, and

$$(N\cap P)\cap(N\cap T)\leqslant N\cap S=1.$$

This implies that $N\cap T$ is a 2-complement of N. Moreover, T is subnormal in G, so $N\cap T$ is subnormal in N by Lemma 1.1④. It follows that $N\cap T$ is a normal 2-complement of N by Lemma 1.1⑤ and therefore, N is 2-nilpotent. Again by Feit-Thompson's Theorem, N is soluble and ⑤ holds.

⑥ The final contradiction.

The results of ② and ④ show that G is soluble. The contradiction completes the proof. □

Theorem 2.3 Let P be a Sylow 2-subgroup of G. If every maximal subgroup of P is Φ^*-embedded in G, then G is soluble.

Proof Assume that the theorem is false and let G be a counterexample of minimal order. Then:

① G is not a non-abelian simple group.

Suppose that G is a non-abelian simple group. By Ref. [15, 10.1.9], $|P|\geqslant 4$ and so P has a non-identity maximal subgroup P_1 which is Φ^*-embedded in G. Similarly as the proof of ① in Theorem 1.2, we have that P_1 is normal in G. Since G is a non-abelian simple group, $P_1=1$, a contradiction. Hence ① holds.

② If N is a non-identity normal subgroup of G, then G/N is soluble.

By ①, G has a non-trivial minimal normal subgroup N. If G/N is of odd order, then G/N is soluble by Feit-Thompson's theorem. Moreover, by Ref. [15, 10.1.9], we should assume that $|G/N|$ is divided by 4. Let M/N be an arbitrary maximal subgroup of PN/N. Then $M=(M\cap P)N$, where $P_1=M\cap P$ is a maximal subgroup of P since $|P:P_1|=|PN:M|=p$. By the hypothesis, G has a subnormal subgroup T such that P_1T is S-quasinormal in G and $P_1\cap T\leqslant S$, where $S\leqslant P_1$ satisfies Φ^*-property in G. We have that TN/N is subnormal in G/N by Lemma 1.1 ①, and $(M/N)(TN/N)=(P_1T)N/N$ is S-quasinormal in G/N by Lemma 1.2③. Moreover $P_1\cap N=P\cap N$ is a Sylow p-subgroup of $P_1T\cap N$ and

$$|P_1T\cap N:T\cap N|=|P_1\cap NT:P_1\cap T|$$

is a power of p, so $P_1T\cap N=(P_1\cap N)(T\cap N)$ by Ref. [1, A, (1.6)(b)]. Then

$$P_1N\cap TN=(P_1\cap T)N$$

by Ref. [1, A, (1.2)] and so

$$(M/N)\cap(TN/N)=(P_1\cap T)N/N\leqslant SN/N,$$

where $SN/N\leqslant M/N$ satisfies Φ^*-property in G/N by Lemma 1.3. It follows that G/N satisfies the hypothesis for G. Hence G/N is soluble.

③ $O_2(G)=O_{2'}(G)=1$.

Since both $O_2(G)$ and $O_{2'}(G)$ are soluble, if one of them is not identity, then G is soluble by ②.

④ Let N be a minimal normal subgroup of G. Then N is the unique minimal normal subgroup of G and $C_G(N)=1$.

By ① and ②, $N>1$ and G/N is soluble. Similarly as the proof of ③ in Theorem 2.1, N is the unique minimal normal subgroup of G and N is non-abelian. Since $C_G(N)$ is normal in G, we have $N\cap C_G(N)=1$. So $C_G(N)=1$ by the uniqueness of N.

⑤ If $S\neq 1$ is a 2-subgroup of G satisfying Φ^*-property in G, then $S\cap N\neq 1$.

Let U be a subnormal subgroup of G such that S is a Sylow 2-subgroup of U. Assume that $S\cap N=1$. Since $S\cap N$ is a Sylow 2-subgroup of $U\cap N$ and $U\cap N$ is subnormal in G by Lemma 1.1②, $U\cap N\leqslant O_{2'}(G)=1$ by Lemma 1.1⑤. It follows from Lemma 1.1③ that $U\leqslant C_G(N)=1$. Hence we have ⑤.

⑥ The final contradiction.

Let P_1 be an arbitrary maximal subgroup of P. Then G has a subnormal subgroup T such that P_1T is S-quasinormal in G and $P_1 \cap T \leqslant S$, where $S \leqslant P_1$ satisfies Φ^*-property in G. If $S \neq 1$, then $S \cap N \neq 1$ by ⑤. Since S satisfies Φ^*-property in G, $|G:N_G(S \cap N)|$ is a power of 2. Then

$$N \leqslant (S \cap N)^G = (S \cap N)^{N_G(S\cap N)P} = (S \cap N)^P \leqslant O_2(G) = 1,$$

a contradiction. Therefore $S=1$ and a Sylow 2-subgroup of T is of order 1 or 2. This implies that T is 2-nilpotent by Lemma 1.5. It follows that $T=1$ by ② and then $P_1 = P_1T$ is S-quasinormal in G. But $O_2(G)=1$, so $P_1=1$ by Lemma 1.2② and Lemma 1.1⑤. Then P is of order 2. This shows that G is 2-nilpotent (see Ref. [15, 10.1.9]) and so G is soluble. The final contradiction completes the proof. □

Remark 2.1 Let p be a prime and

$$(|G|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$$

for some integer $n \geqslant 1$. If all n-maximal subgroups of a Sylow p-subgroup of G are Φ^*-embedded in G, then it can be proved similarly that G is soluble by using Lemma 1.5.

Remark 2.2 Groups satisfying the hypothesis in Theorem 2.3 should not be 2-nilpotent in general. For example, let C_2 and C_3 be cyclic groups of orders 2 and 3, respectively, and $G = C_3 \wr C_2 \wr C_3$, where the wreath products are regular. Then all maximal subgroups of Sylow 2-subgroups of G are Φ^*-embedded in G, but G is not 2-nilpotent.

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