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The maximum Laplacian separators of bicyclic and tricyclic graphs

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Abstract: Let G be an undirected simple graph of order n, $L(G)$ be the Laplacian matrix of G, and $\mu_1(G)\geqslant\mu_2(G)\geqslant\cdots\geqslant\mu_n(G)$ be the eigenvalues of $L(G)$. The Laplacian separator of G is defined as $S_L(G)=\mu_1(G)-\mu_2(G)$. Here the maximum Laplacian separators of bicyclic and tricyclic graphs of a given order were studied, and the corresponding extremal graphs were characterized.

Key words: bicyclic graph; tricyclic graph; Laplacian separator of graph; Laplacian matrix

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双圈图和三圈图的最大拉普拉斯分离度

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摘要:设 G 是一个 n 阶无向简单图,$L(G)$是 G 的拉普拉斯矩阵, 且 $\mu_1(G)\geqslant\mu_2(G)\geqslant\cdots\geqslant\mu_n(G)$是 $L(G)$的特征值. G 的拉普拉斯分离度定义为 $S_L(G)=\mu_1(G)-\mu_2(G)$.研究了给定阶数的双圈图和三圈图的最大拉普拉斯分离度,并刻画了相应的极图.

关键词:双圈图; 三圈图; 图的拉普拉斯分离度;拉普拉斯矩阵

0 Introduction

We consider only finite undirected graphs without loops and multiple edges. Let $G=(V,E)$ be a simple connected graph of order n with vertex set $V=V(G)=\{v_1,v_2,\cdots,v_n\}$ and edge set $E=E(G)=\{e_1,e_2,\cdots,e_m\}$. If $m=n-1+k$, then G is called a k-cyclic graph. Especially, if $k=0$, 1, 2 or 3, G is called a tree, unicyclic graph, bicyclic graph or tricyclic graph, respectively. In general, we denote a star, cycle, path of order n by $K_{1,n-1}$, C_n, P_n, respectively. Denote $K_{1,n-1}+se$ the

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special graph which is obtained from $K_{1,n-1}$ by adding s nonadjacent edges. Denote W_n the special graph which is obtained from $K_{1,n-1}$ by adding 2 adjacent edges.

The degree matrix of G is denoted by $D(G)=\operatorname{diag}(d(v_1), d(v_2), \cdots, d(v_n))$, where $d(v)$ denotes the degree of a vertex v in the graph G. The maximum degree of G is denoted by $\Delta(G)$, or Δ for brevity. The adjacency matrix of G is defined to be a matrix $A(G)=[a_{ij}]$ of order n, where $a_{ij}=1$ if v_i is adjacent to v_j, and $a_{ij}=0$ otherwise. The Laplacian matrix of G is defined by $L(G)=D(G)-A(G)$. The signless Laplacian matrix of G is defined by $Q(G)=D(G)+A(G)$. Obviously, $A(G)$, $L(G)$ and $Q(G)$ are real symmetric matrices. So their eigenvalues are real numbers and can be ordered. Let $\lambda_1(G)\geqslant\lambda_2(G)\geqslant\cdots\geqslant\lambda_n(G)$ be the eigenvalues of $A(G)$. The separator of G is defined as $S_A(G)=\lambda_1(G)-\lambda_2(G)$[1]. Let $\mu_1(G)\geqslant\mu_2(G)\geqslant\cdots\geqslant\mu_n(G)$ be the eigenvalues of $L(G)$. The Laplacian separator of G is defined as $S_L(G)=\mu_1(G)-\mu_2(G)$[1]. Let $q_1(G)\geqslant q_2(G)\geqslant\cdots\geqslant q_n(G)$ be the eigenvalues of $Q(G)$. The signless Laplacian separator of G is defined as $S_Q(G)=q_1(G)-q_2(G)$[2].

Recently, the separator or Laplacian separator or signless Laplacian separator of G has been studied. Li et al.[1] researched the maximum (Laplacian) separators of tree, unicyclic graph. Li et al.[3] characterized the extremal graphs which obtained the largest or second largest Laplacian separators of connected graphs. You et al.[2] presented the extremal graphs which obtained the maximum signless Laplacian separators of tree, unicyclic, bicyclic graphs, tricyclic graphs. Motivated by these researches, we study the maximum Laplacian separators of bicyclic graphs, tricyclic graphs of n order and present the corresponding extremal graphs respectively.

1 Preliminaries

Lemma 1.1[4] Let G be a simple nontrivial graph of order n, $H=G-e$ be the graph which is obtained from G by deleting the edge e. Then

$$\mu_1(G)\geqslant\mu_1(H)\geqslant\mu_2(G)\geqslant \mu_2(H)\geqslant\cdots\geqslant\mu_n(G)\geqslant\mu_n(H).$$

Lemma 1.2[5] Let G be a simple graph of order n, then

$$\mu_1(G)\leqslant\max_{1\leqslant i\leqslant n}\{d(v_i)+m(v_i)\},$$

where $m(v_i)=\dfrac{1}{d(v_i)}\sum\limits_{v_iv_j\in E(G)} d(v_j)$

Suppose we have two graphs G_1 and G_2 (where $V(G_1)$ and $V(G_2)$ are disjoint) with $v_1\in V(G_1)$ and $v_2\in V(G_2)$; the coalescence of G_1 and G_2 with respect to v_1 and v_2 is formed by identifying v_1 and v_2 and is denoted by $G_1\cdot G_2$. In other words, $V(G_1\cdot G_2)=V(G_1)\cup V(G_2)\cup\{v^*\}-\{v_1, v_2\}$, with two vertices in $G_1\cdot G_2$ adjacent if they are adjacent in G_1 or G_2, or if one is v^* and the other is adjacent to v_1 or v_2[6].

Let $L_v(G)$ be the principal sub-matrix of $L(G)$ obtained by deleting the row and column corresponding to the vertex v of G. Denote

$$\Phi(G;x)=\det(xI-L(G)),$$
$$\Phi(L_v(G);x)=\det(xI-L_v(G)).$$

Lemma 1.3[7] Let $G_1\cdot G_2$ be the coalescence of G_1 and G_2 with respect to v_1 and v_2 as defined above. Then we have

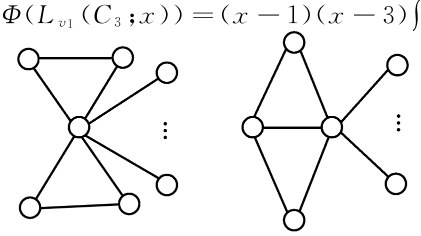
$$\Phi(G_1\cdot G_2;x)=\Phi(G_1;x)\Phi(L_{v2}(G_2);x)+\Phi(G_2;x)\Phi(L_{v1}(G_1);x)-x\Phi(L_{v1}(G_1);x)\Phi(L_{v2}(G_2);x).$$

For convenience, we give some formulas in the following.

$$\left.\begin{aligned}&\Phi(K_{1,n};x)=x(x-n-1)(x-1)^{n-1},\\&\Phi(L_{v2}(K_{1,n};x))=(x-1)^n\end{aligned}\right\}\quad(1)$$

where v_2 is the center of $K_{1,n}$.

$$\left.\begin{aligned}&\Phi(C_3;x)=x(x-3)^2,\\&\Phi(L_{v1}(C_3;x))=(x-1)(x-3)\end{aligned}\right\}\quad(2)$$



$K_{1,n-1}+2e$ $\qquad$ W_n

Fig.1 All bicyclic graphs of order n with $\Delta=n-1$

where v_1 is one vertex of C_3.

Lemma 1.4 Let $K_{1,n-1}+2e$ and W_n be the bicyclic graphs in Fig.1, $n\geqslant 7$, then

$$S_L(W_n)<S_L(K_{1,n-1}+2e)=n-3.$$

Proof ① $K_{1,n-1}+e$ can be seen as the coalescence of C_3 and $K_{1,n-3}$ with respect to v_1 and v_2, where v_1 is one of the vertex of C_3, v_2 is the center of $K_{1,n-3}$. By Eqs.(1), (2) and Lemma 1.3, we get

$$\Phi(K_{1,n-1}+e;x)=x(x-1)^{n-3}(x-3)(x-n) \tag{3}$$

② $K_{1,n-1}+2e$ can be seen as the coalescence of C_3 and $K_{1,n-3}+e$ with respect to v_1 and v_2, where v_1 is one of the vertex of C_3, v_2 is the vertex of $K_{1,n-3}+e$ with $d(v_2)=n-3$. By (3) and simple calculation, we get

$$\left.\begin{aligned}&\Phi(K_{1,n-3}+e;x)=x(x-1)^{n-5}(x-3)(x-n+2),\\&\Phi(L_{v2}(K_{1,n-3}+e);x)=(x-1)^{n-4}(x-3)\end{aligned}\right\}$$

where v_2 is the vertex of $K_{1,n-3}+e$ with $d(v_2)=n-3$.

By Eqs.(2), (3), and the above equations, and Lemma 1.3, we get

$$\Phi(K_{1,n-1}+2e;x)=x(x-3)^2(x-1)^{n-4}(x-n).$$

So, $\mu_1(K_{1,n-1}+2e)=n$, $\mu_2(K_{1,n-1}+2e)=3$, and then $S_L(K_{1,n-1}+2e)=n-3$.

③ W_n can be seen as the coalescence of W_4 and $K_{1,n-4}$ with respect to v_1 and v_2, where v_1 is the vertex of W_4 with $d(v_1)=3$, and v_2 is the center of $K_{1,n-4}$. By simple calculation, we get

$$\left.\begin{aligned}&\Phi(W_4;x)=x(x-2)(x-4)^2,\\&\Phi(L_{v1}(W_4);x)=(x-1)(x-2)(x-4)\end{aligned}\right\}$$

where v_1 is the vertex of W_4 with $d(v_1)=3$.

By (1) and Lemma 1.3, we get

$$\Phi(W_n;x)=x(x-1)^{n-4}(x-2)(x-4)(x-n).$$

So, $\mu_1(W_n)=n$, $\mu_2(W_n)=4$, and then $S_L(W_n)=n-4$. Thus the result follows.

Lemma 1.5 Let G_{1n}, G_{2n}, G_{3n}, G_{4n}, G_{5n} be the tricyclic graphs in Fig.2, $n\geqslant 9$, then

$$S_L(G_{4n})<S_L(G_{3n})<S_L(G_{2n})=S_L(G_{5n})<S_L(G_{1n})=n-3.$$

Proof Let G_1, G_2, G_3, G_4, G_5 be the graphs obtained from G_{1n}, G_{2n}, G_{3n}, G_{4n}, G_{5n} by deleting

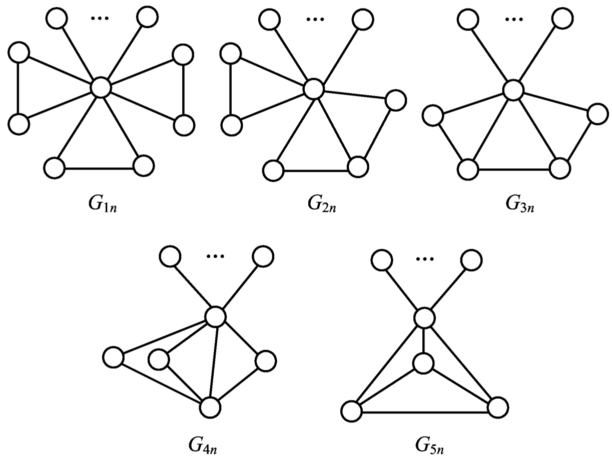


Fig.2 All tricyclic graphs of order n with $\Delta=n-1$

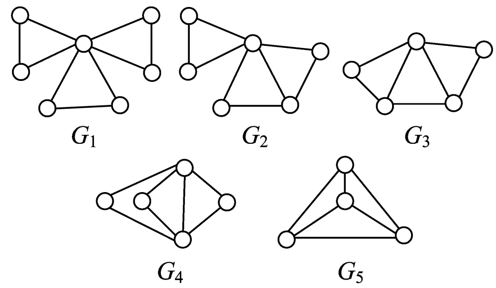


Fig.3 The graphs obtained from Fig.2 by deleting the pendent vertices

pendent vertices (see Fig.3). G_{1n} can be seen as the coalescence of G_1 and $K_{1,n-7}$ with respect to v_1 and v_2, where v_1 is the vertex of G_1 with $d(v_1)=6$, v_2 is the center of $K_{1,n-7}$, namely G_{1n} is $G_1\cdot K_{1,n-7}$. G_{2n} can be seen as the coalescence of G_2 and $K_{1,n-6}$ with respect to v_1 and v_2, where v_1 is the vertex of G_2 with $d(v_1)=5$, v_2 is the center of $K_{1,n-6}$, namely G_{2n} is $G_2\cdot K_{1,n-6}$. G_{3n} can be seen as the coalescence of G_3 and $K_{1,n-5}$ with respect to v_1 and v_2, where v_1 is the vertex of G_3 with $d(v_1)=4$, v_2 is the center of $K_{1,n-5}$, namely G_{3n} is $G_3\cdot K_{1,n-5}$. G_{4n} can be seen as the coalescence of G_4 and $K_{1,n-5}$ with respect to v_1 and v_2, where v_1 is the vertex of G_4 with $d(v_1)=4$, v_2 is the center of $K_{1,n-5}$, namely G_{4n} is $G_4\cdot K_{1,n-5}$. G_{5n} can be seen as the coalescence of G_5 and $K_{1,n-4}$ with respect to v_1 and v_2, where v_1 is the vertex of G_5 with $d(v_1)=3$, v_2 is the center of $K_{1,n-4}$, namely G_{5n} is $G_5\cdot K_{1,n-4}$.

By calculation, we get

$$\Phi(G_1;x)=x(x-1)^2(x-3)^3(x-7),$$

$$\Phi(L_{v_1}(G_1);x)=(x-1)^3(x-3)^3,$$
where v_1 is the vertex of G_1 with $d(v_1)=6$;
$$\Phi(G_2;x)=x(x-1)(x-2)(x-3)(x-4)(x-6),$$
$$\Phi(L_{v_1}(G_2);x)=(x-1)^2(x-2)(x-3)(x-4),$$
where v_1 is the vertex of G_2 with $d(v_1)=5$;
$$\Phi(G_3;x)=x(x-3)(x-5)(x^2-6x+7),$$
$$\Phi(L_{v_1}(G_3);x)=(x-1)(x-3)(x^2-6x+7),$$
where v_1 is the vertex of G_3 with $d(v_1)=4$;
$$\Phi(G_4;x)=x(x-2)^2(x-5)^2,$$
$$\Phi(L_{v_1}(G_4);x)=(x-1)(x-2)^2(x-5),$$
where v_1 is the vertex of G_4 with $d(v_1)=4$;
$$\Phi(G_5;x)=x(x-4)^3,$$
$$\Phi(L_{v_1}(G_5);x)=(x-1)(x-4)^2,$$
where v_1 is the vertex of G_5 with $d(v_1)=3$.

By (1) and Lemma 1.3, we have
$$\Phi(G_{1n};x)=x(x-1)^{n-5}(x-3)^3(x-n),$$
then $\mu_1(G_{1n})=n$, $\mu_2(G_{1n})=3$, $S_L(G_{1n})=n-3$;
$$\Phi(G_{2n};x)=x(x-1)^{n-5}(x-2)(x-3)(x-4)(x-n),$$
then $\mu_1(G_{2n})=n$, $\mu_2(G_{2n})=4$, $S_L(G_{2n})=n-4$;
$$\Phi(G_{3n};x)=x(x-1)^{n-5}(x-3)(x^2-6x+7)(x-n),$$
then $\mu_1(G_{3n})=n$, $\mu_2(G_{3n})=3+\sqrt{2}$, $S_L(G_{3n})=n-(3+\sqrt{2})$;
$$\Phi(G_{4n};x)=x(x-1)^{n-5}(x-2)^2(x-5)(x-n),$$
then $\mu_1(G_{4n})=n$, $\mu_2(G_{4n})=5$, $S_L(G_{4n})=n-5$;
$$\Phi(G_{5n};x)=x(x-1)^{n-4}(x-4)^2(x-n),$$
then $\mu_1(G_{5n})=n$, $\mu_2(G_{5n})=4$, $S_L(G_{5n})=n-4$.

So,
$$S_L(G_{4n})<S_L(G_{3n})<S_L(G_{2n})=S_L(G_{5n})<S_L(G_{1n})=n-3.$$

2 Main results

The girth of graph G is the length of a shortest cycle in G, denoted by $g(G)$. Denote by $B(n)$ and $T(n)$ the sets of bicyclic and tricycle graphs of order n, respectively. G' is a subgraph of a graph G such that $V(G')\subseteq V(G)$, $E(G')\subseteq E(G)$. If $V(G')=V(G)$, we call G' a spanning subgraph of G. Suppose we have two graphs G_1 and G_2 (where $V(G_1)$ and $V(G_2)$ are disjoint), denote $G_1\cup G_2$ the graph with $V(G_1\cup G_2)=V(G_1)\cup V(G_2)$ and $E(G_1\cup G_2)=E(G_1)\cup E(G_2)$.

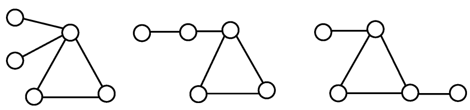


Fig.4 All unicyclic graphs G of order 5 with $g(G)=3$

Theorem 2.1 Let G be a graph in $B(n)$ ($n\geqslant 11$), then $S_L(G)\leqslant S_L(K_{1,n-1}+2e)$, and the equality holds if and only if $G=K_{1,n-1}+2e$.

Proof Let G be a graph in $B(n)$ ($n\geqslant 11$). If $\Delta(G)=n-1$, then $G=K_{1,n-1}+2e$ or W_n. By Lemma 1.4, the result follows.

Next, suppose $\Delta(G)\leqslant n-2$, we will prove that $S_L(G)<S_L(K_{1,n-1}+2e)$.

Assertion 1 If $\Delta(G)\leqslant n-2$,
$$\mu_1(G)\leqslant n-1+\frac{5}{n-2}.$$

Let $u\in V(G)$, and
$$d(u)+m(u)=\max_{v\in V(G)}\{d(v)+m(v)\}.$$

When $d(u)=1$,
$\mu_1(G)\leqslant 1+\Delta(G)\leqslant 1+n-2=n-1$.

When $2\leqslant d(u)\leqslant n-2$. By Lemma 1.2, we get
$$\mu_1(G)\leqslant d(u)+m(u)\leqslant d(u)+\frac{2(n+1)-d(u)-(n-1-d(u))}{d(u)}=d(u)+\frac{n+3}{d(u)}.$$

Let $f(x)=x+\frac{n+3}{x}$. If $x>0$, $f''(x)\geqslant 0$, then $f(x)$ is a convex function. So,
$$\mu_1(G)\leqslant\max\left\{2+\frac{n+3}{2},\ n-2+\frac{n+3}{n-2}\right\}=n-1+\frac{5}{n-2} \tag{4}$$

Thus, if $\Delta(G)\leqslant n-2$, we have
$$\mu_1(G)\leqslant n-1+\frac{5}{n-2}.$$

Assertion 2 If $\Delta(G)\leqslant n-2$, $\mu_2(G)>2.6$.

When $g(G)=3$, one of $F_1\cup(n-5)K_1$, $F_2\cup(n-5)K_1$ and $F_3\cup(n-5)K_1$ must be the spanning subgraph of G, where F_1, F_2, F_3 are given in Fig.4. With calculation, we get

$$\mu_2(F_1)=3,\ \mu_2(F_2)=3,\ \mu_2(F_3)>3.6 \quad (5)$$

By (5) and Lemma 1.1, we have

$$\mu_2(G)\geqslant\min\{\mu_2(F_1\cup(n-5)K_1),\ \mu_2(F_2\cup(n-5)K_1),\mu_2(F_3\cup(n-5)K_1)\}=3 \quad (6)$$

When $g(G)\geqslant 4$, $P_5\cup(n-5)K_1$ must be the spanning subgraph of G. Because $\mu_2(P_5)>2.6$, and by Lemma 1.1, we have

$$\mu_2(G)\geqslant\mu_2(P_5)>2.6.$$

Thus, if $\Delta(G)\leqslant n-2$, we have $\mu_2(G)>2.6$.

According Assertion 1 and Assertion 2, we have that

$$S_L(G)=\mu_1(G)-\mu_2(G)< n-1+\frac{5}{n-2}-2.6<n-3,$$

when $\Delta(G)\leqslant n-2$.

By Lemma 1.4, $S_L(G)<S_L(K_{1,n-1}+2e)$, when $\Delta(G)\leqslant n-2$. Then the result follows.

Theorem 2.2 Let G be a graph in $T(n)$ ($n\geqslant 14$), then $S_L(G)\leqslant S_L(K_{1,n-1}+3e)$, the equality holds if and only if $G=K_{1,n-1}+3e$.

Proof Denote $K_{1,n-1}+3e$ by G_{1n}. Let G be a graph in $T(n)$ ($n\geqslant 14$). If $\Delta(G)=n-1$, then $G\in\{G_{1n}, G_{2n}, G_{3n}, G_{4n}, G_{5n}\}$. By Lemma 1.5, the result follows.

Next, suppose $\Delta(G)\leqslant n-2$, we will prove that $S_L(G)<S_L(G_{1n})$.

Assertion 1 If $\Delta(G)\leqslant n-2$,

$$\mu_1(G)\leqslant n-1+\frac{7}{n-2}.$$

Let $u\in V(G)$, and

$$d(u)+m(u)=\max_{v\in V(G)}\{d(v)+m(v)\}.$$

When $d(u)=1$,

$$\mu_1(G)\leqslant 1+\Delta(G)\leqslant 1+n-2=n-1.$$

When $2\leqslant d(u)\leqslant n-2$. By Lemma 1.2, we get

$$\mu_1(G)\leqslant d(u)+m(u)\leqslant d(u)+\frac{2(n+2)-d(u)-(n-1-d(u))}{d(u)}= d(u)+\frac{n+5}{d(u)}.$$

Let $f(x)=x+\frac{n+5}{x}$. If $x>0$, $f''(x)\geqslant 0$, then $f(x)$ is a convex function. So,

$$\mu_1(G)\leqslant\max\left\{2+\frac{n+5}{2},\ n-2+\frac{n+5}{n-2}\right\}= n-1+\frac{7}{n-2}.$$

Thus, if $\Delta(G)\leqslant n-2$, we have

$$\mu_1(G)\leqslant n-1+\frac{7}{n-2}.$$

Assertion 2 If $\Delta(G)\leqslant n-2$, $\mu_2(G)>2.6$.

When $g(G)=3$, one of $F_1\cup(n-5)K_1$, $F_2\cup(n-5)K_1$ and $F_3\cup(n-5)K_1$ must be the spanning subgraph of G, where F_1, F_2, F_3 are given in Fig.4.

By (5) and Lemma 1.1, we have

$$\mu_2(G)\geqslant\min\{\mu_2(F_1\cup(n-5)K_1),\ \mu_2(F_2\cup(n-5)K_1),\mu_2(F_3\cup(n-5)K_1)\}=3.$$

When $g(G)\geqslant 4$, $P_5\cup(n-5)K_1$ must be the spanning subgraph of G. By $\mu_2(P_5)>2.6$ and Lemma 1.1, we have

$$\mu_2(G)\geqslant\mu_2(P_5)>2.6.$$

Thus, if $\Delta(G)\leqslant n-2$, we have $\mu_2(G)>2.6$.

According Assertion 1 and Assertion 2, we have that

$$S_L(G)=\mu_1(G)-\mu_2(G)< n-1+\frac{7}{n-2}-2.6<n-3,$$

when $\Delta(G)\leqslant n-2$.

By Lemma 1.5,

$$S_L(G)<S_L(G_{1n})=S_L(K_{1,n-1}+2e),$$

when $\Delta(G)\leqslant n-2$. Then the result follows.

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