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A weak-L^p Prodi-Serrin type regularity criterion for electro-hydrodynamics

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Abstract: Regularity criteria for weak solution of the electro-hydrodynamics was studied. It was proved that the solution (u, n, p, Ψ) remains strong on $[0, T]$ if $u \in L^s(0, T; \mathbb{L}^{r,\infty}(\Omega))$ or $\|u\|_{L^{s,\infty}(0,T;L^{r,\infty}(\Omega))} \leqslant C$, where $(3/r)+(2/s)=1$ and $r \in (3, \infty]$, $C>0$ depending only on r and Ω.

Key words: electro-hydrodynamics; regularity criteria; weak solution; strong solution; weak-L^p functions

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electro-hydrodynamics 方程的弱 L^p Prodi-Serrin 型正则准则

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摘要:主要研究了 electro-hydrodynamics 方程弱解的正则准则. 证明了在条件 $u \in L^s(0,T;L^{r,\infty}(\Omega))$或$\|u\|_{L^{s,\infty}(0,T;L^{r,\infty}(\Omega))} \leqslant C$(其中$(3/r)+(2/s)=1$ 且 $r \in (3,\infty]$, $C=C(r,\Omega)>0$)下, 弱解在区间$[0,T]$上也为强解.

关键词:electro-hydrodynamics;正则准则;弱解;强解;弱 L^p 泛函

0 Introduction

Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain with smooth boundary $\partial\Omega$ and let $T>0$ be fixed but arbitrary. We investigate the following form of electro-hydrodynamics in the $\Omega \times (0,T)$

$$\left.\begin{array}{l}\partial_t u+(u \cdot \nabla) u-\triangle u+\nabla \pi=\triangle \Psi \nabla \Psi, \\ \nabla \cdot u=0, \\ \partial_t n+(u \cdot \nabla) n=\operatorname{div}(\nabla n-n \nabla \Psi), \\ \partial_t p+(u \cdot \nabla) p=\operatorname{div}(\nabla p-p \nabla \Psi), \\ -\triangle \Psi=p-n\end{array}\right\} \quad (1)$$

with the initial conditions

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$$(u,n,p)|_{t=0}=(u_0,n_0,p_0)(x),\ x\in\Omega \quad (2)$$

and boundary conditions

$$(u,n,p)|_{\partial\Omega\times(0,T)}=(0,0,0) \quad (3)$$

where u,π,n,p,Ψ represent, respectively, the velocity of the electrolyte fluid, the pressure, anion concentration, cation concentration and electrostatic potential.

Over the past decades, this model has been widely studied by many authors, see Refs.[2-7], to name but a few. A detailed analysis of the model was given by Ryham et al. in Ref. [1]. In particular, Zhao and Bai have given the existence of the strong solution to this model on $[0,T]$ if $u\in L^s(0,T;\mathbb{L}^r(\Omega))$ in Ref. [4], where (r,s) is a Prodi-Serrin pair, namely, $(3/r)+(2/s)=1$ and $r\in(3,\infty]$, $s\in[2,\infty)$. In Ref.[8], when $\phi=0$, the existence of a strong solution to (1) on the whole interval $[0,T]$ is guaranteed if $u\in L^s(0,T;L^r(\mathbb{R}^3))$ with $\frac{2}{s}+\frac{3}{r}=1$, $3<r\leqslant+\infty$ or $\nabla u\in L^s(0,T;L^r(\mathbb{R}^3))$ with $\frac{2}{s}+\frac{3}{r}=1$, $\frac{3}{2}<r\leqslant+\infty$. In this paper, we aim to prove that the weak solution of this model is strong on $[0,T]$ if

$$u\in L^s(0,T;\mathbb{L}^{r,\infty}(\Omega))$$

or

$$\|u\|_{L^{s,\infty}(0,T;L^{r,\infty}(\Omega))}\leqslant C,$$

where $C>0$ depending only on r and Ω.

Our main results are the following.

Theorem 0.1 Let (u,n,p) with

$$u\in L^\infty(0,T;\mathbb{L}^2(\Omega))\cap L^2(0,T;\mathbb{H}^1_{0,\mathrm{div}}(\Omega)),$$
$$(n,p)\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1_0(\Omega))$$

be a weak solution of the system (1) with initial data $u_0\in\mathbb{H}^1_{0,\mathrm{div}}(\Omega)$, $(n_0,p_0)\in L^2(\Omega)$ and $n_0,p_0\geqslant0$. If $u\in L^s(0,T;\mathbb{L}^{r,\infty}(\Omega))$ for some Prodi-Serrin pair (r,s). Then (u,n,p) remains strong on $[0,T]$.

Theorem 0.2 Assume $u_0\in\mathbb{H}^1_{0,\mathrm{div}}(\Omega)$, $(n_0,p_0)\in L^2(\Omega)$ and $n_0,p_0\geqslant0$, and let (u,n,p) be a weak solution of the system (1) with initial data (u_0,p_0,n_0). There exists a constant $C>0$ depending only on r and Ω. such that if

$$\|u\|_{L^{s,\infty}(0,T;L^{r,\infty}(\Omega))}\leqslant C$$

for some Prodi-Serrin pair (r,s), then (u,n,p) remains strong on $[0,T]$.

In Section 1, we provide some preliminaries. Section 2 is devoted to the proof of our main results.

1 Preliminaries

Let Ω be an open subset of $\mathbb{R}^n$. For $p\in[1,\infty)$, we denote the weak-$L^p(\Omega)$ space[10] by $L^{p,\infty}(\Omega)$, and we set

$$\|f\|_{p,\infty}=\sup_{t>0}t[\mu\{x\in\Omega:|f(x)|>t\}]^{1/p},$$

where μ is the Lebesgue measure.

Let $\mathscr{V}=\{v\in\mathbb{C}_0^\infty(\Omega):\mathrm{div}\ v=0\}$, the closures of $\mathscr{V}$ in the norms of $\mathbb{L}^2(\Omega)$ and $\mathbb{H}^1_0(\Omega)$ is denoted by $\mathbb{L}^2{}_{\mathrm{div}}(\Omega)$ and $\mathbb{H}^1_{0,\mathrm{div}}(\Omega)$, respectively. We denote by P the orthogonal projection onto $\mathbb{L}^2_{\mathrm{div}}(\Omega)$ and $A=-P\triangle$ the usual Stokes operator with domain

$$D(A)=\mathbb{H}^1_{0,\mathrm{div}}(\Omega)\cap\mathbb{H}^2(\Omega).$$

We will give the following lemma which is serviceable for us to prove the main results.

Lemma 1.1[9] Let $u\in\mathbb{H}^1_{0,\mathrm{div}}(\Omega)\cap\mathbb{H}^2(\Omega)$ be given, and (r,s) be a Prodi-Serrin pair, namely, $r\in(3,\infty]$ and $s\in[2,\infty)$ satisfy $\frac{3}{r}+\frac{2}{s}=1$. Then

$$\|(u\cdot\nabla)u\|_2\leqslant C_r\|u\|_{r,\infty}\|\nabla u\|_2^{2/s}\|Au\|_2^{1-2/s}.$$

2 Proofs of the results

In this section, we will give the detailed proofs of the main results. For simplicity, C can denote the different positive constants in the sequel.

The proof of Theorem 0.1 Taking the inner product with n (resp. p) in $(1)_3$ (resp. $(1)_4$) and using $(1)_2$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|n\|_2^2+\|\nabla n\|_2^2=\int n\nabla\Psi\cdot\nabla n\mathrm{d}x \quad (4)$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|p\|_2^2+\|\nabla p\|_2^2=-\int p\nabla\Psi\cdot\nabla p\mathrm{d}x \quad (5)$$

According to (4), (5) and $(1)_5$, we can conclude

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|n\|_2^2+\|p\|_2^2)+(\|\nabla n\|_2^2+\|\nabla p\|_2^2)=$$
$$\int n\nabla\Psi\cdot\nabla n\mathrm{d}x-\int p\nabla\Psi\cdot\nabla p\mathrm{d}x=$$
$$\int(-\triangle\Psi)\frac{n^2-p^2}{2}\mathrm{d}x=-\int\frac{p+n}{2}(p-n)^2\mathrm{d}x.$$

Thanks to Ref.[1], we have n, $p\geqslant 0$ in $\Omega\times(0,T)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|n\|_2^2+\|p\|_2^2)+2(\|\nabla n\|_2^2+\|\nabla p\|_2^2)\leqslant 0 \tag{6}$$

Applying Gronwall's inequality to (6), we have

$$\|(n,p)\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))}\leqslant C \tag{7}$$

Applying the standard elliptic regularity theory to $(1)_5$ and using (7), we have

$$\|\Psi\|_{L^\infty(0,T;H^2(\Omega))\cap L^2(0,T;H^3(\Omega))}\leqslant C \tag{8}$$

Applying the operator P to $(1)_1$, we have

$$\partial_t u+Au+P[(u\cdot\nabla)u]=P[\triangle\Psi\nabla\Psi] \tag{9}$$

Multiplying (9) by Au

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_2^2+\|Au\|_2^2=$$
$$-(P[(u\cdot\nabla)u],Au)+(P[\triangle\Psi\nabla\Psi],Au) \tag{10}$$

From Lemma 1.1

$$|(P[(u\cdot\nabla)u],Au)|\leqslant$$
$$\|(u\cdot\nabla)u\|_2\|Au\|_2\leqslant$$
$$C_r\|u\|_{r,\infty}\|\nabla u\|_2^{2/s}\|Au\|_2^{2-2/s}\leqslant$$
$$\frac{C_r}{s}\left(\frac{s}{4(s-1)}\right)^{1-s}\|u\|_{r,\infty}^s\|\nabla u\|_2^2+\frac{1}{4}\|Au\|_2^2 \tag{11}$$

$$|(P[\triangle\Psi\nabla\Psi],Au)|\leqslant$$
$$\|n-p\|_2\|\nabla\Psi\|_\infty\|Au\|_2\leqslant$$
$$C\|\nabla\Psi\|_\infty\|Au\|_2\leqslant$$
$$C\|\nabla\Psi\|_\infty^2+\frac{1}{4}\|Au\|_2^2\leqslant$$
$$C\|\Psi\|_{H^3}^2+\frac{1}{4}\|Au\|_2^2 \tag{12}$$

Using (11)～(12) on (10), we can get the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_2^2+\|Au\|_2^2\leqslant$$
$$\frac{2C_r}{s}\left(\frac{s}{4(s-1)}\right)^{1-s}\|u\|_{r,\infty}^s\|\nabla u\|_2^2+C\|\Psi\|_{H^3}^2 \tag{13}$$

By Gronwall's inequality, we have

$$\|\nabla u(t)\|_2^2\leqslant(\|\nabla u_0\|_2^2+C\|\Psi\|_{L^2(0,T;H^3)})\exp\left(\frac{2C_r}{s}\left(\frac{s}{4(s-1)}\right)^{1-s}\int_0^t\|u(\sigma)\|_{r,\infty}^s\mathrm{d}\sigma\right)\leqslant$$
$$(\|\nabla u_0\|_2^2+C\|\Psi\|_{L^2(0,T;H^3)}\exp\left(\frac{2C_r}{s}\left(\frac{s}{4(s-1)}\right)^{1-s}\int_0^T\|u(\sigma)\|_{r,\infty}^s\mathrm{d}\sigma\right).$$

Due to the fact that $u\in L^s(0,T;\mathbb{L}^{r,\infty}(\Omega))$ and estimate (8), we can get

$$u\in L^\infty(0,T;\mathbb{H}_{0,\mathrm{div}}^1(\Omega)) \tag{14}$$

To prove Theorem 0.1, we argue by contradiction. Suppose that the solution remains strong only on $[0,T')$, with $T'<T$. But if $[0,T')$ is the maximal interval for the existence of the strong solution, then

$$\|\nabla u(t)\|_2+\|n(t)\|_2+\|p(t)\|_2+\|\Psi(t)\|_{H^2}\to\infty,\ \text{as } t\to T'.$$

However, (7), (8) and (14) guarantee that the solution must be strong on $[0,T]$.

The proof is completed.

The proof of Theorem 0.2　The idea of the following proof comes from Ref.[9] (see also Ref.[11]) where the authors studied the weak-L^p Prodi-Serrin type regularity criterion for the Navier-Stokes equations.

For $\epsilon>0$, let

$$r_\epsilon=\frac{3s+3\epsilon(4-s)}{s-2+\epsilon(4-s)},\ s_\epsilon=s+\epsilon(4-s),$$

then (r_ϵ,s_ϵ) is a Prodi-Serrin pair.

Through a standard interpolation

$$\|u\|_{r_\epsilon,\infty}^{s_\epsilon}\leqslant\|u\|_{r,\infty}^{s(1-\epsilon)}\|u\|_{6,\infty}^{4\epsilon}\leqslant$$
$$C^\epsilon\|u\|_{r,\infty}^{s(1-\epsilon)}\|\nabla u\|_2^{4\epsilon}.$$

Using this inequality in (13), for (r_ϵ,s_ϵ) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_2^2\leqslant$$

$$\frac{2C^{\epsilon}C_{r_\epsilon}}{s_\epsilon}\left[\frac{s_\epsilon}{4(s_\epsilon-1)}\right]^{1-s_\epsilon}\|u\|_{r,\infty}^{s(1-\epsilon)}\|\nabla u\|_2^{4\epsilon+2}+$$
$$C\|\Psi\|_{H^3}^2.$$

By Gronwall's inequality, we have

$$\|\nabla u(t)\|_2^2\leqslant\|\nabla u_0\|_2^2+C\|\Psi\|_{L^2(0,T;H^3)}+\frac{2C^{\epsilon}C_{r_\epsilon}}{s_\epsilon}\left[\frac{s_\epsilon}{4(s_\epsilon-1)}\right]^{1-s_\epsilon}\cdot\int_0^t\|u(\sigma)\|_{r,\infty}^{s(1-\epsilon)}\|\nabla u(\sigma)\|_2^{4\epsilon+2}\mathrm{d}\sigma \qquad (15)$$

Set $\kappa(t)=\|u\|_{r,\infty}^{s}$ for $t\in[0,T]$. Then $\kappa(t)\in L^{1,\infty}(0,T)$ and

$$\epsilon\int_0^t\kappa^{1-\epsilon}(\sigma)\mathrm{d}\sigma=$$
$$(1-\epsilon)\int_0^\infty\sigma^{-\epsilon}\mu\{\tau\in[0,T]:\kappa(\tau)>\sigma\}\mathrm{d}\sigma\leqslant$$
$$\epsilon T+\epsilon(1-\epsilon)\|\kappa\|_{1,\infty}\int_1^\infty\frac{1}{\sigma^{1+\epsilon}}\mathrm{d}\sigma=$$
$$\epsilon T+(1-\epsilon)\|\kappa\|_{1,\infty}. \qquad (16)$$

Let

$$\psi(t)\stackrel{\triangle}{=\!=}\|\nabla u_0\|_2^2+C\|\Psi\|_{L^2(0,T;H^3)}+\frac{2C^{\epsilon}C_{r_\epsilon}}{s_\epsilon}\left[\frac{s_\epsilon}{4(s_\epsilon-1)}\right]^{1-s_\epsilon}\int_0^t\kappa^{1-\epsilon}(\sigma)\|\nabla u(\sigma)\|_2^{4\epsilon+2}\mathrm{d}\sigma,$$

then we can conclude $\|\nabla u(\sigma)\|_2^2\leqslant\psi(t)$.

From (15), it follows that

$$\psi'(t)=\frac{2C^{\epsilon}C_{r_\epsilon}}{s_\epsilon}\left[\frac{s_\epsilon}{4(s_\epsilon-1)}\right]^{1-s_\epsilon}\kappa^{1-\epsilon}(t)\|\nabla u(t)\|_2^{4\epsilon+2}\leqslant$$
$$\frac{2C^{\epsilon}C_{r_\epsilon}}{s_\epsilon}\left[\frac{s_\epsilon}{4(s_\epsilon-1)}\right]^{1-s_\epsilon}\kappa^{1-\epsilon}(t)\psi^{1+2\epsilon}(t).$$

Thus, by (16)

$$\psi^{-2\epsilon}(0)-\psi^{-2\epsilon}(t)\leqslant$$
$$\frac{4\epsilon C^{\epsilon}C_{r_\epsilon}}{s_\epsilon}\left[\frac{s_\epsilon}{4(s_\epsilon-1)}\right]^{1-s_\epsilon}\int_0^T\kappa^{1-\epsilon}(\sigma)\mathrm{d}\sigma\leqslant$$
$$\frac{4\epsilon C^{\epsilon}C_{r_\epsilon}}{s_\epsilon}\left[\frac{s_\epsilon}{4(s_\epsilon-1)}\right]^{1-s_\epsilon}T+$$
$$\frac{4(1-\epsilon)\,C^{\epsilon}C_{r_\epsilon}}{s_\epsilon}\left[\frac{s_\epsilon}{4(s_\epsilon-1)}\right]^{1-s_\epsilon}\|\kappa\|_{1,\infty}.$$

Let $\delta>0$ such that

$$\frac{4\,C^{\epsilon}C_{r_\epsilon}}{s_\epsilon}\left[\frac{s_\epsilon}{4(s_\epsilon-1)}\right]^{1-s_\epsilon}\|\kappa\|_{1,\infty}<1-3\delta.$$

Choosing $\epsilon>0$ sufficiently small that $\psi^{-2\epsilon}(0)>1-\delta$, and $\frac{4\epsilon C^{\epsilon}C_{r_\epsilon}}{s_\epsilon}\left[\frac{s_\epsilon}{4(s_\epsilon-1)}\right]^{1-s_\epsilon}T<\delta$.

We have $\psi(t)\leqslant\delta^{-1/2\epsilon}$, $\forall t\in[0,T]$. Due to $\|\nabla u(\sigma)\|_2^2\leqslant\psi(t)$, we have $\|\nabla u(t)\|_2^2\leqslant\delta^{-1/2\epsilon}$, $t\in[0,T]$. Namely, $\|\nabla u(t)\|_2^2$ is bounded on $[0,T]$. Similarly as in the proof of Theorem 0.1, we can conclude that $\|n(t)\|_2$, $\|p(t)\|_2$, $\|\phi(t)\|_{H^2}$ is bounded on $[0,T]$.

The proof is completed.

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