文章编号:0253-2778(2015)04-0259-09

A new estimate of DoA for saturated systems and its applications

SHANG Weike, ZUO Bin, SUN Yuanfeng, ZHANG Jinyuan

(*Research Center of Early Warning & Detection, CETC 38, Hefei* 230088, *China*)

Abstract: A new method for estimating the domain of attraction(DoA) for saturated systems was presented. Compared with the existing results, the advantage of the new result is mainly twofold: ① It does not include any product by the system matrix and the Lyapunov matrix; ② It does not result in heavy computing cost. It will be seen that these features are essentially important in system analysis. For comparison, the new result was extended to uncertain saturated systems, which shows that it leads to less conservativeness. Numerical examples verify the correctness of the conclusion.

Key words: saturated systems; poly-topic uncertain systems; domain of attraction; Lyapunov matrix

CLC number: TP13 Document code: A doi: 10.3969/j.issn.0253-2778.2015.04.001

Citation: SHANG Weike, ZUO Bin, SUN Yuanfeng, et al. A new estimate of DoA for saturated systems and its applications[J]. Journal of University of Science and Technology of China, 2015,45(4):259-267.
尚伟科,左斌,孙元峰,等. 一种新的饱和系统吸引域估计方法及应用[J]. 中国科学技术大学学报,2015,45(4):259-267.

一种新的饱和系统吸引域估计方法及应用

尚伟科,左 斌,孙元峰,张金元

(中国电子科技集团公司第三十八研究所预警探测研发中心,安徽合肥 230088))

摘要:提出了一种估计吸引域的新方法.与已有成果相比,该方法的优势主要体现在以下两个方面:①本文结果中不含有任何系统动态矩阵与Lyapunov矩阵的乘积形式;②所得结果不会引入繁重的计算代价.因此,该结果特别适用于系统分析.为了体现该方法获得结果的优势,将结果推广到不确定饱和系统.结果显示,新的方法对不确定饱和系统吸引域的估计具有较少的保守性.相应的数值仿真验证了结论的正确性.

关键词:饱和系统;多面体不确定系统;吸引域;Lyapunov 矩阵

0 Introduction

Saturated actuators are widely encountered in engineering. In the past decades, it has attracted much attention. In the field of saturated systems, the invariant set or DoA is an important topic.

Received: 2014-09-25; **Revised**: 2014-11-14

Foundation item: National Key Project of Science and Technology Supported Plan(2011BAH24B06).

Biography: SHANG Weike(corresponding author), male, born in 1984, PhD/Engineer. Research field: Stochastic system, radar signal processing. E-mail: wufo@mail.ustc.edu.cn

There has existed a lot of work, for example[1-4] and the reference therein. By placing the saturated control signal into the convex hull of a group of linear ones, Hu et. al. developed a method (Ref. [5] for discrete-time and Ref. [6] for continuous-time cases), and proved that it has the result of Ref. [2] as a special case. By choosing a saturation-dependent Lyapunov function, another result was presented in Ref. [7], and it was claimed that the result was less conservative. Alamo et al.[8] presented an SNS concept to estimate the DoA, and proved that it contains all possible estimations obtained by LDI through a complex recursive procedure. In Lu Ref. [9], a switching anti-windup method was designed to enlarge the DoA of the closed-loop system. As is known, uncertainty is often the potential cause of instability and poor performance, which is inevitable in engineering. Hence, the problem of presenting an applicable method to estimate the DoA of uncertain systems is of vital importance, which has not been solved as seen the references above.

In this note, a new result will be presented, and we shall prove that it is equivalent to both Refs. [5] and [7]. Moreover, the result does not involve any product by Lyapunov matrix and the system dynamics matrix. This 'decoupling' is essentially important for investigating on uncertain systems (see Refs. [10-12]), hence it allows the new result to be extended to uncertain saturated systems. To illustrate the effectiveness, the DoA estimation problems for uncertain saturated systems will be solved by both Hu, et. al.[5] and the new result. It will be seen that the new result has its own advantage in conservativeness reduction. Numerical examples are presented to verify the rightness.

1 A new result on estimation of DOA for saturated systems

1.1 Problem statement

The following discrete-time saturated system is to be investigated in this section.

$$\left.\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\mathrm{Sat}(u(k)) \\ u(k) &= \boldsymbol{F}\boldsymbol{x}(k) \end{aligned}\right\} \tag{1}$$

Here $\boldsymbol{x} \in \mathbb{R}^n$ denotes the system state, $u(k) \in \mathbb{R}^m$ is the control input. $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{F}$ are all constant matrices with compatible dimensions. The function $\mathrm{Sat}(.)$ $\mathbb{R}^m \to \mathbb{R}^m$ is the vector-valued saturation function, which is defined as follows:

$$\mathrm{Sat}(u) = [\mathrm{Sat}(u_1), \mathrm{Sat}(u_2), \cdots, \mathrm{Sat}(u_m)]^{\mathrm{T}}.$$

where,

$$\mathrm{Sat}(u_i) = \mathrm{sgn}(u_i) \min\{1; |u_i|\}.$$

Here, we have slightly abused the notation by using function $\mathrm{Sat}(\cdot)$ to denote both the scalar valued and the vector valued saturation functions. We recall some useful assumptions and results in Hu, et. al.[5].

Let f_j be the jth row of $\boldsymbol{F}$, and define $\mathscr{L}(\boldsymbol{F}) \triangleq \{\boldsymbol{x} \in \mathbb{R}^n : |f_j x| \leqslant 1;\ j \in [1\ m]\}$. Here $[1\ m]$ denotes the set containing all the integers from 1 to m. Note that $\mathscr{L}(\boldsymbol{F})$ denotes the region in which $u(k) = \mathrm{Sat}(\boldsymbol{F}\boldsymbol{x}(k))$ is linear in $\boldsymbol{x}(k)$.

Let $\boldsymbol{P}$ be a positive-definite matrix. For a positive scalar ρ, an ellipsoid $\xi(\boldsymbol{P},\rho) = \{\boldsymbol{x} \in \mathbb{R}^n : x^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x} \leqslant \rho\}$. Moreover, an ellipsoid $\xi(\boldsymbol{P},\rho)$ is inside of $\mathscr{L}(\boldsymbol{F})$ if and only if[4]

$$\rho f_j \boldsymbol{P}^{-1} f_j^{\mathrm{T}} \leqslant 1, \quad \forall j \in [1\ m] \tag{2}$$

Let $\boldsymbol{D}$ be the set of $m \times m$ diagonal matrices whose diagonal elements are either1 or 0. Hence, there are 2^m elements in $\boldsymbol{D}$. Each element of $\boldsymbol{D}$ is labelled $\boldsymbol{D}_j$; $j \in [1\ \ 2^m]$, then $\boldsymbol{D} = \{\boldsymbol{D}_j : j \in [1\ \ 2^m]\}$. Denote $\boldsymbol{D}_j^- = \boldsymbol{I} - \boldsymbol{D}_j$. Clearly, $\boldsymbol{D}_j^-$ is an element of $\boldsymbol{D}$ if and only if $\boldsymbol{D}_j \in \boldsymbol{D}$. In this paper, if not explicitly specified, '$*$' denotes the transpose of corresponding elements introduced by symmetry.

1.2 A new estimate of DoA for saturated systems

We recall the following proposition given in Ref. [5].

Proposition 1.1 Given two feedback matrices $\boldsymbol{F}$, $\boldsymbol{H} \in \mathbb{R}^{m\times n}$, suppose that $|h_j x| \leqslant 1$ for $j \in [1\ m]$, we have that

$$\mathrm{Sat}(\boldsymbol{F}x) \in \mathrm{co}\{\boldsymbol{D}_i\boldsymbol{F}\boldsymbol{x} + \boldsymbol{D}_i^-\boldsymbol{H}\boldsymbol{x} : i \in [1\ \ 2^m]\}.$$

where "co" denotes the convex hull of a set.

It follows

$$\mathrm{Sat}(Fx) = \sum_{i=1}^{2^m} \eta_i(x)(D_iF + D_i^- H)x ,$$

where $\eta_i(x)$ is dependent on x satisfying $\sum_{i=1}^{2^m} \eta_i(x) = 1.$

Based on the proposition, Hu, et. al.[5] presented a criterion to determine whether an ellipsoid is contractively invariant (obviously in the DoA). Then we could choose the "largest" ellipsoid to be an estimate of the DoA. To evaluate the "largest", two kinds of reference sets are usually used: ellipsoids defined as $\boldsymbol{X}_R = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{x} \leqslant 1\}$, $\boldsymbol{R} > 0$ or polyhedrons defined as $\boldsymbol{X}_R = \mathrm{co}\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_s\}$ $\boldsymbol{x}_s \in \mathbb{R}^n$, $k = [1 \quad s]$.

For a set $\boldsymbol{X}_R$ and $\boldsymbol{S} \subset \mathbb{R}^n$, define the largeness of $\boldsymbol{S}$ with respect to $\boldsymbol{X}_R$ as:

$$\lambda_R(\boldsymbol{S}) = \sup\{\lambda : \lambda \boldsymbol{X}_R \subset S\}.$$

Then the following work is to choose the "largest" contractively invariant ellipsoid to be the estimation of the DoA. Hu, et. al.[5] investigated the problem and achieved the following result.

Lemma 1. 1[5] If $\boldsymbol{X}_R = \mathrm{co}\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_s\}$, an estimate of the DoA for system (1) could be expressed as the following optimization problem:

$$\textbf{OP1} \quad \inf_{Q,Z} \ \gamma_1$$

$$\text{s. t. (a1)} \begin{bmatrix} \gamma_1 & \boldsymbol{x}_k^{\mathrm{T}} \\ \boldsymbol{x}_k & \boldsymbol{Q} \end{bmatrix} \geqslant 0, k \in [1 \ s];$$

$$\text{(b1)} \begin{bmatrix} \boldsymbol{Q} & * \\ (\boldsymbol{A}\boldsymbol{Q} + \boldsymbol{B}(\boldsymbol{D}_i\boldsymbol{F}\boldsymbol{Q} + \boldsymbol{D}_i^- \boldsymbol{Z})) & \boldsymbol{Q} \end{bmatrix} > 0,\ i \in [1 \quad 2^m];$$

$$\text{(c1)} \begin{bmatrix} 1 & \boldsymbol{z}_j \\ \boldsymbol{z}_j^{\mathrm{T}} & \boldsymbol{Q} \end{bmatrix} \geqslant 0,\ j \in [1 \ m].$$

Denote the optimal value as γ_1^*, then the size of the largest ellipse with respect to $\boldsymbol{X}_R$ is $\lambda_1^* = 1/\sqrt{\gamma_1^*}$.

The same problem is also investigated in Ref. [7] by choosing a so-called saturation-dependent Lyapunov function, and the following result is presented.

Lemma 1. 2[6] If $\boldsymbol{X}_R = \mathrm{co}\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_s\}$, an estimate of DoA for system (1) could be expressed as:

$$\textbf{OP2} \quad \inf_{Q_i > 0; Q; Z} \ \gamma_2$$

$$\text{s. t. (a2)} \begin{bmatrix} \gamma_2 & * \\ \boldsymbol{x}_k & \boldsymbol{Q}^{\mathrm{T}} + \boldsymbol{Q} - \boldsymbol{Q}_i \end{bmatrix} \geqslant 0, k \in [1 \ s], i \in [1 \quad 2^m];$$

$$\text{(b2)} \begin{bmatrix} \boldsymbol{Q}_i & * \\ (\boldsymbol{A}\boldsymbol{Q} + \boldsymbol{B}(\boldsymbol{D}_i\boldsymbol{F}\boldsymbol{Q} + \boldsymbol{D}_i^- \boldsymbol{Z})) & \boldsymbol{Q}^{\mathrm{T}} + \boldsymbol{Q} - \boldsymbol{Q}_j \end{bmatrix} > 0, \forall i, j \in [1 \quad 2^m];$$

$$\text{(c2)} \begin{bmatrix} 1 & * \\ \boldsymbol{z}_j^{\mathrm{T}} & \boldsymbol{Q}_i \end{bmatrix} \geqslant 0,\ \forall j \in [1 \ m], i \in [1 \quad 2^m].$$

We denote the optimal value as γ_2^*.

In Ref. [7], it was claimed that Lemma 1. 2 is less conservative, as it covers Lemma 1. 1 by letting $\boldsymbol{Q} = \boldsymbol{Q}^{\mathrm{T}} = \boldsymbol{Q}_i$. In the remainder of the subsection, we will present a new result to reduce the complexity. Moreover, we shall prove that our result is equivalent to both Lemma 1. 1 and Lemma 1. 2, which indicates that Ref. [7] did not reduce the conservativeness as it was declared.

Theorem 1. 1 If $\boldsymbol{X}_R = \mathrm{co}\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_s\}$, we conclude that the following optimization problem is equivalent to both Lemma 1. 1 and Lemma 1. 2.

$$\textbf{OP3}\quad \inf_{Q_1>0;Q;Z}\ \gamma_3$$

$$\text{s. t. (a3)}\ \begin{bmatrix}\gamma_3 & * \\ \boldsymbol{x}_k & \boldsymbol{Q}^{\mathrm{T}}+\boldsymbol{Q}-\boldsymbol{Q}_1\end{bmatrix}\geqslant 0, k\in[1\ s];$$

$$\text{(b3)}\ \begin{bmatrix}\boldsymbol{Q}_1 & * \\ (\boldsymbol{AQ}+\boldsymbol{B}(\boldsymbol{D}_i\boldsymbol{FQ}+\boldsymbol{D}_i^-\boldsymbol{Z})) & \boldsymbol{Q}^{\mathrm{T}}+\boldsymbol{Q}-\boldsymbol{Q}_1\end{bmatrix}>0, \forall i\in[1\quad 2^m];$$

$$\text{(c3)}\ \begin{bmatrix}1 & * \\ \boldsymbol{z}_j^{\mathrm{T}} & \boldsymbol{Q}_1\end{bmatrix}\geqslant 0,\ \forall j\in[1\ m].$$

The corresponding optimal value is denoted as γ_3^* .

Proof $\gamma_2^* \leqslant \gamma_1^*$: Note that **OP2** covers **OP1** by letting

$$\boldsymbol{Q}^{\mathrm{T}}=\boldsymbol{Q}=\boldsymbol{Q}_i,\ \forall i\in[1\quad 2^m].$$

That is to say, the constraints of **OP1** are more restrictive. Hence, a less infimum may be obtained in **OP2** , which implies that $\gamma_2^* \leqslant \gamma_1^*$.

$\gamma_3^* \leqslant \gamma_2^*$: Note that **OP3** owns only part of the constraints of **OP2** , hence **OP3** is less conservative, which leads to $\gamma_3^* \leqslant \gamma_2^*$.

$\gamma_1^* \leqslant \gamma_3^*$: For $\boldsymbol{Q}_1 > 0$, we deduce that $\boldsymbol{Q}$ in **OP3** is of full rank. It is noted that

$$\boldsymbol{Q}^{\mathrm{T}}+\boldsymbol{Q}-\boldsymbol{Q}_1\leqslant \boldsymbol{QQ}_1^{-1}\boldsymbol{Q}^{\mathrm{T}} \tag{3}$$

Replacing $\boldsymbol{Q}^{\mathrm{T}}+\boldsymbol{Q}-\boldsymbol{Q}_1$ with $\boldsymbol{QQ}_1^{-1}\boldsymbol{Q}^{\mathrm{T}}$ leads to:

$$\textbf{OP4}\quad \inf_{Q_1>0;Q;Z}$$

$$\text{s. t. (a4)}\ \begin{bmatrix}\gamma_4 & * \\ \boldsymbol{x}_k & \boldsymbol{QQ}_1^{-1}\boldsymbol{Q}^{\mathrm{T}}\end{bmatrix}\geqslant 0, k\in[1\ s];$$

$$\text{(b4)}\ \begin{bmatrix}\boldsymbol{Q}_1 & * \\ (\boldsymbol{AQ}+\boldsymbol{B}(\boldsymbol{D}_i\boldsymbol{FQ}+\boldsymbol{D}_i^-\boldsymbol{Z})) & \boldsymbol{QQ}_1^{-1}\boldsymbol{Q}^{\mathrm{T}}\end{bmatrix}>0, \forall i\in[1\quad 2^m];$$

$$\text{(c4)}\ \begin{bmatrix}1 & * \\ \boldsymbol{z}^{\mathrm{T}}j & \boldsymbol{Q}_1\end{bmatrix}\geqslant 0,\ \forall j\in[1\ m].$$

The optimal value is denoted as γ_4^* . Owing to Eq. (3), we have $\gamma_4^* \leqslant \gamma_3^*$.

Let $\boldsymbol{QQ}_1^{-1}\boldsymbol{Q}^{\mathrm{T}} = \boldsymbol{T} > 0$, Then (a4) is transformed into

$$\text{(aa4)}\ \begin{bmatrix}\gamma_4 & \boldsymbol{x}_k^{\mathrm{T}} \\ \boldsymbol{x}_k & \boldsymbol{T}\end{bmatrix}\geqslant 0,\ k\in[1\ s].$$

Pre- and post-multiplying (b4) with $\begin{bmatrix}\boldsymbol{TQ}^{-\mathrm{T}} & 0 \\ 0 & \boldsymbol{I}\end{bmatrix}$ and it transpose respectively, we have that (b4) is equivalent to

$$\begin{bmatrix}\boldsymbol{T} & * \\ \boldsymbol{AT}+\boldsymbol{B}(\boldsymbol{D}_i\boldsymbol{FT}+\boldsymbol{D}_i^-\boldsymbol{ZQ}^{-1}\boldsymbol{T}) & \boldsymbol{T}\end{bmatrix}>0,\ \forall i\in[1\quad 2^m].$$

Letting $\boldsymbol{ZQ}^{-1}\boldsymbol{T}=\boldsymbol{Y}$, it follows

$$\text{(bb4)}\ \begin{bmatrix}\boldsymbol{T} & * \\ \boldsymbol{AT}+\boldsymbol{B}(\boldsymbol{D}_i\boldsymbol{FT}+\boldsymbol{D}_i^-\boldsymbol{Y}) & \boldsymbol{T}\end{bmatrix}>0,$$
$$\forall i\in[1\quad 2^m].$$

Pre- and post-multiplying (c4) with $\begin{bmatrix}\boldsymbol{I} & 0 \\ 0 & \boldsymbol{TQ}^{-\mathrm{T}}\end{bmatrix}$ and its transpose respectively yields

$$\begin{bmatrix}1 & * \\ (\boldsymbol{z}_j\boldsymbol{Q}^{-1}\boldsymbol{T})^{\mathrm{T}} & \boldsymbol{T}\end{bmatrix}>0,\ \forall j\in[1\ m].$$

Substituting $\boldsymbol{z}_j\boldsymbol{Q}^{-1}\boldsymbol{T}$ with y_j , we have (c4) is equivalent to

$$\text{(cc4)}\ \begin{bmatrix}1 & \boldsymbol{y}_j \\ \boldsymbol{y}_j^T & \boldsymbol{T}\end{bmatrix}\geqslant 0,\ \forall j\in[1\ m].$$

Then **OP4** is transformed into the following equivalent optimization problem **OP4**.

$$\widetilde{\textbf{OP4}}\quad \inf_{T,Y}\ \tilde{\gamma}_4$$
$$\text{s. t. (aa4)}-\text{(cc4)}.$$

The optimal value of $\widetilde{\textbf{OP4}}$ is $\tilde{\gamma}_4^*$, and obviously $\tilde{\gamma}_4^* = \gamma_4^*$.

Note that $\widetilde{\textbf{OP4}}$ is just the same as **OP1** , hence

$\gamma_1^* = \tilde{\gamma}_4^* = \gamma_4^*$. Wing to $r_4^* \leqslant r_3^*$, we have $r_1^* \leqslant r_3^*$.

By now, we have shown $\gamma_1^* = \gamma_2^* = \gamma_3^*$, hence we declare that **OP1**, **OP2** and **OP3** are equivalent to each other.

Remark 1.1 Note that the order of complexity in **OP3** is 2^m, which is only $\frac{1}{2^m}$ of **OP2**, hence the computation cost is obviously reduced, especially for high-order systems.

Remark 1.2 Compared with **OP1**, **OP3** introducs a slack variable $\boldsymbol{Q}$, which is not even required to be symmetric. It doesn't involve any product between Lyapunov matrix and the system dynamics, which may help to construct a parameter-dependent Lyapunov function to reduce conservativeness. And also due to the extra freedom (See de Oliveira et al.[9], Ref. [10] for the discrete-time system case, and Cao et al.[11] for the continuous-time system counterparts), further results may be obtained for uncertain systems.

Remark 1.3 For an ellipse reference set $\boldsymbol{X}_R = \xi(\boldsymbol{R},1)$, we only have to replace (a1) with $\begin{bmatrix} \gamma\boldsymbol{R} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{Q} \end{bmatrix} \geqslant 0$, (a2) with $\begin{bmatrix} \gamma\boldsymbol{R} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{Q}^{\mathrm{T}} + \boldsymbol{Q} - \boldsymbol{Q}_i \end{bmatrix} \geqslant 0$, $i \in [1 \quad 2^m]$, and (a3) with $\begin{bmatrix} \gamma\boldsymbol{R} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{Q}^{\mathrm{T}} + \boldsymbol{Q} - \boldsymbol{Q}_1 \end{bmatrix} \geqslant 0$.

1.3 A numerical example

Example 1.1 Consider the following system dynamics:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0.5 \\ 1.0 \end{bmatrix},$$

$$\boldsymbol{F} = [-0.6167 \quad -1.2703].$$

$\boldsymbol{X}_R = \xi(\boldsymbol{R},1)$ is the reference set, with $\boldsymbol{R} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

Utilizing the Mincx solver in LMI toolbox of Matlab, we solve Lemmas 1.1, 1.2 and Theorem 1.1 and obtain

OP1: $\gamma_1^* = 0.1724$, $\boldsymbol{Z} = [-0.3721 \quad -2.8658]$, and $\boldsymbol{Q} = \begin{bmatrix} 44.9992 & -14.1592 \\ -14.1592 & 13.3806 \end{bmatrix}$.

OP2: $\gamma_2^* = 0.1724$, $\boldsymbol{Z} = [-0.3721 \quad -2.8658]$, and $\boldsymbol{Q} = \boldsymbol{Q}_1 = \boldsymbol{Q}_2 = \begin{bmatrix} 44.9992 & -14.1592 \\ -14.1592 & 13.3806 \end{bmatrix}$.

OP3: $\gamma_3^* = 0.1724$, $\boldsymbol{Z} = [-0.3721 \quad -2.8658]$, and $\boldsymbol{Q} = \boldsymbol{Q}_1 = \begin{bmatrix} 44.9992 & -14.1592 \\ -14.1592 & 13.3806 \end{bmatrix}$.

Obviously $\gamma_1^* = \gamma_2^* = \gamma_3^*$, hence the numerical example shows that **OP1** ~ **OP3** are equivalent to each other.

2 Extension to poly-topic uncertain saturated systems

In this section, the DoA estimation problem is investigated for poly-topic uncertain satuated systems. We will investigate the problem first with Lemma 1.1, and then with Theorem 1.1. we shall see that Theorem 1.1 leads to a less conservative solution when it is applied to uncertain systems.

2.1 Problem statement

Let us consider the following discrete-time uncertain saturated systems.

$$\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{A}(\alpha)\boldsymbol{x}(k) + \boldsymbol{B}Sat(u(k)) \\ u(k) = \boldsymbol{Fx}(k) \end{cases} \tag{4}$$

$\boldsymbol{x} \in \mathbb{R}^n$ denotes the state vector and $u \in \mathbb{R}^m$ is the control input. $\boldsymbol{F}$ is the feedback gain. The dynamic matrix $\boldsymbol{A}(\alpha)$ belongs to a convex poly-topic set defined as:

$$\mathscr{A}: = \left\{\boldsymbol{A}(\alpha) : \boldsymbol{A}(\alpha) = \sum_{i=1}^{M} \alpha_i \boldsymbol{A}_i,\ \sum_{i=1}^{M} \alpha_i = 1,\ \alpha_i \geqslant 0\right\}.$$

In the next subsection, we are aiming at giving an estimate of DoA for Eq. (4).

2.2 Estimation of DoA for uncertain saturated systems

Now, we shall investigate the problem with Lemma 1.1. First of all, we consider the following problem: If $\xi(\boldsymbol{P},\rho)$ is in the DoA of all the vertex systems (the systems whose dynamic matrices are given by the vertices of $\boldsymbol{A}(\alpha)$, then under what conditions is $\xi(\boldsymbol{P},\rho)$ also in the DoA of the entitle system (4)? Should this question be answered, we would take $\xi(\boldsymbol{P},\rho)$ as an estimate of DoA for system (4).

Lemma 2.1 For an ellipse $\xi(\boldsymbol{P},\rho)$, if there exists $\boldsymbol{H} \in \mathbb{R}^{m\times n}$, such that

$$(\boldsymbol{A}_i+\boldsymbol{B}(\boldsymbol{D}_j\boldsymbol{F}+\boldsymbol{D}_j^-\boldsymbol{H}))^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{A}_i+\boldsymbol{B}(\boldsymbol{D}_j\boldsymbol{F}+\boldsymbol{D}_j^-\boldsymbol{H}))-\boldsymbol{P}<0,\ \forall i\in[1\ M],\ j\in[1\quad 2^m] \tag{5}$$

and $\xi(\boldsymbol{P},\rho)\in\mathscr{L}(\boldsymbol{H})$, then we conclude $\xi(\boldsymbol{P},\rho)$ is a contractively invariant set (in the DoA) for Eq. (4).

Proof By Schur complement lemma and convexity, Eq. (5) leads to

$$\sum_{i=1}^{M}\alpha_i\begin{bmatrix}\boldsymbol{P} & *\\ \boldsymbol{A}_i+\boldsymbol{B}(\boldsymbol{D}_j\boldsymbol{F}+\boldsymbol{D}_j^-\boldsymbol{H})) & \boldsymbol{P}^{-1}\end{bmatrix}>0,\ \forall i\in[1\ M],\ j\in[1\quad 2^m] \tag{6}$$

That is

$$\begin{bmatrix}\boldsymbol{P} & *\\ \boldsymbol{A}(\alpha)+\boldsymbol{B}(\boldsymbol{D}_j\boldsymbol{F}+\boldsymbol{D}_j^-\boldsymbol{H})) & \boldsymbol{P}^{-1}\end{bmatrix}>0,\ \forall j\in[1\quad 2^m] \tag{7}$$

which is equivalent to

$$(\boldsymbol{A}(\alpha)+\boldsymbol{B}(\boldsymbol{D}_j\boldsymbol{F}+\boldsymbol{D}_j^-\boldsymbol{H}_i))^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{A}(\alpha)+\boldsymbol{B}(\boldsymbol{D}_j\boldsymbol{F}+\boldsymbol{D}_j^-\boldsymbol{H}))-\boldsymbol{P}<0,\ \forall j\in[1\quad 2^m] \tag{8}$$

Note $\xi(\boldsymbol{P},\rho)\subset\mathscr{L}(\boldsymbol{H})$, therefor $\xi(\boldsymbol{P},\rho)$ is a contractively invariant set for Eq. (4).

Based on Lemma 2.1, we have the following theorem.

Theorem 2.1 Assuming $\boldsymbol{X}_R=\mathrm{co}\{\boldsymbol{x}_1,\cdots,\boldsymbol{x}_s\}$, we could give an estimate of DoA for system (4) by solving the following problem

$$\sup_{P>0,\rho,H}\ \lambda_5$$

s. t. (a) $\lambda_5\boldsymbol{X}_R\subset\xi(\boldsymbol{P},\rho)$;

(b) $(\boldsymbol{A}_i+\boldsymbol{B}(\boldsymbol{D}_j\boldsymbol{F}+\boldsymbol{D}_j^-\boldsymbol{H}))^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{A}_i+\boldsymbol{B}(\boldsymbol{D}_j\boldsymbol{F}+\boldsymbol{D}_j^-\boldsymbol{H}))-\boldsymbol{P}<0,\ \forall i\in[1\ M],\ j\in[1\quad 2^m]$;

(c) $\xi(\boldsymbol{P},\rho)\subset\mathscr{L}(\boldsymbol{H})$.

It could be easily transformed as

OP5 $\inf\limits_{Q,Z}\ \gamma_5$

s. t. (a5) $\begin{bmatrix}\gamma_5 & *\\ \boldsymbol{x}_k & \boldsymbol{Q}\end{bmatrix}\geqslant 0,\ k\in[1\quad s]$;

(b5) $\begin{bmatrix}\boldsymbol{Q} & *\\ (\boldsymbol{A}_i\boldsymbol{Q}+\boldsymbol{B}(\boldsymbol{D}_j\boldsymbol{F}\boldsymbol{Q}+\boldsymbol{D}_j^-\boldsymbol{Z}) & \boldsymbol{Q}\end{bmatrix}>0,\ i\in[1\ M],\ j\in[1\quad 2^m]$;

(c5) $\begin{bmatrix}1 & *\\ \boldsymbol{z}_j^{\mathrm{T}} & \boldsymbol{Q}\end{bmatrix}\geqslant 0,\ j\in[1\quad m]$.

Denote the optimal value of γ_5 as γ_5^*, the corresponding size of the estimate of DoA is given by $\lambda_5^*=1/\sqrt{\gamma_5^*}$

Proof Based on the analysis above, the proof is trivial, hence we omit it here.

The deficiency of the above result lies in choosing a single Lyapunov function to check the stability of the entire system with uncertainty. In other words, we take the common contractively invariant set of all the vertex systems as the contractively invariant set of Eq. (4). As is claimed in the previous section, a less conservative result may be obtained by utilizing Theorem 1.1 due to its ‘decoupling’ characteristics. This inspires us to propose the question: Given the contractively invariant set $\xi(\boldsymbol{P}_i,\rho)$ of every vertex system, could we conclude that there exits the contractively invariant set $\xi(\boldsymbol{P}(\alpha),\rho)$ of the whole uncertain system (4)?

Lemma 2.2 If there exists $\xi(\boldsymbol{P}_i,\rho)$ and $\boldsymbol{H}_i\in\mathbb{R}^{m\times n}$, satisfying $\xi(\boldsymbol{P}_i,\rho)\subset\mathscr{L}(\boldsymbol{H}_i)$, such that

$$\begin{bmatrix}\boldsymbol{P}_i & *\\ \boldsymbol{X}[\boldsymbol{A}_i+\boldsymbol{B}(\boldsymbol{E}_j\boldsymbol{F}+\boldsymbol{E}_j^-\boldsymbol{H}_i)] & \boldsymbol{X}^{\mathrm{T}}+\boldsymbol{X}-\boldsymbol{P}_i\end{bmatrix}>0,\ \forall i\in[1\ M],\ j\in[1\quad 2^m] \tag{9}$$

Then we conclude there exist $\xi(\boldsymbol{P}(\alpha),\rho)$ and $\boldsymbol{H}(\alpha)$, satisfying $\xi(\boldsymbol{P}(\alpha),\rho(\alpha))\subset\mathscr{L}(\boldsymbol{H}(\alpha))$, such that

$$(\boldsymbol{A}(\alpha)+\boldsymbol{B}(\boldsymbol{D}_j\boldsymbol{F}+\boldsymbol{D}_j^-\boldsymbol{H}(\alpha)))^{\mathrm{T}}\boldsymbol{P}(\alpha)(\boldsymbol{A}(\alpha)+\boldsymbol{B}(\boldsymbol{D}_j\boldsymbol{F}+\boldsymbol{D}_j^-\boldsymbol{H}(\alpha)))-\boldsymbol{P}(\alpha)<0,\ \forall j\in[1\quad 2^m] \tag{10}$$

Hence, $\xi(\boldsymbol{P}(\alpha),\rho(\alpha))$ is a contractively the invariant set of system(4).

Proof Note that $\xi(\boldsymbol{P}_i,\rho_i)\subset\mathscr{L}(\boldsymbol{H}_i)$, and also by convexity, we have

$$\sum_{i=1}^{M}\alpha_i\begin{bmatrix}1/\rho & \boldsymbol{h}_{ij}\\ \boldsymbol{h}_{ij}^{\mathrm{T}} & \boldsymbol{P}_i\end{bmatrix}\geqslant 0,\ \forall j\in[1\ m] \tag{11}$$

Denote $\sum_{i=1}^{M}\alpha_i\boldsymbol{P}_i:=\boldsymbol{P}(\alpha)>0$, $\sum_{i=1}^{M}\alpha_i\boldsymbol{H}_i:=\boldsymbol{H}(\alpha)$, and let $\boldsymbol{h}_j(\alpha)$ denote the jth line of $\boldsymbol{H}(\alpha)$. It yields

$$\begin{bmatrix}1/\rho & \boldsymbol{h}_j(\alpha)\\ \boldsymbol{h}_j^{\mathrm{T}}(\alpha) & \boldsymbol{P}(\alpha)\end{bmatrix}\geqslant 0,\ \forall j\in[1\ m] \tag{12}$$

Hence,

$$\xi(\boldsymbol{P}(\alpha),\rho(\alpha))\subset\mathscr{L}(H(\alpha)) \tag{13}$$

(9) leads to

$$\sum_{i=1}^{M}\alpha_i\begin{bmatrix}\boldsymbol{P}_i & *\\ \boldsymbol{X}(\boldsymbol{A}_i+\boldsymbol{B}(\boldsymbol{D}_j\boldsymbol{F}+\boldsymbol{D}_j^-\boldsymbol{H}_i)) & \boldsymbol{X}^{\mathrm{T}}+\boldsymbol{X}-\boldsymbol{P}_i\end{bmatrix}>0,\ \forall j\in[1\quad 2^m] \tag{14}$$

It follows

$$\begin{bmatrix}\boldsymbol{P}(\alpha) & *\\ \boldsymbol{X}(\boldsymbol{A}(\alpha)+\boldsymbol{B}(\boldsymbol{D}_j\boldsymbol{F}+\boldsymbol{D}_j^-\boldsymbol{H}(\alpha))) & \boldsymbol{X}^{\mathrm{T}}+\boldsymbol{X}-\boldsymbol{P}(\alpha)\end{bmatrix}>0,\ \forall j\in[1\quad 2^m] \tag{15}$$

Owing to $\boldsymbol{X}^{\mathrm{T}}+\boldsymbol{X}-\boldsymbol{P}(\alpha)\leqslant\boldsymbol{X}\boldsymbol{P}^{-1}(\alpha)\boldsymbol{X}^{\mathrm{T}}$,
it holds

$$\begin{bmatrix}\boldsymbol{P}(\alpha) & *\\ \boldsymbol{X}(\boldsymbol{A}(\alpha)+\boldsymbol{B}(\boldsymbol{D}_j\boldsymbol{F}+\boldsymbol{D}_j^-\boldsymbol{H}(\alpha))) & \boldsymbol{X}\boldsymbol{P}^{-1}(\alpha)\boldsymbol{X}^{\mathrm{T}}\end{bmatrix}>0,\ \forall j\in[1\quad 2^m] \tag{16}$$

By Schur complement lemma, we have

$$(\boldsymbol{A}(\alpha)+\boldsymbol{B}(\boldsymbol{D}_j\boldsymbol{F}+\boldsymbol{D}_j^-\boldsymbol{H}(\alpha)))^{\mathrm{T}}\boldsymbol{P}(\alpha)(\boldsymbol{A}(\alpha)+\boldsymbol{B}(\boldsymbol{D}_j\boldsymbol{F}+\boldsymbol{D}_j^-\boldsymbol{H}(\alpha)))-\boldsymbol{P}(\alpha)<0,\ \forall j\in[1\quad 2^m] \tag{17}$$

Eq. (13) together with Eq. (17) ensures that $\xi(\boldsymbol{P}(\alpha),\rho)$ is a contractively invariant set of system (4).

Lemma 2.2 tells us the existence of $\xi(\boldsymbol{P}(\alpha),\rho)$. However, it's unlikely to be easily constructed, because we don't know α exactly. Since

$$\xi(\boldsymbol{P}(\alpha),\rho)=\left\{\boldsymbol{x}:\boldsymbol{x}^{\mathrm{T}}\left(\sum_{i=1}^{M}\alpha_i\boldsymbol{P}_i\right)x\leqslant\rho\right\},$$

and

$$\mathscr{L}(\boldsymbol{H}(\alpha))=\left\{\boldsymbol{x}:\left|\left(\sum_{i=1}^{M}\alpha_i h_j\right)x\right|\leqslant 1,\ \forall j\in[1\ m]\right\}.$$

We deduce

$$\bigcap_{i=1}^{M}\xi(\boldsymbol{P}_i,\rho)\subset\xi(\boldsymbol{P}(\alpha),\rho),$$

and

$$\bigcap_{i=1}^{M}\mathscr{L}(\boldsymbol{H}_i)\subset\mathscr{L}(\boldsymbol{H}(\alpha)).$$

Owing to $\xi(\boldsymbol{P}_i,\rho)\subset\mathscr{L}(\boldsymbol{H}_i)$, we also have

$$\bigcap_{i=1}^{M}\xi(\boldsymbol{P}_i,\rho)\subset\bigcap_{i=1}^{M}\mathscr{L}(\boldsymbol{H}_i).$$

Then we have no choice but to choose a smaller set $\bigcap_{i=1}^{M}\xi(\boldsymbol{P}_i,\rho)$ in place of $\xi(\boldsymbol{P}(\alpha),\rho)$) to be the estimation of DoA. However, we will see that it is still less conservative than Theorem 2.1. The optimization problem is transformed as:

$$\sup_{P_i>0,\rho>0,H_i}\ \lambda$$

s. t. (a) $\lambda\boldsymbol{X}_R\subset\xi(\boldsymbol{P}_i,\rho)$, $\forall i\in[1\quad M]$;

(b) $(\boldsymbol{A}_i+\boldsymbol{B}(\boldsymbol{D}_j\boldsymbol{F}+\boldsymbol{D}_j^-\boldsymbol{H}_i))^{\mathrm{T}}P_i(\boldsymbol{A}_i+\boldsymbol{B}(\boldsymbol{D}_j\boldsymbol{F}+\boldsymbol{D}_j^-\boldsymbol{H}_i))-\boldsymbol{P}_i<0$, $\forall i\in[1\ M]$, $j\in[1\quad 2^m]$;

(c) $\xi(\boldsymbol{P}_i,\rho)\subset\mathscr{L}(\boldsymbol{H}_i)$, $\forall i\in[1\quad M]$.

By now, we are able to present a less conservative estimate of DoA for system (4)

Theorem 2.2 Assuming $\boldsymbol{X}_R=\mathrm{co}\{\boldsymbol{x}_1,\cdots,\boldsymbol{x}_s\}$, we could give an estimate of DoA for system (4) by solving the following problem

$$\text{OP6} \quad \inf_{Q_{1i}>0, Q, Z_i, i\in[1\ M]} \gamma_6$$

$$\text{s. t. (a6)} \begin{bmatrix} \gamma_3 & \boldsymbol{x}_k^{\mathrm{T}} \\ \boldsymbol{x}_k & \boldsymbol{Q}^{\mathrm{T}}+\boldsymbol{Q}-\boldsymbol{Q}_{1i} \end{bmatrix} \geqslant 0,\ \forall k\in[1\ s],\ i\in[1\ M];$$

$$\text{(b6)} \begin{bmatrix} \boldsymbol{Q}_{1i} & * \\ (\boldsymbol{A}_i\boldsymbol{Q}+\boldsymbol{B}(\boldsymbol{D}_j\boldsymbol{F}\boldsymbol{Q}+\boldsymbol{D}_j^-\boldsymbol{Z}_i)) & \boldsymbol{Q}^{\mathrm{T}}+\boldsymbol{Q}-\boldsymbol{Q}_{1i} \end{bmatrix} > 0,\ \forall i\in[1\ M],\ j\in[1\quad 2^m];$$

$$\text{(c6)} \begin{bmatrix} 1 & \boldsymbol{z}_{ij} \\ \boldsymbol{z}_{ij}^{\mathrm{T}} & \boldsymbol{Q}_{1i} \end{bmatrix} \geqslant 0,\ i\in[1\quad M],\ j\in[1\quad m].$$

Denote the optimal value of γ_6 as γ_6^* , the corresponding size of the estimate of DoA is given by $\gamma_6^* = 1/\sqrt{\gamma_6^*}$

Proof The proof is omitted for its simplicity.

Remark 2. 1 It is noted that Theorem 2. 2 could cover Theorem 2. 1 by restricting $\boldsymbol{Q}=\boldsymbol{Q}^{\mathrm{T}}=\boldsymbol{Q}_{1i}$, hence Theorem 2. 2 is less conservative. For an ellipse reference set $\boldsymbol{X}_R=\xi(\boldsymbol{R},1)$, we only have to replace (a5) with $\begin{bmatrix} \gamma\boldsymbol{R} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{Q} \end{bmatrix}$, and (a6) with

$$\begin{bmatrix} \gamma\boldsymbol{R} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{Q}^{\mathrm{T}}+\boldsymbol{Q}-\boldsymbol{Q}_i \end{bmatrix} \geqslant 0,\ i\in[1\quad M].$$

2. 3 A numerical example

In this subsection, we consider the following uncertain saturated systems.

$$\left.\begin{aligned} \boldsymbol{x}(k+1) &= \left(\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}+\xi\begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}\right)\boldsymbol{x}(k)+\begin{bmatrix}0.5\\1.0\end{bmatrix}\mathrm{Sat}(u(k)) \\ u(k) &= \boldsymbol{F}\boldsymbol{x}(k) \end{aligned}\right\}$$

where the uncertain parameter ξ is unknown, but belongs to the known range [0, 0. 25]. That is, the vertices of the poly-topic uncertain system are

$$\boldsymbol{A}_1=\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix},\ \boldsymbol{A}_2=\begin{bmatrix}1 & 1.25\\0.25 & 1.25\end{bmatrix}.$$

We might as well choose $\boldsymbol{X}_R$ as a unit ball, i. e. $\boldsymbol{X}_R=\xi(\boldsymbol{I},1)$; $F=[-0.6167\quad -1.2703]$.

By solving Theorem 2. 1, we obtain that $\gamma_5^* = 0.5021$, hence the size of the estimate of DoA is $\lambda_5^8 = 1.4113$. The other parameters are

$$\boldsymbol{Z}=[-0.6721\quad -1.2499],$$

$$\boldsymbol{Q}=\begin{bmatrix}12.8757 & -4.3482\\ -4.3482 & 3.7290\end{bmatrix}.$$

By solving Theorem 2. 2, we obtain that $\gamma_6^* = 0.4183$, which implies the estimate of DoA is given by $\gamma_6^* = 1.5462$. The other parameters are

$$\boldsymbol{Z}_1=[-1.0777\quad -1.0221],$$

$$\boldsymbol{Z}_2=[-1.3645\quad -1.1612],$$

$$\boldsymbol{Q}=\begin{bmatrix}35.8653 & -11.1976\\ -11.1976 & 6.1421\end{bmatrix},$$

$$\boldsymbol{Q}_{11}=\begin{bmatrix}55.2209 & -14.6295\\ -14.6295 & 5.6224\end{bmatrix},$$

$$\boldsymbol{Q}_{12}=\begin{bmatrix}26.8412 & -7.9035\\ -7.9035 & 4.9521\end{bmatrix}.$$

Obviously, $\gamma_6^* > \gamma_5^*$, hence we have obtained a larger estimate of DoA for system (4) by theorem 2. 2.

Finally, to further illustrate the effectiveness of our results, we will consider the problem of uncertainty tolerance. From the information above, both Theorems 2. 1 and 2. 2 conclude that the unit ball $\xi(\boldsymbol{I},1)$ is in the DoA of system ($\gamma_5^* = 1.4113$ and $\gamma_6^* = 1.5462$) . Our aim is to find the max value of ξ to keep the unit ball inside the DoA of system (4). Then the result given by theorem 2. 1 is $\xi^* = 0.44942$, while theorem 2. 2 gives $\xi^* = 0.47625$. Again, we see that the new result obtained in the previous section is more effective in coping with poly-topic uncertain systems with actuator saturation.

3 Conclusion

A new method is presented to estimate the DoA for saturated systems. Due to the fact that the result does not involve any product by Lyapunov matrix and the system dynamic matrix, it provides more ‘freedom’ for system analysis.

As an application, the new result is extended to investigate uncertain saturated systems, and a less conservative result is obtained. Future work is to be focused on further conservativeness reduction.

References

[1] GilbertEG, Tan K T. Linear systems with state and control constraints: The theory and application of maximal output admissible sets[J]. IEEE Transactions on Automatic Control, 1991, 36(9): 1008-1020.

[2] Khalil H K. Nonlinear Systems[M]. Upper Saddle River: Prentice-Hall, 1996.

[3] PittetC, Tarbouriech S, Burdat C. Stability regions for linear systems with saturating controls via circle and Popov criteria[C]// Proceedings of the 36th IEEE Conference on Decision & Control. San Diego, USA: IEEE Press, 1997: 4518-4523.

[4] Hu T S, Lin Z L. Control Systems with Actuator Saturation: Analysis and Design [M]. Berlin: Birkhäuser, 2001.

[5] Hu T S, Lin Z L, Chen B M. Analysis and design for discrete-time linear systems subject to actuator saturation[J]. Systems & Control Letters, 2002, 45(2): 97-112.

[6] Hu T S, Lin Z L, Chen B M. An analysis and design method for linear systems subject to actuator saturation and disturbance[J]. Automatica, 2002, 38(2): 351-359.

[7] Cao YY, Lin Z L. Stability analysis of discrete-time systems with actuator saturation by a saturation-dependent Lyapunov function[J]. Automatica, 2003, 39(7): 1235-1241.

[8] Alamo T, Cepeda A, Limon D, et al. A new concept of invariance for saturated systems[J]. Automatica, 2006, 42(9): 1515-1521.

[9] Lu L, Lin Z L. A switching anti-windup design using multiple Lyapunov functions[J]. IEEE Transactions on Automatic Control, 2009, 55(1): 142-148.

[10] Daafouz J, Bernussou J. Parameter dependent Lyapunov functions for discrete time systems with time varying parametric uncertainties[J]. Systems & Control Letters, 2001, 43(5): 355-359.

[11] CaoY Y, Lin Z L. A descriptor system approach to robust stability analysis and controller synthesis[J]. IEEE Transactions on Automatic Control, 2004, 49(11): 2081-2084.

[12] de Oliveira M C, Bernussou J, Geromel J C. A new discrete-time robust stability condition[J]. Systems & Control Letters, 1999, 37(4): 261-265.