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Hamilton paths and cycles in fault-tolerant varietal hypercubes

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Abstract: The varietal hypercube VQ_n, a variant of the hypercube Q_n, was studied. It was proved that VQ_n contains a fault-free Hamilton cycle provided faulty edges do not exceed $n-2$, and that for two distinct vertices, x and y, there is a fault-free xy-Hamilton path in VQ_n provided faulty edges do not exceed $n-3$ for $n\geqslant 3$. The proof is based on an inductive construction.

Key words: graphs; Hamilton path; Hamilton cycle; varietal hypercube; fault-tolerant networks

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容错变形超立方体的圈和路

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摘要:考虑包含故障边的 $n(n\geqslant 3)$ 维变形超立方体 VQ_n,证明了:如果故障边数不超过 $n-2$,那么 VQ_n 包含非故障边的 Hamilton 圈;如果故障边数不超过 $n-3$,那么对任何两个不同顶点 x 和 y,VQ_n 包含非故障边的 xy-Hamilton 路. 该证明方法采用归纳法.

关键词:图论;Hamilton 圈;Hamilton 路;变形超立方体;容错网络

0 Introduction

As a topology of interconnection networks, the hypercube Q_n is the simplest and most popular since it has many superior properties. The varietal hypercube, VQ_n, which is a variant of Q_n and was proposed in Ref. [1], has many properties similar or superior to Q_n. For example, the connectivity and restricted connectivity of VQ_n and Q_n are the same[2], while, all the diameter and the average distance, fault-diameter and wide-diameter of VQ_n are smaller than that of the hypercube[1,3]. Recently, Ref. [4] has shown that VQ_n is vertex-transitive.

Embedding paths and cycles in various well-

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known networks, such as Q_n, have been extensively investigated in Ref. [5]. Recently, Cao et al.[6] have shown that every edge of VQ_n is contained in cycles of every length from 4 to 2^n except 5, and every pair of vertices with distance d is connected by paths of every length from d to 2^n-1 except 2 and 4 if $d=1$. In this paper, we consider fault-tolerant varietal hypercubes and show that VQ_n contains a fault-free Hamilton cycle provided faulty edges do not exceed $n-2$ for $n\geqslant 3$ and for two distinct vertices, x and y, there is a fault-free xy-Hamilton path in VQ_n provided faulty edges do not exceed $n-3$ for $n\geqslant 3$.

The proofs of these results are in Section 2. The definition and some basic properties of VQ_n are given in Section 1.

1 Definitions and lemmas

We follow Ref. [7] for graph-theoretical terminology and notation not defined here. A graph $G=(V, E)$ always means a simple and connected graph, where $V=V(G)$ is the vertex-set and $E=E(G)$ is the edge-set of G. For $xy\in E(G)$, we call x (resp. y) a neighbor of y (resp. x).

The n-dimensional varietal hypercube VQ_n is the labeled graph defined recursively as follows. VQ_1 is the complete graph of two vertices labeled 0 and 1, respectively. Assume that VQ_{n-1} has been constructed. Let VQ_{n-1}^0 (resp. VQ_{n-1}^1) be a labeled graph obtained from VQ_{n-1} by inserting a zero (resp. 1) in front of each vertex-labeling in VQ_{n-1}. For $n>1$, VQ_n is obtained by joining vertices in VQ_{n-1}^0 and VQ_{n-1}^1, according to the rule: a vertex $x=0x_{n-1}x_{n-2}x_{n-3}\cdots x_2x_1$ in VQ_{n-1}^0 and a vertex $y=1y_{n-1}y_{n-2}y_{n-3}\cdots y_2y_1$ in VQ_{n-1}^1 are adjacent in VQ_n if and only if

① $x_{n-1}x_{n-2}x_{n-3}\cdots x_2x_1=y_{n-1}y_{n-2}y_{n-3}\cdots y_2y_1$ if $n\neq 3k$, or

② $x_{n-3}\cdots x_2x_1=y_{n-3}\cdots y_2y_1$ and $(x_{n-1}x_{n-2}, y_{n-1}y_{n-2})\in I$ if $n=3k$, where $I=\{(00,00),(01,01),(10,11),(11,10)\}$.

Fig. 1 shows the examples of varietal hypercubes VQ_n for $n=1$, 2, 3 and 4.

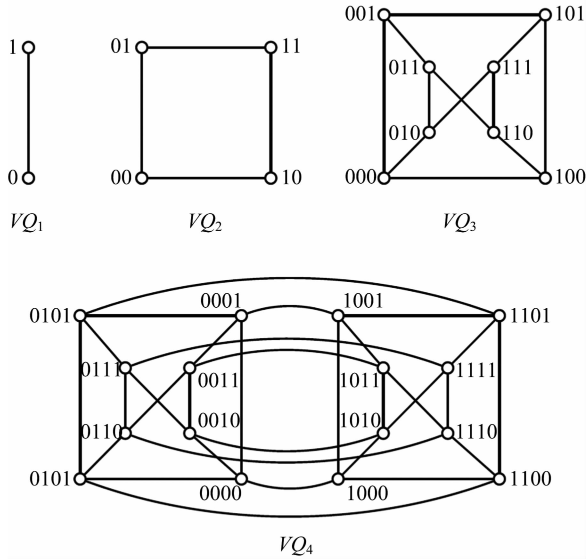


Fig. 1 The varietal hypercubes VQ_1, VQ_2, VQ_3 and VQ_4

An edge xy in VQ_n, where $x=x_nx_{n-1}\cdots x_2x_1$ and $y=y_ny_{n-1}\cdots y_2y_1$, is called the i-transversal edge if $x_n\cdots x_{i-1}=y_n\cdots y_{i-1}$ and $x_i\neq y_i$. For convenience, we express VQ_n as $VQ_n^0\odot VQ_n^1$, where $VQ_n^0\cong VQ_n^1\cong VQ_{n-1}$. Then edges between VQ_n^0 and VQ_n^1 are n-transversal edges. The edges of Type ② are referred to as crossing edges when

$$(x_{n-1}x_{n-2}, y_{n-1}y_{n-2})\in\{(10,11),(11,10)\}.$$

All the other edges are referred to as normal edges.

Let $VQ_n=L\odot R$, where $L=VQ_{n-1}^0$ and $R=VQ_{n-1}^1$, and denote by x_Lx_R the n-transversal edge joining $x_L\in L$ and $x_R\in R$. The recursive structure of VQ_n gives the following simple properties.

Lemma 1.1 Let $VQ_n=L\odot R$ with $n\geqslant 1$. Then VQ_n contains no triangles and every vertex $x_L\in L$ has exactly one neighbor x_R in R. Moreover, $x_Ly_L\in E(L)$ if and only if $x_Ry_R\in E(R)$ for $n\neq 3k$, where x_R and y_R are the neighbors of x_L and y_L in R.

Lemma 1.2 Let $VQ_n=L\odot R$ and xy be an n-transversal edge in VQ_n with $x\in L$ and $y\in R$. For $n\geqslant 3$, let $x=0ab\beta$, where $\beta=x_{n-3}\cdots x_1$. Then $y=1a'b'\beta$, where $ab=a'b'$ if xy is a normal edge, and $(ab, a'b')=(1b, 1\bar{b})$ if xy is a crossing edge, where $\bar{b}=\{0,1\}\backslash b$.

Let VQ_{n-2}^{ab} be a labeled graph obtained from VQ_{n-2} by inserting ab in front of each vertex-labeling in VQ_{n-2}, where $a,b\in\{0,1\}$. Clearly, $VQ_{n-2}^{ab}\cong VQ_{n-2}$. By definition, $VQ_{n-1}^{0}=VQ_{n-2}^{00}\odot VQ_{n-2}^{01}$ and $VQ_{n-1}^{1}=VQ_{n-2}^{10}\odot VQ_{n-2}^{11}$. By Lemma 1.2, it is easy to see that when VQ_n is expressed as $(VQ_{n-2}^{00}\odot VQ_{n-2}^{01})\odot(VQ_{n-2}^{10}\odot VQ_{n-2}^{11})$, VQ_n is of the recursive structure shown as Fig. 2.

Let x and y be two distinct vertices in a graph G. An xy-path is a sequence of adjacent vertices, written as $(x_0, x_1, x_2, \cdots, x_m)$, in which $x=x_0$, $y=x_m$ and all the vertices $x_0, x_1, x_2, \cdots, x_m$ are different from each other. For a path

$$P=(x_0,\cdots,x_i,\ x_{i+1},\ \cdots,\ x_m),$$

we can write $P=P(x_0,x_i)+x_ix_{i+1}+P(x_{i+1},x_m)$, and the notation $P-x_ix_{i+1}$ denotes the subgraph obtained from P by deleting the edge x_ix_{i+1}. An xy-path P is called a cycle if $x=y$; a cycle is called a Hamilton cycle if it contains all vertices in G. An xy-path P is called an xy-Hamilton path if it contains all vertices in G. A graph G is Hamiltonian if it contains a Hamilton cycle, and is called Hamilton-connected if it contains an xy-Hamilton path for any two vertices x and y in G. Clearly, if G is Hamilton-connected, then it certainly is Hamiltonian.

Lemma 1.3[6] For $n\geqslant 2$, every edge of VQ_n is contained in a Hamilton cycle. For $n\geqslant 3$, VQ_n is Hamilton-connected.

Faults of some processors and/or communication lines in a large-scale system are inevitable. However, the presence of faults gives rise to a large number of problems that have to be considered for some applications. Ref. [2] showed that VQ_n is n-connected. This fact implies that for any set of faults $F\subset E(VQ_n)$ with $|F|<n$, the remainder network VQ_n-F is still connected. However, one does not know whether VQ_n-F still remains Hamilton-connected or not.

Let $F\subset E(G)$ be a set of edge-faults of G. A subgraph H of G is called fault-free if H contains no edges in F, and G is called t-edge-fault-tolerant Hamiltonian (resp. t-edge-fault-free Hamilton-connected) if $G-F$ contains a Hamilton cycle (resp. is Hamilton-connected) for any $F\subset E(G)$ with $|F|\leqslant t$.

The n-dimensional crossed cube CQ_n is such a graph: its vertex-set is the same as VQ_n, two vertices $x=x_n\cdots x_2x_1$ and $y=y_n\cdots y_2y_1$ are linked by an edge if and only if there exists some j $(1\leqslant j\leqslant n)$ such that ① $x_n\cdots x_{j+1}=y_n\cdots y_{j+1}$, ② $x_j\neq y_j$, ③ $x_{j-1}=y_{j-1}$ if j is even, and ④ $(x_{2i}x_{2i-1},\ y_{2i}y_{2i-1})\in I$ for each $i=1,2,\cdots,\left\lceil \frac{1}{2}j\right\rceil-1$, where

$$I=\{(00,00),(01,01),(10,11),(11,10)\}.$$

By definition, $VQ_n\cong CQ_n$ for each $n=1,2,3$. The following results on CQ_n are used in the proofs of our main results for $n=3$.

Lemma 1.4[8-9] CQ_n is $(n-2)$-edge-fault-tolerant Hamiltonian for $n\geqslant 3$.

Lemma 1.5[10] If each vertex is incident to at least two fault-free edges, then CQ_n is $(2n-5)$-edge-fault-tolerant Hamiltonian.

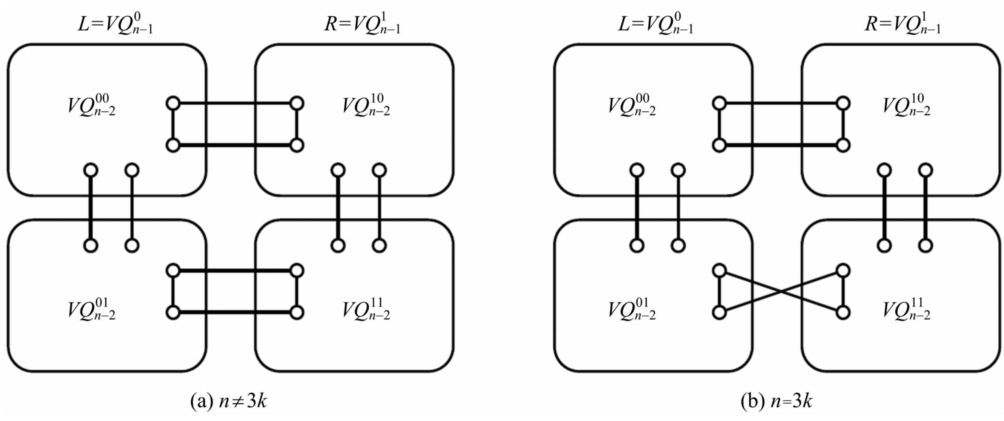


Fig. 2 The recursive structure of VQ_n

2 Main results

Theorem 2.1 VQ_n is $(n-3)$-edge-faulty-tolerant Hamilton-connected for $n \geqslant 3$.

Proof We proceed by induction on $n \geqslant 3$.

By Lemma 1.3, the conclusion is true for $n=3$. Suppose now that $n \geqslant 4$ and the result holds for any integer less than n. Let $F \subset E(VQ_n)$ with $|F| \leqslant n-3$, x and y be two distinct vertices in VQ_n. We need to prove that $VQ_n - F$ contains an xy-Hamilton path. By Lemma 1.3, we can assume $|F| \geqslant 1$. Let $VQ_n = L \odot R$, and let

$$L = VQ_{n-2}^{00} \odot VQ_{n-2}^{01},\ R = VQ_{n-2}^{10} \odot VQ_{n-2}^{11},$$
$$F_L = F \cap E(L),\ F_R = F \cap E(R),$$
$$F_n = F \setminus (F_L \cup F_R).$$

By symmetry of structure of VQ_n, we can assume $|F_L| \geqslant |F_R|$.

Case 1 $|F_L| \leqslant n-4$.

By the assumption, $|F_R| \leqslant n-4$ and $n \geqslant 5$.

Subcase 1.1 $x, y \in L$ or $x, y \in R$. Without loss of generality, assume $x, y \in R$.

Since $|F_R| \leqslant n-4 = (n-1)-3$, by the induction hypothesis $R - F_R$ contains an xy-Hamilton path, say P_R. Since $\varepsilon(P_R) = 2^{n-1}-1 > 2(n-3) \geqslant 2|F|$, there is an edge $u_R v_R$ in P_R such that the edges $u_R u_L$ and $v_R v_L$ are not in F, where u_L and v_L are neighbors of u_R and v_R in L. Since $|F_L| \leqslant n-4 = (n-1)-3$, by the induction hypothesis $L - F_L$ contains a $u_L v_L$-Hamilton path, say P_L. Thus, $P_R - u_R v_R + u_R u_L + v_R v_L + P_L$ is an xy-Hamilton path in $VQ_n - F$ (see Fig. 3(a)).

Subcase 1.2 $x \in L$ and $y \in R$.

Since there are 2^{n-1} edges between L and R and $2^{n-1}-2 > n-3 \geqslant |F|$, there is an edge $u_L u_R \notin F_n$ such that $u_L \neq qx$ and $u_R \neq qy$. By the induction hypothesis, let P_L be an xu_L-Hamilton path in $L - F_L$, and P_R be a yu_R-Hamilton path in $R - F_R$. Then $P_L + u_L u_R + P_R$ is an xy-Hamilton path in $VQ_n - F$ (see Fig. 3a(b)).

Case 2 $|F_L| = n-3$.

In this case, $|F_R| = |F_n| = 0$. Let

$$F_{00} = F_L \cap E(VQ_{n-2}^{00}),\ F_{01} = F_L \cap E(VQ_{n-2}^{01}).$$

Without loss of generality, we can assume $F_{00} \neq \emptyset$.

Subcase 2.1 $x, y \in L$.

Arbitrarily take $e = u_L v_L \in F_L$. Since $|F_L - e| = n-4 = (n-1)-3$, by the induction hypothesis $L-(F_L - e)$ contains an xy-Hamilton path, say P_L. Without loss of generality, assume $e \in E(P_L)$. Let u_R and v_R be neighbors of u_L and v_L in R, respectively. By Lemma 1.3, R contains a $u_R v_R$-Hamilton path, say P_R. Then $P_L - u_L v_L + u_L u_R + v_L v_R + P_R$ is an xy-Hamilton path in $VQ_n - F$.

Subcase 2.2 $x \in L$ and $y \in R$.

If $n=4$, then $L \cong R \cong VQ_3 \cong CQ_3$. Since $|F_L| = 1$, by Lemma 1.5 $L - F_L$ contains a Hamilton cycle, say C_L. Choose a neighbor u_L of x in C_L such that its neighbor u_R in R is not y. By Lemma 1.3, R contains a yu_R-Hamilton path, say P_R. Then, $C_L - xu_L + u_L u_R + P_R$ is an xy-Hamilton path in $VQ_4 - F$.

Assume now $n \geqslant 5$, that is, $n-2 \geqslant 3$.

(a) $y \in VQ_{n-2}^{11}$ (see Fig. 4(a)).

Arbitrarily take $z_{11} \in VQ_{n-2}^{11}$ with $z_{11} \neq qy$, and let z_{01} be the neighbor of z_{11} in L. By Lemma 1.3,

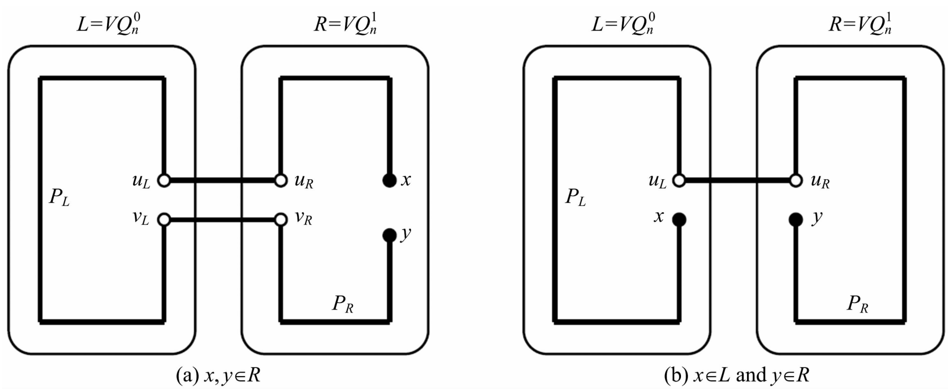


Fig. 3 Illustrations of Case 1 in the proof of Theorem 2.1

VQ_{n-2}^{11} contains a $z_{11}y$-Hamilton path, say P_{11}. Arbitrarily take $e\in F_{00}$. Since $n\geqslant 5$, by the induction hypothesis $L-(F_L-e)$ contains an xz_{01}-Hamilton path, say P_L. Without loss of generality, assume $e=u_{00}v_{00}$ in P_L. Let u_{10} and v_{10} be neighbors of u_{00} and v_{00} in VQ_{n-2}^{10}, respectively. By Lemma 1.3, VQ_{n-2}^{10} contains a $u_{10}v_{10}$-Hamilton path, say P_{10}. Then $P_{10}+u_{00}u_{10}+v_{00}v_{10}+P_L-u_{00}v_{00}+z_{01}z_{11}+P_{11}$ is an xy-Hamilton path in VQ_n-F (see Fig. 4(a)).

(b) $y\in VQ_{n-2}^{10}$ (see Fig. 4(b)).

Arbitrarily take a vertex z_{01} in VQ_{n-2}^{01} with $z_{01}\neq x$. Let z_{11} be the neighbor of z_{01} in VQ_{n-2}^{11}, $e=u_{00}v_{00}\in F_{00}$ such that their neighbors u_{10} and v_{10} in VQ_{n-2}^{10} do not contain y. Since $n\geqslant 5$, by the induction hypothesis $L-(F_L-e)$ contains an xz_{01}-Hamilton path, say P_L. Without loss of generality, assume that e is in P_L. By Lemma 1.3, VQ_{n-2}^{10} contains a $u_{10}v_{10}$-Hamilton path, say P_{10}. Since $y\in P_{10}$, we can write $P_{10}=P_{10}(v_{10},y)+yw_{10}+P_{10}(w_{10},u_{10})$. Let w_{11} be the neighbor of w_{10} in VQ_{n-2}^{11}. By Lemma 1.3, VQ_{n-2}^{11} contains a $z_{11}w_{11}$-Hamilton path, say P_{11}. Then $P_L-u_{00}v_{00}+u_{00}u_{10}+v_{00}v_{10}+P_{10}-yw_{10}+w_{10}w_{11}+P_{11}+z_{01}z_{11}$ is an xy-Hamilton path in VQ_n-F (see Fig. 4(b)).

Subcase 2.3 $x, y\in R$.

If $n=4$, then $L\cong R\cong VQ_3\cong CQ_3$. By Lemma 1.3, R contains an xy-Hamilton path, say P_R. Since $L\cong CQ_3$ and $|F_L|=1$, by Lemma 1.5 $L-F_L$ contains a Hamilton cycle, say C_L. Since L and R are 3-regular and isomorphic, there is an edge u_Rv_R in P_R which is not incident with x and y such that the corresponding edge e_L in L is contained in C_L. Since $n=4$, by Lemma 1.1 $e=u_Lv_L$, where u_L and v_L be neighbors of u_R and v_R in L, respectively. Thus, $P_R-u_Rv_R+u_Lu_R+v_Lv_R+C_L$ is an xy-Hamilton path in VQ_4-F (as a reference, see Fig. 3(a)).

Assume $n\geqslant 5$ below, that is, $n-2\geqslant 3$.

(a) $x,y\in VQ_{n-1}{}^{11}$ (see Fig. 5(a)).

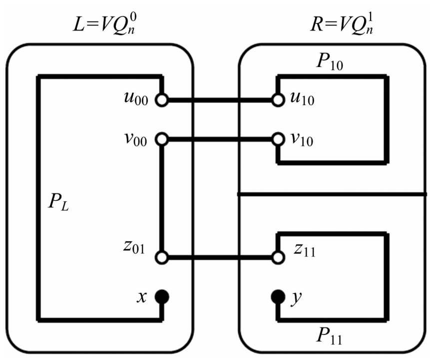
By Lemma 1.3, VQ_{n-2}^{11} contains an xy-Hamilton path, say P_{11}. Take $u_{11}v_{11}\in E(P_{11})$, and let u_{01} and v_{01} be neighbors of u_{11} and v_{11} in VQ_{n-2}^{01}, respectively. Take $e=w_{00}z_{00}\in F_{00}$. By the induction hypothesis, $L-(F_L-e)$ contains a $u_{01}v_{01}$-Hamilton path, say P_L. Without loss of generality, assume that e is in P_L, and let w_{10} and z_{10} be neighbors of w_{00} and z_{00} in VQ_{n-2}^{10}, respectively. By Lemma 1.3, VQ_{n-2}^{10} contains a $w_{10}z_{10}$-Hamilton path, say P_{10}. Thus, $P_{10}+w_{00}w_{10}+z_{00}z_{10}+P_L-w_{00}z_{00}+P_{11}-u_{11}v_{11}+u_{01}u_{11}+v_{01}v_{11}$ is an xy-Hamilton path in VQ_n-F (see Fig. 5(a)).

(b) $x\in VQ_{n-1}{}^{11}$ and $y\in VQ_{n-2}^{10}$ (see Fig. 5(b)).

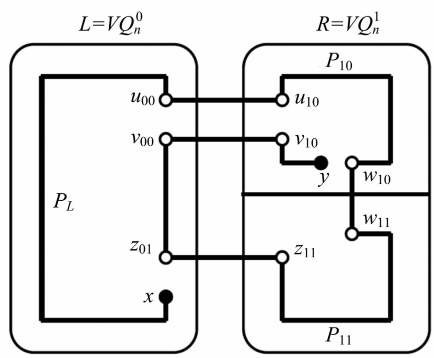
Choose $e=u_{00}v_{00}\in F_{00}$ such that their neighbors u_{10} and v_{10} in VQ_{n-2}^{10} do not contain y. By the induction hypothesis, $L-(F_L-e)$ contains a $u_{00}v_{00}$-Hamilton path, say P_L. Without loss of generality, assume e in P_L. By Lemma 1.3, VQ_{n-2}^{10} contains a $u_{10}v_{10}$-Hamilton path, say P_{10}. Since $y\in P_{10}$, we can write

$$P_{10}=P_{10}(v_{10},y)+yw_{10}+P_{10}(w_{10},u_{10}).$$

Let w_{11} be the neighbor of w_{10} in VQ_{n-2}^{11}. By Lemma 1.3, VQ_{n-2}^{11} contains an xw_{11}-Hamilton



(a) $x\in L$ and $y\in VQ_{n-2}^{11}$



(b) $x\in L$ and $y\in VQ_{n-2}^{10}$

Fig. 4 Illustrations of Subcase 2.2 in the proof of Theorem 2.1

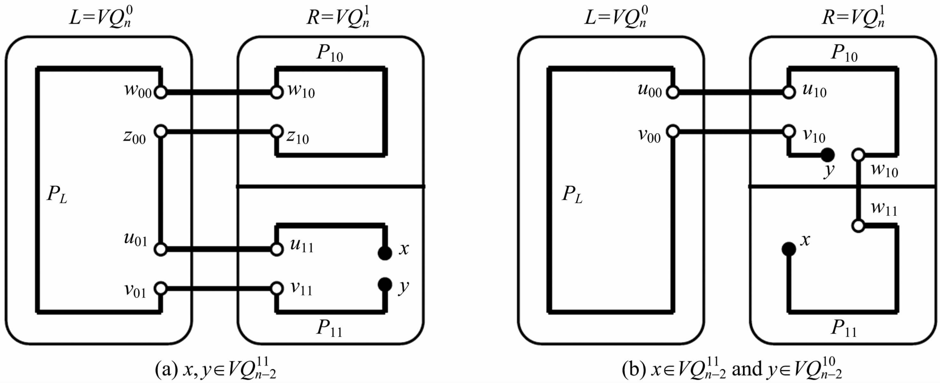


Fig. 5 Illustrations of Subcase 2.3 in the proof of Theorem 2.1

path, say P_{11}. Then $P_L - u_{00}v_{00} + u_{00}u_{10} + v_{00}v_{10} + P_{10} - yw_{10} + w_{10}w_{11} + P_{11}$ is an xy-Hamilton path in $VQ_n - F$ (see Fig. 5(b)).

(c) $x, y \in VQ_{n-2}^{10}$ (see Fig. 6(a)).

By Lemma 1.3, VQ_{n-2}^{10} contains an xy-Hamilton path, say P_{10}.

(c1) $|F_{01}| \neq 0$.

Take $w_{10}z_{10} \in E(P_{10})$, and let w_{00} and z_{00} be neighbors of w_{10} and z_{10} in VQ_{n-2}^{00}, respectively. Take $e = u_{01}v_{01} \in F_{01}$. By the induction hypothesis, $L-(F_L - e)$ contains a $w_{00}z_{00}$-Hamilton path, say P_L. Without loss of generality, assume that e is in P_L, and let u_{11} and v_{11} be neighbors of u_{01} and v_{01} in VQ_{n-2}^{11}, respectively. By Lemma 1.3, VQ_{n-2}^{11} contains a $u_{11}v_{11}$-Hamilton path, say P_{11}. Thus, $P_{10} - w_{10}z_{10} + w_{00}w_{10} + z_{00}z_{10} + P_L - u_{01}v_{01} + u_{01}u_{11} + v_{01}v_{11} + P_{11}$ is an xy-Hamilton path in $VQ_n - F$ (see Fig. 6(a)).

(c2) $|F_{01}| = 0$. Then $|F_{00}| = n-3 \geqslant 2$ since $n \geqslant 5$.

Let P'_{00} be an $x_{00}y_{00}$-Hamilton path in VQ_{n-2}^{00} that corresponds to P_{10} obtained from P_{10} by changing the left-most coordinate 1 of every vertex into 0, where x_{00} (resp. y_{00}) is a vertex corresponding to x (resp. y). Arbitrarily take an edge $u_{00}v_{00}$ in P'_{00}. Let u_{10} and v_{10} be neighbors of u_{00} and v_{00} in VQ_{n-2}^{10}, respectively. Then $u_{10}v_{10}$ is an edge in P_{10}, $u_{00}u_{10}$ and $v_{00}v_{10}$ are edges in VQ_n (see Fig. 2).

If P'_{00} contains at most one edge in F_{00}. Without loss of generality, take $e_1 = u_{00}v_{00} \in F_{00}$, and let $P_{00} = P'_{00}$.

If P'_{00} contains exactly two edges e_1 and e_2 in F_{00} and $n=5$. Let $e_1 = u_{00}v_{00}$. By Lemma 1.3, there is a Hamilton cycle C_L in VQ_3^{00} containing the edge e_1. Without loss of generality, assume that $e_2 = x_{00}y_{00}$ is in C_R, and let $P_{00} = C_L - e_2$.

If P'_{00} contains at least three edges in F_{00}. Then $n-3 = |F| = |F_{00}| \geqslant 3$, that is, $n-2 \geqslant 4$. Let $e_1 = u_{00}v_{00}$ and $e_2 = x_{00}y_{00}$ be two edges in F_{00}. Since $|F_{00} - e_1 - e_2| = (n-2) - 3$, by the induction hypothesis $VQ_{n-2}^{00} - (F_{00} - e_1 - e_2)$ contains an

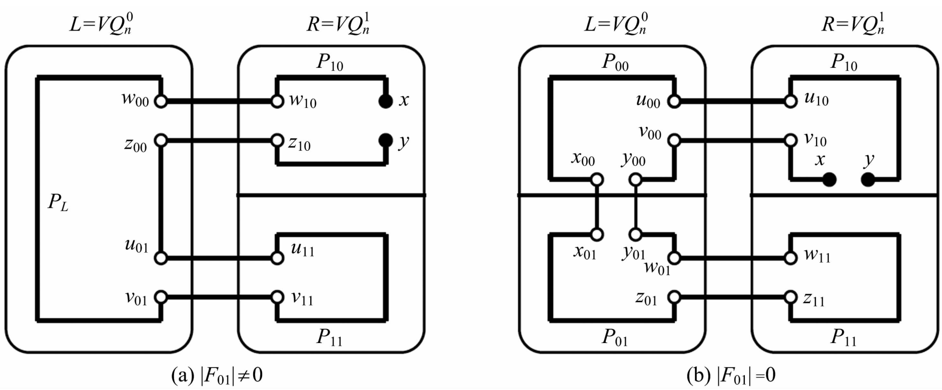


Fig. 6 Other illustrations of Subcase 2.3 in the proof of Theorem 2.1

$x_{00}y_{00}$-Hamilton path, say P_{00}. Without loss of generality, assume that e_1 is in P_{00}.

Let x_{01} and y_{01} be neighbors of x_{00} and y_{00} in VQ_{n-2}^{01}, respectively. By Lemma 1.3, VQ_{n-2}^{01} contains an $x_{01}y_{01}$-Hamilton path, say P_{01}. Arbitrarily take an edge $w_{01}z_{01}$ in P_{01}. Let w_{11} and z_{11} be neighbors of w_{01} and z_{01} in VQ_{n-2}^{11}, respectively. By Lemma 1.3, VQ_{n-2}^{11} contains a $w_{11}z_{11}$-Hamilton path, say P_{11}. Then $P_{10}-u_{10}v_{10}+u_{00}u_{10}+v_{00}v_{10}+P_{00}-u_{00}v_{00}+x_{00}x_{01}+y_{00}y_{01}+P_{01}-w_{01}z_{01}+w_{01}w_{11}+z_{01}z_{11}+P_{11}$ is an xy-Hamilton path in VQ_n-F (see Fig. 6(b)).

The theorem follows. □

Theorem 2.2 VQ_n is $(n-2)$-edge-fault-tolerant Hamiltonian for $n\geqslant 3$.

Proof We proceed by induction on $n\geqslant 3$.

Since $VQ_3\cong CQ_3$, by Lemma 1.4, the conclusion is true for $n=3$. Assume the induction hypothesis for $n-1$ with $n\geqslant 4$. Let $F\subset E(VQ_n)$ with $|F|\geqslant 1$, $VQ_n=L\odot R$, and let

$$F_L=F\cap L, F_R=F\cap R, F_n=F\setminus(F_L\cup F_R).$$

Without loss of generality, assume $|F_L|\geqslant|F_R|$.

If $|F_L|=n-2$, then $|F_R|=|F_n|=0$. For any $e\in F_L$, $|F_L-e|=n-3$. By the induction hypothesis, $L-(F_L-e)$ contains a Hamilton cycle, say C_L. Without loss of generality, assume that $e=u_Lv_L$ is in C_L. Let u_R and v_R be neighbors of u_L and v_L in R, respectively. Then $u_Rv_R\in E(L)$. By Theorem 2.1, R contains a u_Rv_R-Hamilton path, say P_R. Then, $C_L-u_Lv_L+u_Lu_R+v_Lv_R+P_R$ is a Hamilton cycle in VQ_n-F.

We now assume $|F_L|\leqslant n-3$. If $n=4$, then $|F_R|+|F_n|\leqslant 1$. Since $L\cong R\cong CQ_3$, by Lemma 1.4 both L and R contain fault-free Hamilton cycles, say C_L and C_R, respectively. Since CQ_3 is 3-regular, any two Hamilton cycles have at least one edge in common. Thus, assume $u_Lv_L\in E(C_L)$ and $u_Rv_R\in E(C_R)$. Then $C_L\cup C_R-u_Lv_L-u_Rv_R+u_Lu_R+v_Lv_R$ is a Hamilton cycle in VQ_4-F.

We now assume $n\geqslant 5$. Since $|F_L|\leqslant n-3$, $|F_R|\leqslant n-4$, otherwise $n-2\geqslant|F|\geqslant|F_L|+|F_R|\geqslant 2n-6$, which contradicts the hypothesis of $n\geqslant 5$. By the induction hypothesis, $L-F_L$ contains a Hamilton cycle, say C_L. Choose $e=u_Lv_L\in C_L$ such that it is not incident with any edge in F_n if $F_n\neq\emptyset$. Let u_R and v_R be the neighbors of u_L and v_L in R. By Theorem 2.1, R contains a u_Rv_R-Hamilton path, say P_R. Then, $C_L-u_Lv_L+u_Lu_R+v_Lv_R+P_R$ is a Hamilton cycle in VQ_n-F.

The theorem follows. □

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