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位置参数变点估计的收敛速度

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摘要:对至多一个位置参数变点的模型,基于反对称核函数,运用累积和(cumulative sum, CUSUM)方法给出了变点的U统计量型估计量,并研究了变点估计量的相合性和收敛速度,同时也给出了局部对立条件下估计量的相合性.最后进行了不同样本容量和参数设置下变点估计的模拟分析,从直方图可以看出:变点位置越靠近中间位置,估计越精确.模拟结果显示所给出的估计量的有效性.

关键词:变点;U统计量;累积和;相合性;收敛速度

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Convergence rate of location parameter change-point estimator

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Abstract: For the location parameter change point model allowing at most one change, an antisymmetric kernel function was constructed and a change point estimator was proposed with the help of the CUSUM method. The consistency and convergence rate of the change point estimator were studied under normal conditions and its consistency was given under the condition of local alternative hypothesis. Finally, simulation was performed under different sample sizes and parameter settings. From the histograms, the closer the change-point location gets to the middle, the more accurate the estimate is. The results show that our methods are effective.

Key words: change point; U-statistics; cumulative sum; consistency; convergence rate

0 引言

设 $X_1, X_2, \cdots, X_n$ 为一列独立的随机变量序列,满足

$$X_1, X_2, \cdots, X_{[n\tau_0]}\ \text{i.i.d.} \sim F(x),$$
$$X_{[n\tau_0]+1}, \cdots, X_n\ \text{i.i.d.} \sim F(x-\theta).$$

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其中，τ_0，θ 为未知参数，τ_0 称之为变点，θ 称之为位置参数，我们称上述变点模型为位置参数变点模型.

关于位置参数变点的估计，Bhattacharya 等[1]通过序列的秩，研究了误差为正态条件下的序列变点问题；缪柏其[2]运用滑窗方法，构造了基于反对称核的 Wilcoxon 检验统计量进行检测；Gombay 等[3]运用 U 统计量方法考虑了位置参数变点的检验和估计；Gombay[4]研究了对立假设下用于检验变点存在性的 U 统计量型估计量的渐近分布；Csörgö 等[5,Chapter 2.4]基于对称或反对称核函数，构造了变点的 U 统计量型检验统计量，并研究了检验统计量的极限分布.

变点问题自 20 世纪 70 年代以来一直是统计的热门课题之一，目前已广泛应用于工业质量控制、经济、金融、生物统计等领域. 关于独立随机变量序列变点的统计推断问题，在参数模型和非参数模型假设下已有大量的文献，如文献[5-9]等. 关于相依随机变量序列变点的统计推断，也有大量的文献.

在变点的检验和估计中，常常利用 U 统计量来检测变点，并对变点作出估计. 如文献[3,9-11]等运用 U 统计量方法考虑了刻度参数变点的检验和估计；Gombay[4]研究了对立假设下用于检验变点存在性的 U 统计量的渐近分布，Horváth 等[12]通过样本的置换(permutations)方法，提出了随机向量序列中变点的 U 统计量检验方法.

本文基于缪柏其[2]所提出的反对称核函数，应用累积和(cumulative sum，CUSUM)方法，给出了位置参数变点的 U 统计量型估计量，并研究了变化度为常数时变点估计的相合性和收敛速度以及局部对立假设条件下的变点估计的收敛速度. 定义反对称核函数

$$h(x,y)=I(x<y)-\frac{1}{2}=\begin{cases}\frac{1}{2}, & x<y;\\ -\frac{1}{2}, & x\geqslant y\end{cases} \tag{1}$$

和

$$T_{n1}(k)=\binom{k}{2}^{-1}\sum_{1\leqslant i<j\leqslant k}h(X_i,X_j);$$

$$T_{n2}(k)=\binom{n-k}{2}^{-1}\sum_{k+1\leqslant i<j\leqslant n}h(X_i,X_j);$$

$$Q(k)=T_{n1}(k)-T_{n2}(k).$$

注意到 $k=k_0$ 时，$T_{n1}(k)$ 和 $T_{n2}(k)$ 都是由独立同分布的随机变量构成，且 $h(x,y)$ 为反对称函数，有 $ET_{n1}(k_0)=0$，$ET_{n2}(k_0)=0$，故 $EQ(k)=0$；当 $k\neq k_0$ 时，$T_{n1}(k)$ 和 $T_{n2}(k)$ 中有一个是由独立但不同分布的随机变量序列构成，故 $EQ(k)\neq 0$. 因此定义变点的估计为

$$\begin{aligned}\hat{k}=&\arg\min_k|T_{n1}(k)-T_{n2}(k)|=\\&\arg\min_k|Q(k)|\end{aligned} \tag{2}$$

为方便起见，记 $c,c_0,c_1,c_2,\cdots$ 为与 n 无关的常数，每次出现可以不同.

1　变点估计的强相合性和强收敛速度

定理 1　独立随机变量序列 $X_1,X_2,\cdots,X_n$ 满足

$$X_1,X_2,\cdots,X_{[n\tau_0]}\ \text{i.i.d.}\sim F(x);$$
$$X_{[n\tau_0]+1},\cdots,X_n\ \text{i.i.d.}\sim F(x-\theta),\theta\neq 0$$

为位置参数，F 为连续分布. 假定

$$k_0=[n\tau_0],\ \hat{k}=[n\hat{\tau}],\ k=[n\tau],\ 0<\tau<1.$$

则当位置参数 θ 为常数时，则有 $\hat{\tau}\xrightarrow{P}\tau_0$；且 $\forall\,\varepsilon>0$，

$$P(g(n)\,|\hat{\tau}-\tau_0|>\varepsilon)\to 0,\ n\to\infty,$$

其中，$g(n)=n^{\frac{1}{2}}l^{-1}(n)$，$l(n)$ 为满足 $\lim\limits_{n\to\infty}l(n)=\infty$ 的慢变函数.

注　在定理 1 中位置参数为常数，若位置参数不是常数而是满足 $\theta\to 0$，此时记其为 θ_n，称之为局部对立条件，我们仍可得到变点估计量的相合性. 记 $\rho=E[F(X_{k_0+1})]-\frac{1}{2}$.

定理 2　在定理 1 的条件下，若位置参数 θ 不为常数，而满足 $\theta=\theta_n\to 0$ 时，即 $\rho\triangleq\rho_n\to 0$，且 ρ_n 满足 $n\rho_n^2\to\infty$，则仍有 $\hat{\tau}\xrightarrow{P}\tau_0$.

为证明上述定理，先给出下面的几个引理：

引理 1　设 $Y_1,Y_2,\cdots,Y_n$ 相互独立的随机变量序列，记

$$U_n=\binom{n}{2}^{-1}\sum_{1\leqslant i<j\leqslant n}\phi(Y_i,Y_j),$$

其中，$\phi(Y_i,Y_j)$，$1\leqslant i<j\leqslant n$，满足

$$E[\phi(Y_i,Y_j)\mid Y_i]=E[\phi(Y_i,Y_j)\mid Y_j]=0,$$

若存在常数 $b>0$，使 $|\phi(Y_i,Y_j)|\leqslant b$，则 $\forall\,\varepsilon>0$，

$$P(|U_n|\geqslant\varepsilon)\leqslant 2\exp\left\{-\frac{\varepsilon^2 n}{4b^2}\right\}.$$

证明　参见文献[13].　□

引理 2(Bennett 不等式) 假设 $|Z_i|\leqslant b$, $E(Z_i)=0, i=1,2,\cdots,n$. 记

$$S_j=\sum_{i=1}^{j}Z_i,\ \sigma_i^2=E(Z_i^2),\ \sigma^2=\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2.$$

则对任意的 $x>0$,

$$P\left\{\frac{1}{n}\mid S_n\mid\geqslant x\right\}\leqslant 2\exp\left\{-\frac{nx^2}{2(\sigma^2+bx)}\right\};$$

$$P(\max_{1\leqslant j\leqslant n}\mid S_j\mid\geqslant nx)\leqslant$$

$$2\exp\left\{-\frac{nx}{b}\left[\left(1+\frac{\sigma}{bx}\right)\log\left(1+\frac{bx}{\sigma^2}\right)-1\right]\right\}.$$

证明 参见文献[14]. □

引理 3(Rényi-Hájek-Chow 不等式) 设$\{X_i\}$, $i=1,2,\cdots,n$,为一鞅差序列. 记

$$\sigma_i^2=EX_i^2, S_j=\sum_{i=m}^{j}X_i, 1\leqslant m\leqslant n.$$

则对 $\forall x>0$,有

$$P(\max_{m\leqslant j\leqslant n}a_j\parallel S_j\parallel\geqslant x)\leqslant\frac{1}{x^2}\sum_{j=m}^{n}a_j^2\sigma_j^2.$$

其中,$\{a_n\}$ 满足 $a_1\geqslant a_2\geqslant\cdots\geqslant a_n>0$.

证明 参见文献[15]. □

因为当 $i,j\leqslant k_0$ 或 $i,j>k_0$ 时,X_i,X_j 独立同分布,故 $E[h(X_i,X_j)]=0$;但当 $i\leqslant k_0, j>k_0$ 时, X_i, X_j 独立但不同分布, 故 $E[h(X_i,X_j)]\neq 0$. 注意到 $\rho=E[F(X_{k_0+1})]-\frac{1}{2}$,且

$$\rho=E[F(X_{k_0+1})]-\frac{1}{2}=$$

$$E\{E[I(X_{k_0}<X_{k_0+1})\mid X_{k_0+1}]\}-\frac{1}{2}=$$

$$E[I(X_{k_0}<X_{k_0+1})]-\frac{1}{2}=$$

$$E\{E[I(X_{k_0}<X_{k_0+1})\mid X_{k_0}]\}-\frac{1}{2}=$$

$$E\{E[1-P(X_{k_0+1}\leqslant X_{k_0})\mid X_{k_0}]\}-\frac{1}{2}=$$

$$E\{E[1-P(X_{k_0+1}+\theta\leqslant X_{k_0}+\theta)\mid X_{k_0}]\}-\frac{1}{2}=$$

$$\frac{1}{2}-E[F(X_{k_0}+\theta)],$$

因此,

$$E[h(X_i,X_j)]=\begin{cases}0, & i,j\leqslant k_0\text{ 或 } i,j>k_0;\\ \rho, & i\leqslant k_0,\ j>k_0\end{cases}\tag{3}$$

定理 1 证明 当 $k>k_0$ 时,对 $Q(k)$分解可得

$$Q(k)=\binom{k}{2}^{-1}\Big[\sum_{1\leqslant i<j\leqslant k_0}h(X_i,X_j)+\sum_{k_0<i<j\leqslant k}h(X_i,X_j)+\sum_{i=1}^{k_0}\sum_{j=k_0+1}^{k}h(X_i,X_j)\Big]-\binom{n-k}{2}^{-1}\sum_{k<i<j\leqslant n}h(X_i,X_j);$$

计算可得

$$EQ(k)=\binom{k}{2}^{-1}\Big[\sum_{1\leqslant i<j\leqslant k_0}Eh(X_i,X_j)+\sum_{k_0<i<j\leqslant k}Eh(X_i,X_j)+\sum_{i=1}^{k_0}\sum_{j=k_0+1}^{k}Eh(X_i,X_j)\Big]-\binom{n-k}{2}^{-1}\sum_{k<i<j\leqslant n}Eh(X_i,X_j)=$$

$$\binom{k}{2}^{-1}\Big[\sum_{i=1}^{k_0}\sum_{j=k_0+1}^{k}Eh(X_i,X_j)\Big]=\frac{2k_0(k-k_0)}{k(k-1)}\rho\tag{4}$$

当 $k\leqslant k_0$ 时,类似分解计算可得

$$EQ(k)=\binom{n-k}{2}^{-1}\Big[\sum_{i=k+1}^{k_0}\sum_{j=k_0+1}^{n}Eh(X_i,X_j)\Big]=\frac{2(k_0-k)(n-k_0)}{(n-k)(n-k-1)}\rho\tag{5}$$

综合式(4)和(5)得

$$EQ(k)=\begin{cases}\dfrac{2(k-k_0)(n-k_0)}{(n-k)(n-k-1)}\rho, & k\leqslant k_0;\\[2mm] \dfrac{2k_0(k-k_0)}{k(k-1)}\rho, & k>k_0;\end{cases}=$$

$$\begin{cases}(\tau-\tau_0)2\rho\dfrac{(1-\tau_0)}{(1-\tau)\left(1-\tau-\dfrac{1}{n}\right)}, & k\leqslant k_0;\\[2mm] (\tau-\tau_0)2\rho\dfrac{\tau_0}{\tau\left(\tau-\dfrac{1}{n}\right)}, & k>k_0\end{cases}\tag{6}$$

不妨假设 τ_0 满足 $\delta_0\leqslant\tau_0\leqslant 1-\delta_0$,其中,$\delta_0$ 为满足 $0<\delta_0<\frac{1}{2}$的常数, 则有

$$\frac{\tau_0}{\hat{\tau}^2}\geqslant\frac{\tau_0}{(1-\delta_0)^2};\ \frac{1-\tau_0}{(1-\hat{\tau})^2}\geqslant\frac{1-\tau_0}{(1-\delta_0)^2}.$$

记 $\tau^*=\min\{\tau_0,1-\tau_0\}$. 由三角不等式有

$$\mid\hat{\tau}-\tau_0\mid\frac{2\rho\tau^*}{(1-\delta_0)^2}\leqslant\mid EQ(\hat{k})\mid\leqslant$$

$$\mid Q(\hat{k})-EQ(\hat{k})\mid+\mid Q(\hat{k})\mid=$$

$$\mid Q(\hat{k})-EQ(\hat{k})\mid+\mid Q(\hat{k})\mid+$$

$$|Q(k_0)-EQ(k_0)|-|Q(k_0)|\leqslant 2\max_{\delta_0 n\leqslant k\leqslant n-\delta_0 n}|Q(k)-EQ(k)|,$$

即

$$\frac{\rho\tau^*}{(1-\delta_0)^2}|\hat{\tau}-\tau_0|\leqslant\max_{\delta_0 n\leqslant k\leqslant n-\delta_0 n}|Q(k)-EQ(k)| \tag{7}$$

令 $g(n)$满足$\lim\limits_{n\to\infty}g(n)=\infty$，$\forall\,\varepsilon>0$，所以有

$$\begin{aligned}
&P(g(n)|\hat{\tau}-\tau_0|>\varepsilon)\leqslant\\
&P\left(\max_{[\delta_0 n]\leqslant k\leqslant n-[\delta_0 n]}|Q(k)-EQ(k)|>\frac{\varepsilon}{g(n)}\frac{\rho\tau^*}{(1-\delta_0)^2}\right)\leqslant\\
&P\left(\max_{[\delta_0 n]\leqslant k\leqslant k_0}|Q(k)-EQ(k)|>\frac{\varepsilon}{g(n)}\frac{\rho\tau^*}{(1-\delta_0)^2}\right)+\\
&P\left(\max_{k_0\leqslant k\leqslant n-[\delta_0 n]}|Q(k)-EQ(k)|>\frac{\varepsilon}{g(n)}\frac{\rho\tau^*}{(1-\delta_0)^2}\right)\triangleq A_1+A_2
\end{aligned} \tag{8}$$

仅考虑 A_2，A_1 类似可得．在 $k_0\leqslant k\leqslant n-[n\delta_0]$时，有

$$\begin{aligned}
Q(k)-EQ(k)=&\binom{k}{2}^{-1}\sum_{1\leqslant i<j\leqslant k_0}h(X_i,X_j)+\binom{k}{2}^{-1}\sum_{k_0<i<j\leqslant k}h(X_i,X_j)+\\
&\binom{k}{2}^{-1}\sum_{i=1}^{k_0}\sum_{j=k_0+1}^{k}[h(X_i,X_j)-Eh(X_i,X_j)]-\binom{n-k}{2}^{-1}\sum_{k<i<j\leqslant n}h(X_i,X_j)\triangleq\\
&B_1+B_2+B_3+B_4
\end{aligned} \tag{9}$$

记

$\Phi_1(X_i,X_j)=h(X_i,X_j)-E[h(X_i,X_j)\mid X_i]-E[h(X_i,X_j)\mid X_j]$，$i,j\leqslant k_0$；

$\Phi_2(X_i,X_j)=h(X_i,X_j)-E[h(X_i,X_j)\mid X_i]-E[h(X_i,X_j)\mid X_j]$，$i,j>k_0$；

$\Phi_{12}(X_i,X_j)=h(X_i,X_j)-E[h(X_i,X_j)\mid X_i]-E[h(X_i,X_j)\mid X_j]+E[h(X_i,X_j)]$，$i\leqslant k_0$，$j>k_0$.

由 $h(X_i,X_j)$定义，知$|\Phi_1(X_i,X_j)|\leqslant 4$，$|\Phi_2(X_i,X_j)|\leqslant 4$，$|\Phi_{12}(X_i,X_j)|\leqslant 4\ \forall\,i,j$，且有

$E[\Phi_1(X_i,X_j)]=0$，$E[\Phi_1(X_i,X_j)\mid X_i]=0$，$E[\Phi_1(X_i,X_j)\mid X_j]=0$，$i,j\leqslant k_0$；

$E[\Phi_2(X_i,X_j)]=0$，$E[\Phi_2(X_i,X_j)\mid X_i]=0$，$E[\Phi_2(X_i,X_j)\mid X_j]=0$，$i,j>k_0$；

$E[\Phi_{12}(X_i,X_j)]=0$，$E[\Phi_{12}(X_i,X_j)\mid X_i]=0$，$E[\Phi_{12}(X_i,X_j)\mid X_j]=0$，$i\leqslant k_0$，$j>k_0$.

对 B_1 分解得

$$\begin{aligned}
B_1=&\binom{k}{2}^{-1}\Big[\sum_{1\leqslant i<j\leqslant k_0}[Eh(X_i,X_i)\mid X_i]+\sum_{1\leqslant i<j\leqslant k_0}[Eh(X_i,X_i)\mid X_j]+\\
&\sum_{1\leqslant i<j\leqslant k_0}\{h(X_i,X_j)-E[h(X_i,X_j)\mid X_i]-E[h(X_i,X_j)\mid X_j]\}\Big]=\\
&\binom{k}{2}^{-1}\sum_{i=1}^{k_0-1}(k_0-i)E[h(X_i,X_j)\mid X_i]+\binom{k}{2}^{-1}\sum_{j=2}^{k_0}(j-1)E[h(X_i,X_j)\mid X_j]+\\
&\binom{k}{2}^{-1}\sum_{1\leqslant i<j\leqslant k_0}\Phi_1(X_i,X_j)\triangleq B_{11}+B_{12}+B_{13}
\end{aligned} \tag{10}$$

B_2，B_4 类似分解得

$$\begin{aligned}
B_2=&\binom{k}{2}^{-1}\sum_{i=k_0+1}^{k-1}(k-i)E[h(X_i,X_j)\mid X_i]+\\
&\binom{k}{2}^{-1}\sum_{i=k_0+2}^{k}(j-k_0-1)E[h(X_i,X_j)\mid X_j]+\binom{k}{2}^{-1}\sum_{k_0<i<j\leqslant k}\Phi_2(X_i,X_j)\triangleq\\
&B_{21}+B_{22}+B_{23}
\end{aligned} \tag{11}$$

$$B_4 = \binom{n-k}{2}^{-1} \sum_{i=k+1}^{n-1} (n-i) E[h(X_i, X_j) \mid X_i] +$$
$$\binom{n-k}{2}^{-1} \sum_{j=k+2}^{n} (j-k-1) E[h(X_i, X_j) \mid X_j] - \binom{n-k}{2}^{-1} \sum_{k \leqslant i < j \leqslant n} \Phi_2(X_i, X_j) \triangleq$$
$$B_{41} + B_{42} + B_{43} \tag{12}$$

对 B_3 部分分解得

$$B_3 = \binom{k}{2}^{-1} \sum_{i=1}^{k_0} (k-k_0) \{ E[h(X_i, X_j) \mid X_i] - Eh(X_i, X_j) \} +$$
$$\binom{k}{2}^{-1} \sum_{j=k_0+1}^{k} k_0 \{ E[h(X_i, X_j) \mid X_j] - Eh(X_i, X_j) \} +$$
$$\binom{k}{2}^{-1} \sum_{i=1}^{k_0} \sum_{j=k_0+1}^{k} \{ h(X_i, X_j) - E[h(X_i, X_j) \mid X_i] - E[h(X_i, X_j) \mid X_j] + E[h(X_i, X_j)] \} =$$
$$\frac{k_0(k-k_0)}{\binom{k}{2}} \cdot \frac{1}{k_0} \sum_{i=1}^{k_0} \{ E[h(X_i, X_j) \mid X_i] - Eh(X_i, X_j) \} +$$
$$\frac{k_0(k-k_0)}{\binom{k}{2}} \cdot \frac{1}{k-k_0} \sum_{j=k_0+1}^{k} \{ E[h(X_i, X_j) \mid X_j] - Eh(X_i, X_j) \} +$$
$$\frac{k_0(k-k_0)}{\binom{k}{2}} \cdot \frac{1}{k_0} \cdot \frac{1}{k-k_0} \sum_{i=1}^{k_0} \sum_{j=k_0+1}^{k} \Phi_{12}(X_i, X_j) \triangleq B_{31} + B_{32} + B_{33} \tag{13}$$

综合式(9)～(13)有

$$A_2 = P\left(\max_{k_0 \leqslant k \leqslant n-\delta_0 n} | Q(k) - EQ(k) | > \frac{\varepsilon}{g(n)} \cdot \frac{\rho \tau^*}{(1-\delta_0)^2} \right) =$$
$$P\left(\max_{k_0 \leqslant k \leqslant n-\delta_0 n} | B_{11} + B_{12} + \cdots + B_{41} + B_{42} + B_{43} | > \frac{\varepsilon}{g(n)} \cdot \frac{\rho \tau^*}{(1-\delta_0)^2} \right) \leqslant$$
$$P\left(\max_{k_0 \leqslant k \leqslant n-\delta_0 n} | B_{11} | > \frac{1}{12} \frac{\varepsilon}{g(n)} \cdot \frac{\rho \tau^*}{(1-\delta_0)^2} \right) + \cdots + P\left(\max_{k_0 \leqslant k \leqslant n-\delta_0 n} | B_{43} | > \frac{1}{12} \frac{\varepsilon}{g(n)} \cdot \frac{\rho \tau^*}{(1-\delta_0)^2} \right) \triangleq$$
$$D_1 + D_2 + \cdots + D_{12} \tag{14}$$

下面分别证明之. 因为

$$D_1 = P\left(\max_{k_0 \leqslant k \leqslant n-\delta_0 n} \binom{k}{2}^{-1} \left| \sum_{i=1}^{k_0-1} (k_0-i) E[h(X_i, X_j) \mid X_i] \right| > \frac{1}{12} \frac{\varepsilon}{g(n)} \cdot \frac{\rho \tau^*}{(1-\delta_0)^2} \right) \leqslant$$
$$P\left(\max_{k_0 \leqslant k \leqslant n-\delta_0 n} \left| \sum_{i=1}^{k_0-1} \frac{2(k_0-i)}{k(k-1)} E[h(X_i, X_j) \mid X_i] \right| > \frac{1}{12} \frac{\varepsilon}{g(n)} \cdot \frac{\rho \tau^*}{(1-\delta_0)^2} \right) \leqslant$$
$$P\left(\frac{1}{k_0} \left| \sum_{i=1}^{k_0-1} E[h(X_i, X_j) \mid X_i] \right| > \frac{1}{24} \frac{\varepsilon}{g(n)} \cdot \frac{\rho \tau^*}{(1-\delta_0)^2} \right) \overset{\text{Bennett}}{\leqslant}$$
$$2\exp\left\{ - \frac{k_0 \left[\frac{\varepsilon}{g(n)} \cdot \frac{\rho \tau^*}{24(1-\delta_0)^2} \right]^2}{2\left(\frac{1}{12} + \frac{\varepsilon}{g(n)} \cdot \frac{\rho \tau^*}{24(1-\delta_0)^2} \right)} \right\} \leqslant 2\exp\left\{ - \frac{c_1 \, n\varepsilon \rho^2}{g(n)} \right\} \tag{15}$$

$$D_2 = P\left(\max_{k_0 \leqslant k \leqslant n-\delta_0 n} \binom{k}{2}^{-1} \left| \sum_{j=2}^{k_0} (j-1) E[h(X_i, X_j) \mid X_j] \right| > \frac{1}{12} \frac{\varepsilon}{g(n)} \cdot \frac{\rho \tau^*}{(1-\delta_0)^2} \right) \leqslant$$

$$P\left\{\frac{1}{k_0}\left|\sum_{j=2}^{k_0}E[h(X_i,X_j)\mid X_j]\right|>\frac{1}{24}\frac{\varepsilon}{g(n)}\cdot\frac{\rho\tau^*}{(1-\delta_0)^2}\right\}\leqslant$$
$$2\exp\left\{-\frac{c_2\ n\varepsilon\rho^2}{g(n)}\right\} \tag{16}$$

注意到 $k>k_0$，有 $\binom{k}{2}^{-1}\leqslant\binom{k_0}{2}^{-1}$，$\binom{k-k_0}{2}\leqslant\binom{k}{2}$，由引理 1 可得

$$D_3=P\left\{\max_{k_0\leqslant k\leqslant n-\delta_0 n}|B_{13}|>\frac{1}{12}\frac{\varepsilon}{g(n)}\cdot\frac{\rho\tau^*}{(1-\delta_0)^2}\right\}\leqslant$$
$$P\left\{\binom{k_0}{2}^{-1}\left|\sum_{1\leqslant i<j\leqslant k_0}\Phi_2(X_i,X_j)\right|>\frac{1}{12}\frac{\varepsilon}{g(n)}\cdot\frac{\rho\tau^*}{(1-\delta_0)^2}\right\}\overset{\text{Lemma 1}}{\leqslant}$$
$$2\exp\left\{-\frac{k_0\cdot(\tau^*)^2\cdot\varepsilon^2\cdot\rho^2}{4\times144\times(1-\delta_0)^4[g(n)]^2}\right\}\leqslant 2\exp\left\{-\frac{c_3\ n\varepsilon^2\rho^2}{g^2(n)}\right\} \tag{17}$$

$$D_4=P\left\{\max_{k_0\leqslant k\leqslant n-\delta_0 n}|B_{21}|>\frac{1}{12}\frac{\varepsilon}{g(n)}\cdot\frac{\rho\tau^*}{(1-\delta_0)^2}\right\}=$$
$$P\left\{\max_{k_0\leqslant k\leqslant n-\delta_0 n}\left|\binom{k}{2}^{-1}\sum_{i=k_0+1}^{k}(k-i)E[h(X_i,X_j\mid X_i)]\right|>\frac{1}{12}\frac{\varepsilon}{g(n)}\cdot\frac{\rho\tau^*}{(1-\delta_0)^2}\right\}\leqslant$$
$$P\left\{\max_{k_0\leqslant k\leqslant n-\delta_0 n}\left|\frac{1}{k}\cdot\sum_{i=k_0+1}^{k}E[h(X_i,X_j\mid X_i)]\right|>\frac{1}{12}\frac{\varepsilon}{g(n)}\cdot\frac{\rho\tau^*}{(1-\delta_0)^2}\right\}\overset{\text{R-H-C}}{\leqslant}$$
$$\frac{144\cdot(1-\delta_0)^4\cdot[g(n)]^2}{\varepsilon^2\cdot(\tau^*)^2\cdot\rho^2}\frac{1}{n}\cdot\left(\frac{1}{\tau^*}-\frac{1}{1-\delta_0}\right)\leqslant c_4\frac{g^2(n)}{n\rho^2}\frac{1}{\varepsilon} \tag{18}$$

$$D_5=P\left\{\max_{k_0\leqslant k\leqslant n-\delta_0 n}|B_{22}|>\frac{1}{12}\frac{\varepsilon}{g(n)}\cdot\frac{\rho\tau^*}{(1-\delta_0)^2}\right\}\leqslant c_5\frac{g^2(n)}{n\rho^2}\frac{1}{\varepsilon} \tag{19}$$

记 $\mathscr{A}_k=\sum_{k_0\leqslant i<j\leqslant k}\Phi_2(X_i,X_j)$，因为 $E[\mathscr{A}_{k+1}\mid\mathscr{A}_k]=\mathscr{A}_k$，故 $\mathscr{A}_k$ 为鞅序列，由引理 3 可得

$$D_6=P\left\{\max_{k_0\leqslant k\leqslant n-\delta_0 n}|B_{23}|>\frac{1}{12}\frac{\varepsilon}{g(n)}\cdot\frac{\rho\tau^*}{(1-\delta_0)^2}\right\}=$$
$$P\left\{\max_{k_0\leqslant k\leqslant n-\delta_0 n}\left|\frac{\binom{k-k_0}{2}}{\binom{k}{2}}\cdot\frac{1}{\binom{k-k_0}{2}}\sum_{k_0<i<j\leqslant k}\Phi_2(X_i,X_j)\right|>\frac{1}{12}\frac{\varepsilon}{g(n)}\cdot\frac{\rho\tau^*}{(1-\delta_0)^2}\right\}\leqslant$$
$$P\left\{\max_{k_0\leqslant k\leqslant n-\delta_0 n}\left|\binom{k-k_0}{2}^{-1}\sum_{k_0<i<j\leqslant k}\Phi_2(X_i,X_j)\right|>\frac{1}{12}\frac{\varepsilon}{g(n)}\cdot\frac{\rho\tau^*}{(1-\delta_0)^2}\right\}\overset{\text{Lemma 3}}{\leqslant}$$
$$\sum_{k=k_0+1}^{n}\frac{k-k_0-1}{144\ k^2(k-1)^2}\frac{4\times144\ (1-\delta_0)^2g^2(n)}{\varepsilon^2(\tau^*)^2\rho^2}\leqslant$$
$$\frac{g^2(n)}{\varepsilon^2\rho^2}\frac{4(1-\delta_0)^2}{(\tau^*)^2}\sum_{k=k_0+1}^{n}\frac{1}{k_0k(k-1)}=$$
$$\frac{g^2(n)}{\varepsilon^2\ n\rho^2}\frac{4(1-\delta_0)^2}{(\tau^*)^3}\left(\frac{1}{\tau^*}-1\right)\leqslant c_6\frac{g^2(n)}{n\rho^2}\frac{1}{\varepsilon^2} \tag{20}$$

$$D_7=P\left\{\max_{k_0+1\leqslant k\leqslant n-\delta_0 n}|B_{31}|>\frac{1}{12}\frac{\varepsilon}{g(n)}\cdot\frac{\rho\tau^*}{(1-\delta_0)^2}\right\}\leqslant$$
$$P\left\{\frac{2}{k_0}\left|\sum_{i=1}^{k_0}\{E[h(X_i,X_j)\mid X_i]-Eh(X_i,X_j)\}\right|>\frac{1}{12}\frac{\varepsilon}{g(n)}\cdot\frac{\rho\tau^*}{(1-\delta_0)^2}\right\}\overset{\text{R-H-C}}{\leqslant}$$

$$\frac{4\times 144\times(1-\delta_0)^4[g(n)]^2}{(\tau^*)^3\cdot\varepsilon^2\cdot n\cdot\rho^2}\leqslant c_7\frac{g^2(n)}{n\rho^2}\frac{1}{\varepsilon^2} \tag{21}$$

$$D_8=P\left(\max_{k_0+1\leqslant k\leqslant n-\delta_0 n}|B_{32}|>\frac{1}{12}\frac{\varepsilon}{g(n)}\cdot\frac{\rho\tau^*}{(1-\delta_0)^2}\right)=$$

$$P\left(\max_{k_0+1\leqslant k\leqslant n-\delta_0 n}\frac{2k_0}{k(k-1)}\left|\sum_{j=k_0+1}^{k}\{E[h(X_i,X_j)\mid X_j]-Eh(X_i,X_j)\}\right|>\frac{1}{12}\frac{\varepsilon}{g(n)}\cdot\frac{\rho\tau^*}{(1-\delta_0)^2}\right)\leqslant$$

$$P\left(\max_{k_0+1\leqslant k\leqslant n-\delta_0 n}\frac{2}{k_0}\left|\sum_{j=k_0+1}^{k}\{E[h(X_i,X_j)\mid X_j]-Eh(X_i,X_j)\}\right|>\frac{1}{12}\frac{\varepsilon}{g(n)}\cdot\frac{\rho\tau^*}{(1-\delta_0)^2}\right)\leqslant$$

$$\frac{4\times 144\cdot(1-\delta_0)^4\cdot[g(n)]^2}{\varepsilon^2\cdot(\tau^*)^2\cdot n\cdot\rho^2}\leqslant c_8\frac{g^2(n)}{n\rho^2} \tag{22}$$

记 $\mathscr{F}_k=\sum_{i=1}^{k_0}\sum_{j=k_0+1}^{k}\Phi_{12}(X_i,X_j)$，其满足

$$E[\mathscr{F}_{k+1}\mid\mathscr{F}_k]=E\left[\sum_{i=1}^{k_0}\sum_{j=k_0+1}^{k}\Phi_{12}(X_i,X_j)+\sum_{i=1}^{k_0}E[\Phi_{12}(X_i,X_{k+1})\mid\mathscr{F}_k]\right]=\mathscr{F}_k$$

$\mathscr{F}_k$ 为鞅，由引理 3，有

$$D_9=P\left(\max_{k_0+1\leqslant k\leqslant n-\delta_0 n}|D_{33}|>\frac{1}{12}\frac{\varepsilon}{g(n)}\cdot\frac{\rho\tau^*}{(1-\delta_0)^2}\right)=$$

$$P\left(\max_{k_0+1\leqslant k\leqslant n-\delta_0 n}\frac{2}{k(k-1)}\left|\sum_{i=1}^{k_0}\sum_{j=k_0+1}^{k}\Phi_{12}(X_i,X_j)\right|>\frac{1}{12}\frac{\varepsilon}{g(n)}\cdot\frac{\rho\tau^*}{(1-\delta_0)^2}\right)\overset{\text{Lemma 3}}{\leqslant}$$

$$\frac{[24\cdot(1-\delta_0)^2\cdot g(n)]^2}{(\varepsilon\cdot\tau^*\cdot\rho)^2}\sum_{k=k_0+1}^{(1-\delta_0)n}\frac{k^*(k-k^*)}{[k(k-1)]^2}\leqslant$$

$$c_9\frac{g^2(n)}{n\rho^2}\cdot\frac{1}{\varepsilon^2}\left(\frac{1}{\tau^*}-\frac{1}{1-\delta_0}\right) \tag{23}$$

记 $l=(n-k)$，类似 D_4，由 R-H-C 不等式有

$$D_{10}=P\left(\max_{k_0\leqslant k\leqslant n-\delta_0 n}|B_{41}|>\frac{1}{12}\frac{\varepsilon}{g(n)}\cdot\frac{\rho\tau^*}{(1-\delta_0)^2}\right)\leqslant$$

$$P\left(\max_{k_0\leqslant k\leqslant n-\delta_0 n}\left|\frac{2}{n-k}\sum_{i=k+1}^{n}E[h(X_i,X_j)\mid X_i]\right|>\frac{1}{12}\frac{\varepsilon}{g(n)}\cdot\frac{\rho\tau^*}{(1-\delta_0)^2}\right),\ j>k_0$$

$$=P\left(\max_{\delta_0 n\leqslant l\leqslant n-k_0}\left|\frac{2}{l}\sum_{i=1}^{l}E[h(X_i,X_j)\mid X_i]\right|>\frac{1}{12}\frac{\varepsilon}{g(n)}\cdot\frac{\rho\tau^*}{(1-\delta_0)^2}\right),\ j>k_0$$

$$\leqslant\frac{144(1-\delta_0)^2g^2(n)}{\varepsilon^2\cdot(\tau^*)^2\rho^2}\sum_{l=\delta_0 n}^{\tau^* n}\frac{1}{l^2}\leqslant c_{10}\frac{g^2(n)}{n\rho^2}\frac{1}{\varepsilon} \tag{24}$$

$$D_{11}=P\left(\max_{k_0\leqslant k\leqslant n-\delta_0 n}|B_{42}|>\frac{1}{12}\frac{\varepsilon}{g(n)}\cdot\frac{\rho\tau^*}{(1-\delta_0)^2}\right)\leqslant c_{10}\frac{g^2(n)}{n\rho^2}\frac{1}{\varepsilon} \tag{25}$$

类似 D_6 有

$$D_{12}=P\left(\max_{k_0\leqslant k\leqslant n-\delta_0 n}|B_{43}|>\frac{1}{12}\frac{\varepsilon}{g(n)}\cdot\frac{\rho\tau^*}{(1-\delta_0)^2}\right)=$$

$$P\left(\max_{k_0\leqslant k\leqslant n-\delta_0 n}\left|\binom{n-k}{2}^{-1}\sum_{k+1\leqslant i<j\leqslant n}\Phi_2(X_i,X_j)\right|>\frac{1}{12}\frac{\varepsilon}{g(n)}\cdot\frac{\rho\tau^*}{(1-\delta_0)^2}\right)\overset{\text{Lemma 3}}{\leqslant}$$

$$\sum_{k=k_0+1}^{n}\frac{k-k_0-1}{144k^2(k-1)^2}\frac{4\times 144(1-\delta_0)^2g^2(n)}{\varepsilon^2(\tau^*)^2\rho^2}\leqslant c_{12}\frac{g^2(n)}{n\rho^2}\frac{1}{\varepsilon^2} \tag{26}$$

综合式(15)～(26)得,当位置参数 θ 为常数时,$A_2\to 0$,其中,$g(n)=n^{\frac{1}{2}}l^{-1}(n)$,$l(n)$为满足$\lim\limits_{n\to\infty}=\infty$的慢变函数,即得定理 1. □

定理 2 证明　由式(15)～(26),有

$$D_1\leqslant 2\exp\left\{-\frac{c_1 n\varepsilon\rho_n^2}{g(n)}\right\},\ D_2\leqslant 2\exp\left\{-\frac{c_2 n\varepsilon\rho_n^2}{g(n)}\right\},$$

$$D_3\leqslant 2\exp\left\{-\frac{c_3 n\varepsilon^2\rho_n^2}{g^2(n)}\right\},\ D_4\leqslant c_4\frac{g^2(n)}{n\rho_n^2}\frac{1}{\varepsilon},$$

$$D_5\leqslant c_5\frac{g^2(n)}{n\rho_n^2}\frac{1}{\varepsilon},\ D_6\leqslant c_6\frac{g^2(n)}{n\rho_n{}^2}\frac{1}{\varepsilon^2},$$

$$D_7\leqslant c_7\frac{g^2(n)}{n\rho_n{}^2}\frac{1}{\varepsilon^2},\ D_8\leqslant c_8\frac{g^2(n)}{n\rho_n{}^2}\frac{1}{\varepsilon^2},$$

$$D_9\leqslant c_9\frac{g^2(n)}{n\rho_n^2}\cdot\frac{1}{\varepsilon^2}\left[\frac{1}{\tau^*}-\frac{1}{1-\delta_0}\right],$$

$$D_{10}\leqslant c_{10}\frac{g^2(n)}{n\rho_n^2}\frac{1}{\varepsilon},\ D_{11}\leqslant c_{10}\frac{g^2(n)}{n\rho_n^2}\frac{1}{\varepsilon},$$

$$D_{12}\leqslant c_{12}\frac{g^2(n)}{n\rho_n{}^2}\frac{1}{\varepsilon^2},$$

在局部对立条件下,若位置参数 $\theta=\theta_n\to 0$,即 $\rho_n\to 0$,且 $n\rho_n^2\to\infty$,有 $D_1\to 0,\cdots,D_{12}\to 0$,即得定理 2. □

2　数据模拟

为了验证节 1 中变点估计的相合性和收敛速度,我们选取正态分布样本数据进行模拟分析. 为此,选取样本容量 n=1 000,真实变点 $\tau_0=k_0/n$ 分别取为:0.25,0.4,0.5,0.75.重复次数为 500 次.

分别考虑下列 3 种类型的参数设置:

M_1:$X_1,\cdots,X_{k_0}$ 来自 $N(0,1)$,$X_{k_0+1},\cdots,X_n$ 来自 $N(5,1)$;

M_2:$X_1,\cdots,X_{k_0}$ 来自 $N(0,3)$,$X_{k_0+1},\cdots,X_n$ 来自 $N(5,3)$;

M_3:$X_1,\cdots,X_{k_0}$ 来自 $N(0,5)$,$X_{k_0+1},\cdots,X_n$ 来自 $N(5,5)$.

由样本容量 n=1 000时 500 次重复模拟实验数据所得变点估计值的直方图如图 1～3 所示.图中各参数含义分别为:σ^2 表示正态样本的方差,μ_1 表示变化前均值,μ_2 表示变化后均值,τ_0 表示真实变点.

由直方图可以看出:

① 变点的位置对变点的估计值有一定影响;变点位置越靠近中间位置时,估计值越精确,靠近两端时估计值偏小,这与常见的累积和型估计量有同样的性质.

② 比较图 1～3 可得:正态样本的方差越小,估计越精确,这是由于方差较大时数据波动性较大,导致估计的误差增大;另外,可以看出在每一种情形下,所得变点的估计会出现异常值,即接近 0 或者 1 的估计值.

另外,对于样本容量 n=500,我们也进行了模

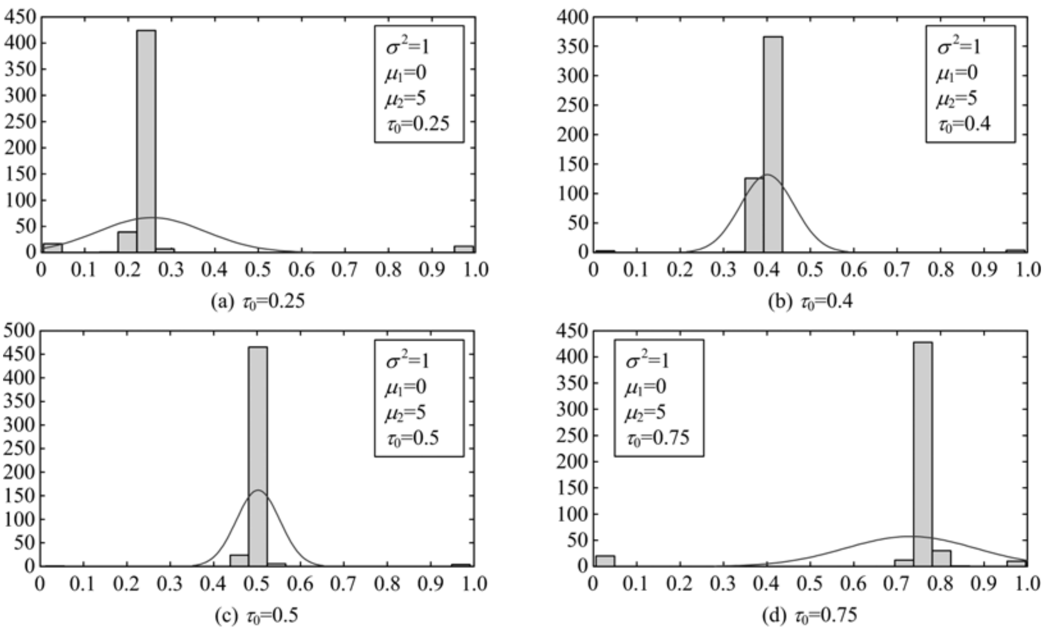

图 1　样本容量 $n=1\ 000,\sigma^2=1,\mu_1=0,\mu_2=5$**,所得到的变点估计的直方图**

Fig. 1　Histograms of change-point estimator with $n=1\ 000,\sigma^2=1,\mu_1=0,\mu_2=5$

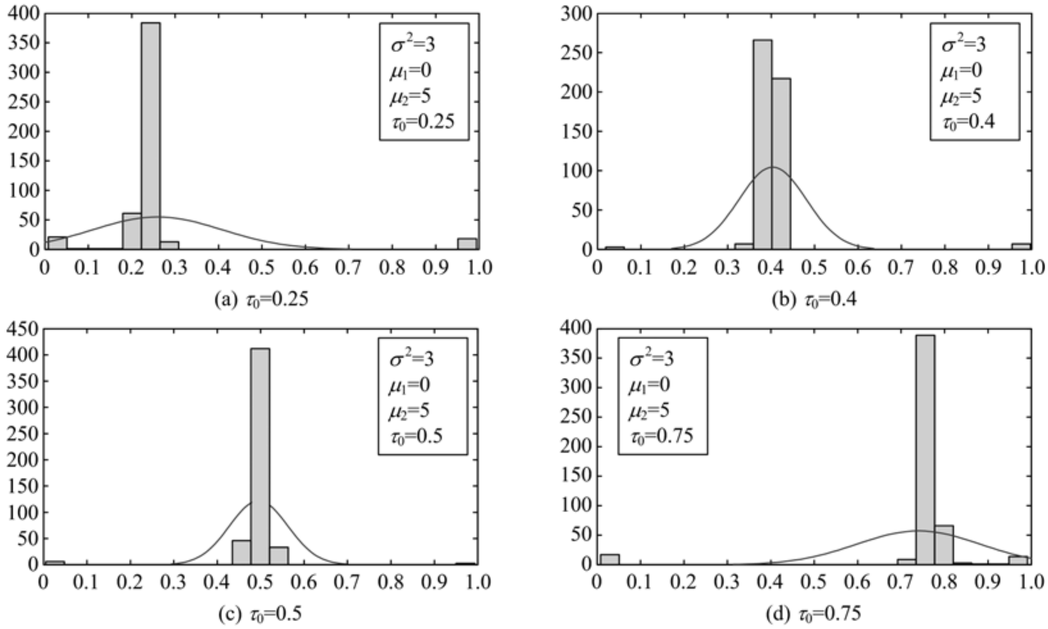

图 2　样本容量 $n=1\ 000$, $\sigma^2=3$, $\mu_1=0$, $\mu_2=5$, 所得到的变点估计的直方图

Fig. 2　Histograms of change-point estimator with $n=1\ 000$, $\sigma^2=3$, $\mu_1=0$, $\mu_2=5$

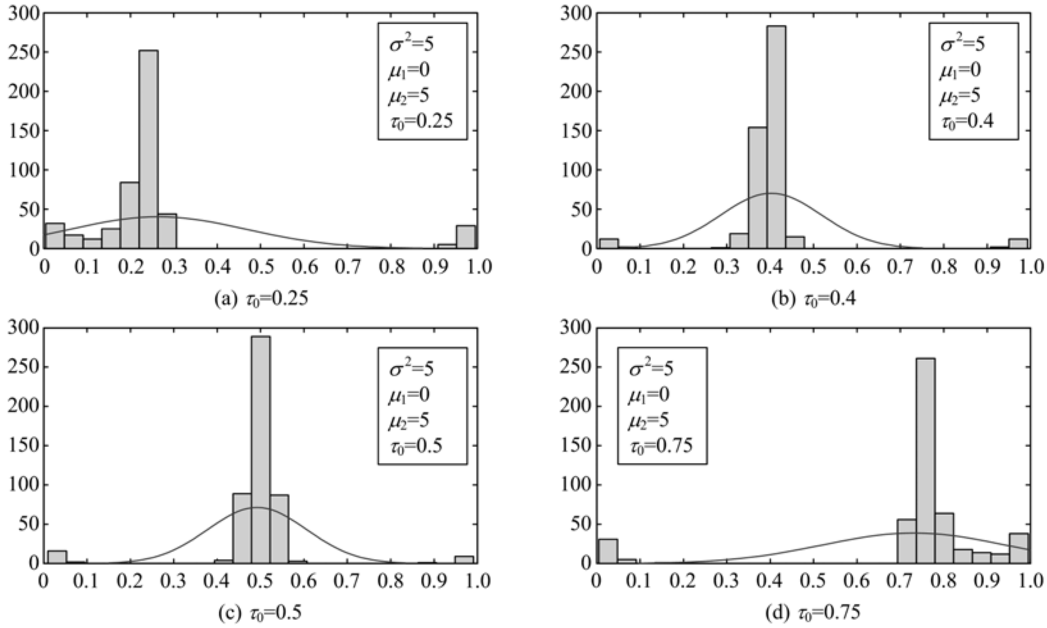

图 3　样本容量 $n=1\ 000$, $\sigma^2=5$, $\mu_1=0$, $\mu_2=5$, 所得到的变点估计的直方图

Fig. 3　Histograms of change-point estimator with $n=1\ 000$, $\sigma^2=5$, $\mu_1=0$, $\mu_2=5$

拟画图，与 $n=1\ 000$ 时有类似结论，在此不一一列出.

3　结论

本文应用所提出的反对称核函数，应用 CUSUM 方法，提出了位置参数变点估计的估计量，并研究了变点前后变化度为常数时变点估计的相合性和收敛速度，同时也给出了局部对立假设条件下变点估计的收敛速度. 模拟结果显示所给出的估计量的有效性. 关于变点估计量的渐近分布研究，由于涉及较深入的概率极限理论知识，我们将在今后的工作中作一步的研究.

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