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一类非线性扰动发展方程的渐近解

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摘要:研究了一类非线性发展方程. 首先作行波变换,讨论了在非扰动情况下的非线性方程,利用双曲函数待定系数方法,求得了相应方程的孤立子精确解. 然后利用广义变分迭代方法,求出了原非线性扰动发展方程渐近孤立子行波解. 最后通过举例,说明了利用本方法求出的渐近孤立子解简单可行,并有良好的精度.

关键词:发展方程;非线性;扰动

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The asymptotic solutions to a class of nonlinear disturbed evolution equations

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Abstract: A class of nonlinear evolution equations was considered. Firstly, introducing a travelling wave transform, the non-disturbance case was discussed by employing the undetermined coefficient method of the hyperbolic functions and solitary exact solution to the corresponding nonlinear equation was obtained. Then, the solitary travelling wave asymptotic solution to the original nonlinear disturbed evolution equation was founded by using the generalized variational iteration method. Finally, an example was given to show the simplicity and feasibility of the asymptotic solitary solution.

Key words: evolution equation; nonlinear; destabilization

0 引言

非线性发展方程在数学物理、理论物理、凝聚态,生物数学、流体力学等许多领域中有广泛的应用,是当前学术界关注的对象. 近来许多学者在激波理论[1-3]、光波散射[4]、神经网络[5]、大气物理[6-8]

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等方面都作了许多研究. 非线性孤立子理论的各种解法也不断地在创新. 孤立子理论的一种研究方法是孤立子扰动理论的渐近方法. 它主要是用扰动理论渐近展开式将非线性孤立子方程转化为易求解的方程来求解[9,10]. 它的优点在于思路简明,计算方便,能得到较高近似度的解,并且,得到的渐近解保留了相应的解析运算特性,故对得到的结果还能进一步分析. 本方法具有较广泛的研究前景.

莫嘉琪等利用微分不等式、不动点原理、同伦映射等方法讨论了一系列非线性问题[11-28]. 例如,文献[21]用同伦映射得到了一类 KdV 方程的近似解,文献[27]用奇异摄动方法构造了一类二次方程在转向点附近具有尖层性质的渐进解.目前, 对于非线性发展方程已有所研究,它代表的是相应自然现象的精简和浓缩. 但从进一步研究的意义来说,它们已经不能完全适应当前科学发展的需要. 故有必要来研究更能代表真实自然现象的非线性扰动发展方程. 本文就是在这样背景下提出来的.

1 扰动发展方程

讨论如下一类非线性扰动发展方程：

$$u_{xx}-u_{tt}+mu^2+puu_x+quu_t=f(u,u_x,u_t) \tag{1}$$

式中, m,p,q 为常数, $f(x,y,z)$ 为扰动项,设它是关于其变量在所讨论的范围内为充分光滑的有界函数.

众所周知,非线性方程(1)一般没有有限项初等函数形式的精确解. 为此我们将采用特殊的方法来求得方程(1)渐近解.

首先,作行波变换：

$$s=ax-ct \tag{2}$$

式中, a,c 为常数. 将式(2)代入式(1), 得

$$(a^2-c^2)u_{ss}+mu^2+(ap-cq)uu_s=f(u,au_x,-cu_s) \tag{3}$$

当 $f=0$ 时, 对应于方程(3)的无扰动情况下的典型方程为

$$(a^2-c^2)u_{ss}+mu^2+(ap-cq)uu_s=0 \tag{4}$$

现在来求出上述典型方程的孤立子精确解. 设非线性方程(4)有如下的形式的解：

$$u=D_0+D_1v+D_2v^2+D_3(1-v^2)^{1/2} \tag{5}$$

式中, $D_i(i=0,1,2,3)$ 为待定常数, 而 v 是方程

$$\frac{\mathrm{d}v}{\mathrm{d}s}=-v(1-v^2)^{1/2} \tag{6}$$

的解. 显然方程(6)有解为

$$v=\operatorname{sech}s \tag{7}$$

由式(5)～(7),有

$$u=D_0+D_1\operatorname{sech}s+D_2\operatorname{sech}^2s+D_3\tanh s \tag{8}$$

$$u_s=D_3\operatorname{sech}^2s-D_1\operatorname{sech}s\tanh s-2D_2\operatorname{sech}^2s\tanh s \tag{9}$$

$$u_{ss}=D_1\operatorname{sech}s+4D_2\operatorname{sech}^2s-2D_1\operatorname{sech}^3s-6D_2\operatorname{sech}^4s-2D_3\operatorname{sech}^2s\tanh s \tag{10}$$

$$\begin{aligned}uu_s={}&D_1D_3\operatorname{sech}s+(D_0D_3-2D_2D_3)\operatorname{sech}^2s+3D_2D_3\operatorname{sech}^4s-D_0D_1\operatorname{sech}s\tanh s+\\&(D_3^2-2D_0D_2-D_1^2)\operatorname{sech}^2s\tanh s+(D_1D_2-2D_1D_2)\operatorname{sech}^3s\tanh s-2D_2^2\operatorname{sech}^4s\tanh s\end{aligned} \tag{11}$$

将式(8)～(11)代入式(4), 合并同类函数项的系数, 可得

$$\begin{aligned}&[(a^2-c^2)D_1+2m^2D_0D_1+(ap-cq)D_1D_3]\operatorname{sech}s+\\&[4(a^2-c^2)D_2+m^2(D_1^2+2D_0D_2-D_3^2)+(ap-cq)(D_0D_3-2D_2D_3)]\operatorname{sech}^2s+\\&[2m^2D_1D_2-2(a^2-c^2)D_1]\operatorname{sech}^3s+\\&[-6(a^2-c^2)D_2+3(ap-cq)D_2D_3]\operatorname{sech}^4s+\\&[2m^2D_1D_2]\tanh s+\\&[2m^2D_1D_3-(ap-cq)D_0D_1]\operatorname{sech}s\tanh s+\\&[2m^2D_2D_3-2(a^2+c^2)D_3+(ap-cq)(D_3^2-2D_0D_2-D_1^2)]\operatorname{sech}^2s\tanh s+\\&(ap-cq)(D_1D_2-2D_1D_2)\operatorname{sech}^3s\tanh s-\\&(ap-cq)D_2^2\operatorname{sech}^4s\tanh s=0.\end{aligned}$$

因此

$$((a^2-c^2)+2m^2D_0+(ap-cq)D_3)D_1=0 \tag{12}$$

$$4(a^2-c^2)D_2+m^2(D_1^2+2D_0D_2-D_3^2)+(ap-cq)(D_0D_3-2D_2D_3)=0 \tag{13}$$

$$(m^2D_2-(a^2-c^2))D_1=0 \tag{14}$$

$$(-2(a^2-c^2)+(ap-cq)D_3)D_2=0 \tag{15}$$

$$m^2D_1D_2=0 \tag{16}$$

$$(2n^2D_3-(ap-cq)D_0)D_1=0 \tag{17}$$

$$2(n^2D_2-(a^2-c^2))D_3+(ap-cq)(D_3^2-2D_0D_2-D_1^2)=0 \tag{18}$$

$$(ap-cq)D_1D_2=0 \tag{19}$$

$$(ap-cq)D_2^2=0 \tag{20}$$

由式(12)～(20), 可得

$$\left.\begin{aligned} D_0 &= \frac{2m^2(a^2-c^2)}{(ap-cq)^2},\ D_1 = D_2 = 0, \\ D_3 &= \frac{2(a^2-c^2)}{ap-cq} \end{aligned}\right\} \tag{21}$$

将式(7),(21)代入式(5)，便得到非线性方程(4)的孤立子精确解：

$$u(s) = \frac{2(a^2-c^2)}{ap-cq}\left(\frac{m^2}{ap-cq} + \tanh s\right) \tag{22}$$

取 $a=2, c=m=p=q=1$ 时，非线性方程(4)对应的孤立子精确解曲线如图 1 所示.

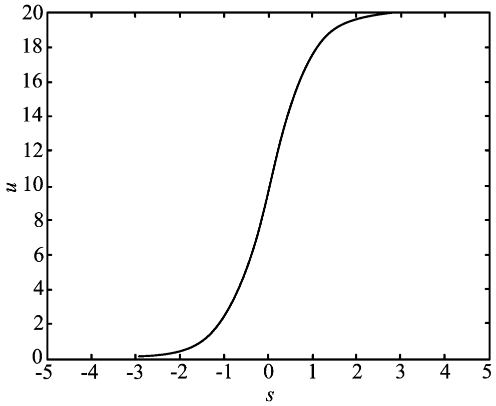

图 1　非线性方程(4)的孤立子精确解曲线，$a=2, c=m=p=q=1$

Fig.1　Curve of the solitary exact solution $u(s)$ as $a=2, c=m=p=q=1$

将行波变换(2)代入(22)，我们便得到非线性典型方程：

$$u_{tt} - u_{xx} + mu^2 + puu_x + quu_t = 0 \tag{23}$$

的精确孤立子行波解 $u(t,x)$：

$$u(t,x) = \frac{2(a^2-c^2)}{ap-cq}\left(\frac{m^2}{ap-cq} + \tanh(ax-ct)\right) \tag{24}$$

2　扰动方程渐近解

先讨论非线性扰动方程(3). 引入如下函数 $u(s)$ 的泛函 $F[u]$：

$$F[u] = u(s) - \int_0^s \lambda(\eta)[(a^2+c^2)u_{\eta\eta} + m\bar{u}^2 + (ap-cq)\bar{u}\,\bar{u}_\eta - \bar{f}(\bar{u}, a\bar{u}_\eta, -c\bar{u}_\eta)]\mathrm{d}\eta \tag{25}$$

式中，$\bar{u}, \bar{f}$ 分别为 u, f 的限制变量[29,30]，λ 为对应的 Lagrange 算子.

下面来计算泛函 $F[u]$ 的变分 δF，并令其为零.

$$\begin{aligned} \delta F &= \delta u - \delta\int_0^s \lambda(\eta)[(a^2-c^2)u_{\eta\eta} + m\bar{u}^2 + (ap-cq)\bar{u}\,\bar{u}_\eta - \bar{f}(\bar{u}, a\bar{u}_\eta, -c\bar{u}_\eta)]\mathrm{d}\eta = \\ &\delta u - (a^2-c^2)[\lambda(\eta)\delta u_\eta]_{\eta=s} + (a^2-c^2)[\lambda'(\eta)\delta u]_{\eta=s} - \\ &(a^2-c^2)\int_0^s \lambda''(\eta)\delta u\,\mathrm{d}\eta = 0. \end{aligned}$$

于是有

$$\frac{\mathrm{d}^2\lambda}{\mathrm{d}\eta^2} = 0 \tag{26}$$

$$\lambda(\eta)\big|_{\eta=s} = 0,\ \lambda'(\eta)\big|_{\eta=s} = -\frac{1}{(a^2-c^2)} \tag{27}$$

显然问题(26),(27)的解为

$$\lambda(\eta) = -\frac{1}{(a^2-c^2)}(\eta - s) \tag{28}$$

由式(25)和(28)，我们构造如下非线性扰动方程(3)解的广义变分迭代式：

$$\begin{aligned} u_{n+1}(s) &= u_n(s) + \\ &\frac{1}{(a^2-c^2)}\int_0^s (\eta - s)\Big[(a^2-c^2)\frac{\mathrm{d}^2 u_n}{\mathrm{d}\eta^2} + mu_n^2 + (ap-cq)u_n\frac{\mathrm{d}u_n}{\mathrm{d}\eta} - \\ &f\left(u_n, a\frac{\mathrm{d}u}{\mathrm{d}\eta}, -c\frac{\mathrm{d}u}{\mathrm{d}\eta}\right)\Big]\mathrm{d}\eta \end{aligned} \tag{29}$$

式中，$u_0(s)$ 为解的初始迭代，$n=0,1,\cdots$.

选取初始迭代 $u_0(s)$ 为非线性方程(4)的孤立子解(22). 即

$$u_0(s) = \frac{2(a^2-c^2)}{ap-cq}\left(\frac{m^2}{ap-cq} + \tanh s\right) \tag{30}$$

于是由初始迭代(30)，便可得到广义变分迭代式(29)的一次渐近解 $u_1(s)$：

$$\begin{aligned} u_1(s) &= \frac{2(a^2-c^2)}{ap-cq}\left(\frac{m^2}{ap-cq} + \tanh s\right) - \\ &\frac{1}{(a^2-c^2)}\int_0^s (\eta - s)\left[f\left(u_0, a\frac{\mathrm{d}u_0}{\mathrm{d}\eta}, -c\frac{\mathrm{d}u_0}{\mathrm{d}\eta}\right)\right]\mathrm{d}\eta \end{aligned} \tag{31}$$

式中，$u_0(s)$ 由式(30)表示.

由一次迭代(31)和广义变分迭代式(29)可得二次渐近解 $u_2(s)$：

$$\begin{aligned} u_2(s) &= \frac{2(a^2-c^2)}{ap-cq}\left(\frac{m^2}{ap-cq} + \tanh s\right) + \\ &\frac{1}{(a^2-c^2)}\int_0^s (\eta - s)\Big[(a^2-c^2)\frac{\mathrm{d}^2 u_1}{\mathrm{d}\eta^2} + \\ mu_1^2 &+ (ap-cq)u_1\frac{\mathrm{d}u_1}{\mathrm{d}\eta} - \left(f\left(u_0, a\frac{\mathrm{d}u_0}{\mathrm{d}\eta}, -c\frac{\mathrm{d}u_0}{\mathrm{d}\eta}\right) + \right. \end{aligned}$$

$$f\left(u_1, a\frac{du_1}{d\eta}, -c\frac{du_1}{d\eta}\right)\Big]d\eta \tag{32}$$

式中，$u_0(s)$，$u_1(s)$ 分别由式(30)，(31)表示.

同样可依次得到非线性扰动方程(3)的 n 次孤立子渐近解 $u_n(s)$：

$$u_n(s)=\frac{2(a^2-c^2)}{ap-cq}\left(\frac{m^2}{ap-cq}+\tanh s\right)+\frac{1}{(a^2-c^2)}\int_0^s(\eta-s)\cdot\sum_{i=1}^{n-1}\left[(a^2-c^2)\frac{d^2u_i}{d\eta^2}+mu_i^2+(ap-cq)u_i\frac{du_i}{d\eta}\right]-\sum_{i=0}^{n-1}f\left(u_i, a\frac{du_i}{d\eta}, -c\frac{du_i}{d\eta}d\eta\right) \tag{33}$$

式中，$u_i(s)(i\leqslant n-1)$ 分别为逐次已知的函数，$n=3,4,\cdots$.

再由行波变换(2)代入式(31)～(33)的 $u_n(s)$，便得到非线性扰动发展方程(1)的 n 次孤立子渐近行波解 $u_n(t,x)$：

$$u_n(t,x)=\frac{2(a^2-c^2)}{ap-cq}\left(\frac{m^2}{ap-cq}+\tanh(ax-c)\right)+\frac{1}{(a^2+c^2)}\int_0^{ax-ct}(\eta-s)\cdot\sum_{i=1}^{n-1}\left[(a^2-c^2)\frac{d^2u_i}{d\eta^2}+mu_i^2+(ap-cq)u_i\frac{du_i}{d\eta}\right]-\sum_{i=0}^{n-1}f\left(u_i, a\frac{du_i}{d\eta}, -c\frac{du_i}{d\eta}d\eta\right),\ n=1,2,\cdots.$$

3 举例

为简单起见，取 $a=2, c=m=p=q=1$. 讨论如下一个特殊的非线性扰动发展方程：

$$u_{xx}-u_{tt}+u^2+uu_x+uu_t=\varepsilon\exp(-u^2) \tag{34}$$

式中，ε 为小的正参数. 由式(2)，作行波变换：$s=2x-t$，对应于发展方程(34)为

$$3u_{ss}+u^2+2uu_s=\varepsilon\exp(-u^2) \tag{35}$$

方程(35)相应的无扰动情形为

$$3u_{ss}+u^2+2uu_s=0 \tag{36}$$

由式(22)知非线性方程(36)有如下孤波精确解，并取它为扰动方程(35)的广义变分迭代的初始近似：

$$u_0(s)=6(1+\tanh s) \tag{37}$$

由式(31)，非线性方程(35)的一次孤立子渐近解 $u_1(s)$ 为

$$u_1(s)=6(1+\tanh s)+\frac{1}{3}\int_0^s(\eta-s)[-\varepsilon\exp(-100(1+\tanh\eta)^2)]d\eta \tag{38}$$

选取 $\varepsilon=0.2$，非线性方程(35)的零次、一次孤立子渐近解 $u_0(s)$，$u_1(s)$ 的曲线如图 2 所示.

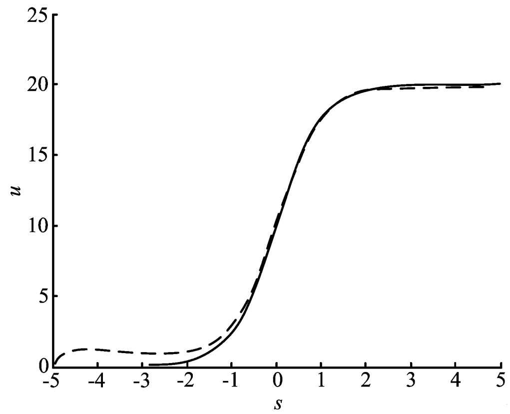

图 2 非线性方程(34)在 $\varepsilon=0.2$ 时的零次(实线)、一次(虚线)孤立子渐近解曲线

Fig.2 The zeroth times (real line) and first times (imaginary line) asymptotic cures of solitary solution as $\varepsilon=0.2$

由图 2 可以看到一次迭代解 $u_1(s)$ 零次近似解 $u_0(s)$ 曲线的校正情况.

由式(32)，非线性方程(35)的二次孤立子渐近解 $u_2(s)$ 为

$$u_2(s)=6(1+\tanh s)+\frac{1}{3}\int_0^s(\eta-s)\left[\left(3\frac{d^2u_1}{d\eta^2}+u_1^2+u_1\frac{du_1}{d\eta}-\varepsilon(\exp(-u_0)-\exp(-u_1)\right)\right]d\eta \tag{39}$$

式中，$u_0(s)$，$u_1(s)$ 分别由式(37)，(38)表示.

同样可依次得到非线性扰动方程(35)的 n 次孤立子渐近解 $u_n(s)$：

$$u_n(s)=6(1+\tanh s)+\frac{1}{3}\int_0^s(\eta-s)\left[3\sum_{i=1}^{n-1}\left[\frac{d^2u_i}{d\eta^2}+u_i^2+u_i\frac{du_i}{d\eta}-\varepsilon\sum_{i=0}^{n-1}\left[\exp\left(-\sum_{i=1}^{n-1}u_i^2\right]d\eta\right.\right.\right. \tag{40}$$

式中，$u_i(s)(i\leqslant n-1)$ 分别为逐次已知的函数，$n=3,4,\cdots$.

由摄动理论[31,32]还可得知非线性扰动发展方程(34)对应的孤立子解 $u(s)$ 与第 n 次迭代渐近解 $u_n(s)$ 之间在有限的区域上具有如下的渐近估计式：

$$u(s)=u_n(s)+O(\varepsilon^{n+1}),\ n=1,2,\cdots,0<\varepsilon\ll 1$$

再由行波变换(2)代入式(37)～(40),便得到非线性扰动发展方程(34)的 n 次孤立子渐近行波解 $u_n(t,x)$：

$$u_n(t,x)=6(1+\tanh(2x-t))+\frac{1}{3}\int_0^{2x-t}(\eta-s)\Big[5\sum_{i=1}^{n-1}\Big[\frac{d^2u_i}{d\eta^2}+u_i^2+u_i\frac{du_i}{d\eta}-\varepsilon\sum_{i=0}^{n-1}\Big[\exp\Big(-\sum_{i=1}^{n-1}u_i^2\Big]d\eta,\ n=1,2,\cdots$$

4 结论

本文使用的广义变分迭代方法来求非线性扰动发展方程的渐近解简单、可行. 选取的初始迭代 u_0 是采用对应的非扰动发展孤立子精确解. 它保证了对应的非线性扰动发展方程较快地求得在要求的精度范围内的孤立子渐近解. 这样的解更接近模型的真实现象且所得的结果更加简单实用.

本文采用的广义变分迭代方法，它不同于通常的数值方法作简单的模拟. 用广义变分迭代方法得到解的表示式能够继续进行解析运算.所以由相应的渐近解的表示式,我们还能够用微分、积分等手段来继续研究方程解各种性态.

参考文献(References)

[1] MCPHADEN M J, ZHANG D. Slowdown of the meridional overturning circulation in the upper Pacific Ocean [J]. Nature, 2002, 415: 603-608.

[2] GU Daifang, PHILANDER S G H. Interdecadal climate fluctuations that depend on exchanges between the tropics and extratropics [J]. Science, 1997, 275 (7): 805-807.

[3] 刘式适，赵强，付遵涛，等. 一类非线性方程的新周期解[J]. 物理学报，2002，51(1)：10-14.
LIU Shikuo, ZHAO Qiang, FU Zuntao, et al. New periodic solutions to a kind of nonlinear wave equations [J]. Acta Phys Sin, 2002, 51(1): 10-14.

[4] 左伟明，潘留仙，颜家壬. Landau-Ginzburg-Higgs 方程的微扰理论[J]. 物理学报，2005，54 (1)：1-5.
ZUO Weiming, PAN Liuxian, YAN Jiaren. The theory of the perturbation for Landau-Ginzburg-Higgs equation[J]. Acta Phys Sin, 2005, 54 (1): 1-5.

[5] 陈观伟，马世旺. 带有边界势和一般时间频率的非周期离散非线性 Schrodinger 方程:无穷多个孤立子[J]. 中国科学:数学，2014，44(8)：843-856.
CHEN Guanwei, MA Shiwang. Non-periodic discrete nonlinear Schrodinger equations with unbounded potential and general temporal frequencies: Infinitely many solitons [J]. Scientia Sinica (Mathematica), 2014, 44(8): 843-856.

[6] LIN Yihua, ZENG Qingcun. Kelvin waves in a simple air-sea coupled model in the tropics [J]. Progress in Natural Sci, 1999, 9 (3): 211-215.

[7] LIN Yihua, JI Zhongzhen, ZENG Qingcun. Meridional wind forced low frequency disturbances in the tropical ocean [J]. Progress in Natural Sci, 1999, 9 (7): 532-538.

[8] 封国林，戴新刚，王爱慧，等. 混沌系统中可预报性研究[J]. 物理学报，2001，50(4)：606-611.
FENG Guolin, DAI Xingang, WANG Aihui, et al. On numerical predictability in the chaos system [J]. Acta Phys Sin, 2001, 50(4): 606-611.

[9] PAN Liuxian, YAN Jiaren, ZHOU Cuanghui. Direct approach for soliton perturbations based on the Green's function[J]. Chin Phys, 2003, 10 (7): 594-598.

[10] 范恩贵，张鸿庆. 非线性波动方程的孤波解[J]. 物理学报，1997，46 (7)：1254-1258.
FAN Engui, ZHANG Hongqing. The solitary wave solutions for a class of nonlinear wave equations [J]. Acta Phys Sin, 1997, 46 (7): 1254-1258.

[11] MO Jiaqi. Singular perturbation for a class of nonlinear reaction diffusion systems[J]. Science in China, Ser A, 1989, 32(11): 1306-1315.

[12] MO Jiaqi, LIN Wantao, ZHU Jiang. The perturbed solution of sea-air oscillator for ENSO model [J]. Progress in Natural Sci, 2004, 14(6): 550-552.

[13] MO Jiaqi, LIN Wantao. The variational iteration solving method for sea-air oscillator model of interdecadal climate fluctuations [J]. Chin Phys B, 2005, 14 (5): 875-878.

[14] MO Jiaqi, WANG Hui, LIN Wantao, et al. Varitional iteration method for solving the mechanism of the equatorial Eastern Pacific El Niño-Southern Oscillation [J]. Chin Phys B, 2006, 15(4): 671-675.

[15] MO Jiaqi. The singularly perturbed problem for combustion reaction diffusion[J]. Acta Math Appl Sin, 2001, 17(2): 255-259.

[16] MO Jiaqi, SHAO S S L. The singularly perturbed boundary value problems for higher-order semilinear elliptic equations[J]. Advances in Math, 2001, 30 (2): 141-148.

[17] MO Jiaqi. A class of nonlocal singularly perturbed problems for nonlinear hyperbolic differential equation [J]. Acta Math Appl Sin, 2001, 17(4): 469-474.

[18] MO Jiaqi, WANG Hui. A class of nonlinear nonlocal singularly perturbed problems for reaction diffusion equations [J]. J Biomathematics, 2002, 17 (2): 143-148.

[19] MO Jiaqi. Homotopiv mapping solving method for gain fluency of a laser pulse amplifier[J]. Science in China, Ser G, 2009, 59 (7): 1007-1010.

[20] MO Jiaqi, LIN Sutong. The homotopic mapping solution for the solitary wave for a generalized nonlinear evolution equation[J]. Chin Phys B, 2009, 18(9): 3628-3631.

[21] MO Jiaqi. Approximate solution of homotopic mapping to solitary wave for generalized nomlinear KdV system [J]. Chin Phys Lett, 2009, 26 (1): 010204.

[22] MO Jiaqi. Generalized variational iteration solution of soliton for disturbed KdV equation[J]. Commun Theor Phys, 2010, 53 (3): 440-442.

[23] MO Jiaqi, CHEN Xianfeng. Homotopic mapping method of solitary wave solutions for generalized complex Burgers equation[J]. Chin Phys B, 2010, 19 (10): 100203.

[24] MO Jiaqi. A singularly perturbed reaction diffusion problem for the nonlinear boundary condition with two parameters[J]. Chin Phys B, 2010, 18 (1): 010203.

[25] 莫嘉琪. 一类非线性扰动发展方程的广义迭代解[J]. 物理学报, 2011, 60 (2): 020202.
MO Jiaqi. The variational iteration solution method for a class of nonlinear disturbed evolution equations[J]. Acta Phys Sin, 2011, 60 (2): 020202.

[26] FENG Yihu, MO Jiaqi. The shock asymptotic solution for nonlinear elliptic equation with two parameters[J]. Math Appl, 2015, 27(3): 579-585.

[27] FENG Yihu, LIU Shude. Spike layer solutions of some quadratic singular perturbation problems with high-order turning points[J]. Math Appl, 2014, 27 (1): 50-51.

[28] 冯依虎, 石兰芳, 汪维刚, 等. 一类广义非线性强阻尼扰动发展方程的行波解[J]. 应用数学和力学, 2015, 36(3): 315-324.
FENG Yihu, SHI Lanfang, WANG Weigang, et al. The traveling wave solution for a class of generalized nonlinear strong damping disturbed evolution equations [J]. Appl Math Mech, 2015, 36(3): 315-324.

[29] 何吉欢. 工程和科学计算中的近似非线性分析方法[M]. 郑州: 河南科学技术出版社, 2002.

[30] HE J H, WU X H. Construction of solitary solution and compacton-like solution by variational iteration method[J]. Chaos Solitions & Fractals, 2006, 29 (1): 108-113.

[31] DE JAGER E M, JIANG Furu. The Theory of Singular Perturbation[M]. Amsterdam, Netherlands: North- Holland Publishing Co,1996.

[32] BARBU L, MOROSANU G. Singularly Perturbed Boundary-Value Problems [M]. Basel, Switzerland: Birkhauser Verlag AG, 2007.

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[4] CVETKOVIĆ D M, DOOB M, SACKS H. Spectra of Graphs: Theory and Applications [M]. New York: Academic Press, 1979.

[5] MERRIS R. A note on Laplacian graph eigenvalues[J]. Linear Algebra Appl, 1998, 285: 33-35.

[6] CVETKOVIĈD M, DOOB M, SACHS H. Spectra of Graphs: Theory and Application [M]. Heidelberg: Johann Ambrosius Barth Verlag, 1995.

[7] YUAN X Y. A note on the Laplacian spectral radii of bicyclic graphs[J]. Advances in Mathematics, 2010, 6: 703-708.