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On GI-flat and GF-torsion modules

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Abstract: Two classes of modules were studied: GI-flat and GF-torsion modules, where GI stands for Gorenstein injective modules and GF for Gorenstein flat modules. Two homological dimensions for a ring were investigated, the supremum of the flat dimension of Gorenstein injective modules and the supremum of the GF-torsion dimension of all modules. The relation between these classes of modules and the homological dimensions was also studied.

Key words: Gorenstein injective modules; Gorenstein flat modules; GI-flat modules; GF-torsion modules

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关于 GI 平坦模和 GF 挠模

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摘要:研究了两类模:GI 平坦模和 GF 挠模,其中,GI 表示 Gorenstein 内射模,GF 表示 Gorenstein 平坦模;刻画了环的两个同调维数,即 Gorenstein 内射模的最大平坦维数和模的最大 GF 挠维数.同时也研究了这些模类和同调维数之间的关系.

关键词:Gorenstein 内射模;Gorenstein 平坦模;GI 平坦模;GF 挠模

0 Introduction

Throughout this paper, R denotes an associative ring with identity and all modules are unitary. We use Mod-R (resp. R-Mod) to denote the category of left (resp. right) R-modules, and $pd_R(M)$, $id_R(M)$, $fd_R(M)$ to denote, respectively, the projective, injective and flat dimensions of a

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module M in Mod-R (resp. R-Mod). Recall that a module $M\in R$-Mod is called Gorenstein injective if there exists an exact sequence in R-Mod with all terms injective: $\cdots\to I_1\to I_0\to I^0\to I^1\to\cdots$, such that $M=\mathrm{Im}(I_0\to I^0)$ and the sequence is still exact after applying the functor $\mathrm{Hom}_R(I,-)$ for any injective right R-module I. A module $M\in R$-Mod is called Gorenstein flat if there exists an exact sequence in R-Mod with all terms flat: $\cdots\to F_1\to F_0\to F^0\to F^1\to\cdots$, such that $M=\mathrm{Im}(F_0\to F^0)$ and the sequence is still exact after applying the functor $-\otimes I$ for any injective left R-module I (See Ref. [1]). The Gorenstein injective and flat dimensions of R-module M, denoted by $Gid_R(M)$ and $Gfd_R(M)$, are defined in terms of resolutions by Gorenstein injective and flat modules respectively. Bennis and Mahdou showed that Gorenstein injective modules are not necessarily projective (flat) over QF-rings (see Ref. [2, Example 2.5]). Recently, Gao[3] introduced the notion of GI-injetive modules and discussed the GI-injective dimension of modules and rings. One easily finds the fact that every Gorenstein injective R-module is flat implies R is left IF (i. e., rings satisfying every injective left R-module is flat), but the converse is not true in general. Inspired by Refs. [3-4], we will study GI-flat and GF-torsion modules and characterize the GI-flat and GF-torsion dimension of modules and rings.

In Section 1, we study a right dimension $r.GIFD(R)$. Firstly, we discuss the relations between $r.GIFD(R)$ and several known dimensions for a ring R. Then we give some properties of $r.GIFD(R)$ in terms of GI-flat and generalize several results of Refs. [3, 5] (see Theorem 1.7). Finally, defining a ring to be $rightGIF$ if $r.GIFD(R)=0$, we find that

{semisimple rings} ⊂ {$rightGIF$ rings} ⊂ {IF rings}

and the inclusions are strict.

In Section 2, we introduce the definition GF-torsion modules, and give some characterizations of GF-torsion dimensions and $l.GFFD(R)$. For any integer $n\geqslant 0$ and $N\in R$-Mod, it is shown that (Theorem 2.4)

$$Gftd_R(N)\leqslant n\Leftrightarrow Tor_i^R(Q,N)=0$$

for any Gorenstein flat right R-module Q and all $i\geqslant n+1\Leftrightarrow$ if the sequence $0\to N_n\to N_{n-1}\to\cdots\to N_0\to N\to 0$ is exact with $N_0,\cdots,N_{n-1}$ are GF-torsion modules, then N_n is GF-torsion. General background materials can be found in Refs. [6-9].

1 GI-Flat modules and dimensions

Our aim, in this section, is to investigate the global dimension, i. e., $r.GIFD(R)$, which will strength the study of $GIFD(R)$.

Definition 1.1 A left R-module N is called a GI-flat if $Tor_1^R(E,N)=0$ for any Gorenstein injective right R-module E.

Set

$$r.GIFD(R)=\sup\{fd(N)\mid N\in R\text{-Mod with N Gorenstein injective}\}.$$

A ring R is said to be $rightGIF$ if $GIFD(R)=0$. Similarly, with the left modules, we can define the $l.GIFD(R)$. When R is a commutative ring, we drop the unneeded letters r and l.

Remark 1.2 ① In fact, the analogous concepts, $l.GIFD(R)$, $leftGIF$ rings and right GI-flat modules have been defined and studied in Refs. [3,5]. Generally, $l.GIFD\neq r.GIFD$, and if R is any ring and $wD(R)<\infty$, then

$$l.GIFD(R)=r.GIFD(R)=wD(R),$$

see Ref. [3].

② Clearly, a $rightGIF$ ring is right IF ring.

③ It is clear that the class of GI-flat left R-modules is closed under extensions, direct products and direct summands, and it is also clear that every flat left R-module is GI-flat.

Definition 1.3 Let R be a ring. Set

$$r.GIGFD(R)=\sup\{Gfd(N)\mid N\in R\text{-Mod with N Gorenstein injective}\}.$$

Remark 1.4[7] For any ring R, let

$$r.IFD(R)=\sup\{fd(N)\mid N\in R\text{-Mod with N injective}\}.$$

Then R is an IF ring if and only if $IFD(R)=0$; if and only if $GIGFD(R)=0$; if and only if $wGgldim(R)=0$ (see Ref. [10, Proposition 2.3]).

Note that

$wGgldim(R) = \sup\{Gfd_R(M) \mid M \text{ is an } R\text{-module}\}$, which is called the weak Gorenstein global dimension of R. Similarly, we can define $r.\,wGgldim(R)$ and $l.\,wGgldim(R)$.

Example 1.5[3, Example 3.4.1] Consider the quasi-Frobenius local ring $R=k[X]/(X^2)$ where k is a field, and denote by $\overline{X}$ the residue class in R of X. Then

① R is a right IF ring.

② The idea $\overline{X}$ is Gorenstein injective but not flat by Ref. [2, Examples 2.5 and 3.3]. Hence, a right IF ring is not necessarily *rightGIF*. Generally,

$$r.\,GIFD(R) \neq r.\,GIGFD(R).$$

Recall that the right finitistic projective dimension $RightFPD(R)$ of R is defined as

$$RightFPD(R) = \sup\{pd_R(M) \mid M \text{ is a right } R\text{-module with } pd_R(M) < \infty\}.$$

The (left) finitistic Gorenstein dimension of the ring R is defined by

$$FGFD(R) = \sup\{Gfd_R(M) \mid M \text{ is a (left) } R\text{-module with } Gfd_R(M) < \infty\}.$$

And $FFD(R)$ denotes the usual (left) finitistic flat dimension, see Refs. [8,11].

Proposition 1.6 ① Assume that R is left and right coherent with finite $RightFPD(R)$,

$$r.\,IFD(R) \leqslant r.\,GIGFD(R) \leqslant r.\,GIFD(R).$$

② Assume that R is right coherent, then $FGFD \leqslant l.\,IFD(R)$.

Proof ① It is clear that $r.\,GIGFD(R) \leqslant r.\,GIFD(R)$.

Assume that R is left and right coherent with finite $RightFPD(R)$. For any injective right R-modules E, since $fd_R(E)=Gfd_R(E)$[11, Theorem 2.6], we have

$$fd_R(E) \leqslant r.\,GIGFD(R).$$

Hence

$$r.\,IFD(R) \leqslant r.\,GIGFD(R).$$

② Note that

$$FGFP(R) = FFD(R)$$

by Ref. [8, Theorem 3.24]. In order to prove that

$$FGFD(R) \leqslant l.\,IFD(R),$$

we need only to show $FFD(R) \leqslant l.\,IFD(R)$. In fact, for any left R-module N, we choose a short exact sequence $0 \to N \to E \to L \to 0$ with E injective. Then, for any right R-module M with $fd_R(M)=n<\infty$, we obtain the exact sequence

$$0 \to Tor_n^R(M,N) \to Tor_n^R(M,E).$$

If $n > l.\,IFD(R)$, then the last term is zero. Hence $Tor_n^R(M,N)=0$, a contradiction. □

Let $0 \to F_n \xrightarrow{d_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0 \to N \to 0$ be a flat resolution of left R-module N, then $Y_n = \ker d_n$, $n \geqslant 1$ is said to be the nth yoke of N. Next we give a functional description of $GIFD(R)$ in terms of GI-flat left R-modules.

The proof of ①⇒②⇒③ on the following theorem is analogous that of Ref. [3, Theorem 3.4.4]. For completeness, we give a proof here.

Theorem 1.7 Let R be a ring and n be a positive integer, then the following four conditions are equivalent:

① $r.\,GIFD(R) \leqslant n$.

② If $0 \to Q_n \to \cdots \to Q_1 \to Q_0 \to N \to 0$ is an exact sequence such that for any Gorenstein injective right R-module E,

$$Tor_n^R(E,Q_0) = Tor_{n-i+1}^R(E,Q_i) = Tor_{n-i}^R(E,Q_i) = 0,\ i=1,2,\cdots,n-1,$$

then Q_n is GI-flat.

③ Every left R-module N has a GI-flat $(n-1)$th yoke.

④ For every left ideal I, R/I has a GI-flat $(n-1)$th yoke.

Proof ①⇒② Consider the following short exact sequence:

$0 \to K_{i+1} \to Q_i \to K_i \to 0$ for $i=0,1,2,\cdots,n-1$, where

$$K_0 = Q_{-1} = N,\ K_i = \ker(Q_{i-1} \to Q_{i-2}),$$

and $K_n = Q_n$. For any Gorenstein injective right R-module E, $fd_R(E) \leqslant n$ by ①, so

$$Tor_{n+1}^R(E,Q_0) = Tor_{n+1}^R(E,K_0) = 0.$$

By hypothesis, we have

$$Tor_{n-i+1}^R(E,Q_i) = Tor_{n-i}^R(E,Q_i) = 0,\quad i=1,2,\cdots,n-1.$$

Consider the sequence $0 \to K_{i+1} \to Q_i \to K_i \to 0$, we have

$$0 = Tor^R_{n-i+1}(E, Q_i) \to Tor^R_{n-i+1}(E, K_i) \to Tor^R_{n-i}(E, K_{i+1}) \to Tor^R_{n-i}(E, Q_i) = 0.$$

That is equivalent to saying

$$Tor^R_{n-i+1}(E, K_i) \cong Tor^R_{n-i}(E, K_{i+1}).$$

Hence

$$Tor^R_1(E, Q_n) \cong Tor^R_2(E, k_{n-1}) \cong \cdots \cong Tor^R_{n+1}(E, Q_0) = 0,$$

which implies that Q_n is GI-flat.

②⇒③ Let $\cdots \to F_{n-1} \to \cdots \to F_1 \to F_0 \to N \to 0$ be a flat resolution of left R-module N and Y_n the $(n-1)$th yoke. Since F_i is flat for any $i=0,1,\cdots,n-1$, then any Gorenstein injective right R-module E, we have

$$Tor^R_n(E, F_0) = Tor^R_{n-i+1}(E, F_i) = Tor^R_{n-i}(E, F_i) = 0,\ i = 1, 2, \cdots, n-1,$$

thus, Y_{n-1} is GI-flat by ②.

③⇒④ Clearly.

④⇒① Let $\cdots \to F_{n-1} \to \cdots \to F_1 \to F_0 \to R/I \to 0$ be a flat resolution of R/I. By ④, the $(n-1)$th yoke Y_{n-1} is GI-flat, thus

$$Tor^R_{n+1}(E, R/I) \cong Tor^R_1(E, K_n) = 0$$

for any Gorenstein injective right R-module E, that is to say $r.GIFD(R) \leqslant n$. □

Considering the case $n=0$ and $n=1$ in the above Theorem 1.7, we can obtain some useful characterizations.

Corollary 1.8 Let R be a ring. Then the following three conditions are equivalent:

① R is *rightGIF*.

② Every left R-module N is GI-flat.

③ For every left ideal I, R/I is GI-flat.

Corollary 1.9 Let R be a ring. Then the following four conditions are equivalent:

① $r.GIFD(R) \leqslant 1$.

② Every submodule of a GI-flat left R-module is GI-flat.

③ Every submodule of a flat left R-module is GI-flat.

④ For every left ideal of R is GI-flat.

Proof ①⇒② Suppose K is a submodule of a GI-flat left R-module F. Then we have an exact sequence

$$0 \to K \to F \to F/K \to 0.$$

For any Gorenstein injective right R-module E, we obtain the following exact sequence

$$Tor_2{}^R(E, F/K) \to Tor^R_1(E, K) \to Tor^R_1(E, F).$$

The first term is zero since $fd_R(E) \leqslant 1$ and the last term is zero since F is GI-flat. Thus

$$Tor^R_1(E, K) = 0.$$

②⇒③⇒④ are trivial.

④⇒① For every left ideal I, there exists a short exact sequence $0 \to I \to R \to R/I \to 0$, so R/I has a GI-flat $1th$ yoke by ④. Thus $r.GIFD(R) \leqslant 1$ by Theorem 1.7. □

The following corollary is the version of GI-flat left R-module of Ref. [5, Proposition 3.14].

Corollary 1.10[3, Proposition 3.4.2] Let R be a ring, then $WD(R) \leqslant 1$ if and only if $r.GIFD(R) \leqslant 1$ and every GI-flat left R-module is flat.

The following proposition generalizes the results of Ref. [5, Theorem 3.11].

Proposition 1.11 Let R be a commutative ring and n a nonnegative integer. Then $GIFD(R) \leqslant n$ if and only if $fd_R(E \otimes F) \leqslant n$ for all flat R-modules F and for all Gorenstein injective R-modules E.

Proof (⇐) follows with $F=R$.

(⇒) For any Gorenstein injective R-module E with $fd_R(E) \leqslant n$, we choose an exact sequence

$$0 \to Q_n \to \cdots \to Q_1 \to E \to 0$$

with every Q_i flat. Then, for any flat R-module F,

$$0 \to Q_n \otimes F \to \cdots \to Q_1 \otimes F \to E \otimes F \to 0$$

is exact and $Q_i \otimes F$ is flat. Thus,

$$fd_R(E \otimes F) \leqslant n.$$ □

Corollary 1.12 Let R be a commutative ring and n a nonnegative integer. Then R is *rightGIF* if and only if $E \otimes F$ is flat for all flat R-modules F and for all Gorenstein injective R-modules E.

2 GF-torsion modules

Definition 2.1 A left R-module N is called a GF-torsion if $Tor^R_1(Q, N) = 0$ for any Gorenstein flat right R-module Q.

Definition 2.2 Let R be a ring and $N \in R$-Mod. The GF-torsion dimension of a module N,

$Gftd_R(N)$, is defined to be the smallest integer $n \geqslant 0$ such that

$$Tor_{n+1}^R(Q, N) = 0$$

for any Gorenstein flat right R-module Q. If there is no such n, set $Gftd_R(N) = \infty$. And we define $l.GFFD(R)$ as the supremum of the GF-torsion dimension of all left R-modules.

Note ① Every flat left R-module is GF-torsion.

② It is clear that the class of GF-torsion left R-modules is closed under (arbitrary) direct sums. If R is right coherent, then the class of GF-torsion left R-modules is projectively resolving and closed under direct summands.

The following lemma is the version of right R-modules of Ref. [12, Lemma 3.3].

Lemma 2.3 Let $0 \to N \to F \to N' \to 0$ be a short exact sequence of right R-modules. If N' is Gorenstein flat and F is flat. Then N is Gorenstein flat.

Theorem 2.4 Let $N \in R$-Mod and for any integer $n \geqslant 0$, the following statements are equivalent.

① $Gftd_R(N) \leqslant n$;

② $Tor_i^R(Q, N) = 0$ for any Gorenstein flat right R-module Q and all $i \geqslant n+1$;

③ If the sequence $0 \to N_n \to N_{n-1} \to \cdots \to N_0 \to N \to 0$ is exact with $N_0, \cdots, N_{n-1}$ are GF-torsion modules, then N_n is GF-torsion.

Proof ①⇔② holds by definition.

③⇒① Consider any exact sequence with the form $0 \to K \to F_{n-1} \to \cdots \to F_0 \to N \to 0$, where every F_i is flat left R-module. Since every flat module is a GF-torsion module, K is a GF-torsion module by ③. So, for all Gorenstein flat right R-modules Q, we have

$$Tor_{n+1}^R(Q, N) = Tor_1^R(Q, K) = 0.$$

Thus, $Gftd_R(N) \leqslant n$.

②⇒③ Let $0 \to N_n \to N_{n-1} \to \cdots \to N_0 \to N \to 0$ be an exact sequence with $N_0, \cdots, N_{n-1}$ are GF-torsion modules. And let Q be any Gorenstein flat right R-module. A standard argument shows that

$$Tor_1^R(Q, N_n) = Tor_{n+1}^R(Q, N) = 0.$$

Thus, N_n is a GF-torsion module. □

Proposition 2.5 Let R be a ring, then the following statements are equivalent:

① $l.GFFD(R) \leqslant 1$;

② Every submodule of any flat left R-module is a GF-torsion module;

③ Every submodule of any GF-torsion module is a GF-torsion module.

Proof ①⇒② Let F be any flat left R-module and G a submodule of F. Then $0 \to G \to F \to F/G \to 0$ induces the sequence

$$Tor_2^R(Q, F/G) \to Tor_1^R(Q, G) \to Tor_1^R(Q, F) = 0$$

where Q is any Gorenstein flat right R-module. By Theorem 2.4,

$$Tor_2^R(Q, F/G) = 0$$

since $Gftd_R(F/G) \leqslant 1$. So we have

$$Tor_1^R(Q, G) = 0.$$

②⇒① Let N be any left R-module and choose a short exact sequence $0 \to K \xrightarrow{i} F \to N \to 0$, where F is flat and i is the inclusion map. Since K is GF-torsion module,

$$Tor_2^R(Q, N) \to Tor_1^R(Q, K) = 0$$

for any Gorenstein flat right R-module Q. Hence $Gftd_R(N) \leqslant 1$, as desired.

③⇒② is trivial since every flat left R-module is a GF-torsion module.

②⇒③ Let N be any GF-torsion module and L any submodule of N, then there exists a short exact sequence $0 \to L \to N \to N/L \to 0$. Choose a short exact sequence $0 \to K \xrightarrow{i} F \to N/L \to 0$ where F is flat. Consider the following pullback diagram of $F \to N/L$ and $N \to N/L$:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & \to & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & L & \to & H & \to & F & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L & \to & N & \to & N/L & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

By ③, K is GF-torsion. Furthermore, in the short exact sequence $0 \to K \to H \to N \to 0$, both K and N are GF-torsion, then so is H. Consider the sequence $0 \to L \to H \to F \to 0$. Let Q be any right R-module. By the exact sequence,

$$0 = Tor_2^R(Q,F) \to Tor_1^R(Q,L) \to Tor_1^R(Q,H) = 0,$$

we get $Tor_1^R(Q,L)=0$, as desired. □

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