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A note on optimistic strongly regular graphs

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Abstract: It is known that for a fixed integer $\alpha \geqslant 2$, all but finitely many coconnected ones, the strongly regular graphs with smallest eigenvalue $-\alpha$ fall into two infinite families. Graham and Lovász raised the question of whether optimistic graphs exist and it was answered positively by Azarija. Here strongly regular graphs were classified with smallest eigenvalue -3, and the optimistic ones among them were determined.

Key words: coconnected graph; strongly regular graphs; eigenvalue; optimistic graphs

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关于最优强正则图的一个注记

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摘要:人们已经知道,最小特征值为$-\alpha$的强正则图,除了有限多个补图连通的强正则图外,分成两个无限类,其中α是一个不小于2的整数.在Graham和Lovász提出最优图类的存在性问题后,Azarija对这个问题给出了肯定的回答.这里刻画了最小特征值为-3的强正则图,而且确定了其中的最优图类.

关键词:补图连通图;强正则图;特征值;最优图

0 Introduction

In this paper, we classify strongly regular graphs with smallest eigenvalue -3 and determine the optimistic ones among them.

Strongly regular graphs were introduced by Bose[1]. Neumaier[2] showed that for a fixed integer $\alpha \geqslant 2$, all but finitely many coconnected strongly regular graphs with smallest eigenvalue $-\alpha$ fall into two infinite families.

Theorem 0.1[2] Let $\alpha \geqslant 2$ be a fixed integer. Then there exists a constant $C(\alpha)$ such that any coconnected strongly regular graph Γ with smallest eigenvalue $-\alpha$ having more than $C(\alpha)$ vertices has

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one of the following parameters:

① $((\gamma+1)(\gamma(\alpha-1)+\alpha)/\alpha, \gamma\alpha; \gamma-1+(\alpha-1)^2, \alpha^2)$, where γ is a positive integer;

② $((\gamma+1)^2, \gamma\alpha; \gamma-1+(\alpha-2)(\alpha-1), \alpha(\alpha-1))$, where γ is a positive integer.

Graham and Lovász[3] raised the question of whether optimistic graphs exist (although they did not use the term). This question was answered positively by Azarija in Ref.[4], where the term optimistic was introduced. In Ref.[4], it is shown that conference graphs of order at least 13 are optimistic and also that the strongly regular graphs with parameters $(t^2, 3(t-1); t, 6)$ are optimistic for $t \geqslant 5$.

We classify all strongly regular graphs with smallest eigenvalue -3 in Theorem 2.1. And in Theorem 3.1, all optimistic strongly regular graphs with smallest eigenvalue -3 are determined.

1 Definitions

All the graphs considered in this paper are finite undirected and simple (for unexplained terminology and more details, see Ref.[5]). Suppose that Γ is a connected graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, where $E(\Gamma)$ consists of unordered pairs of two adjacency vertices. The distance $d(x, y)$ between any two vertices x and y of Γ is the length of a shortest path connecting x and y. We denote v as the number of vertices of Γ and define the diameter D of Γ as the maximum distance in Γ. For a vertex $x \in V(\Gamma)$, define $\Gamma_i(x)$ to be the set of vertices at distance precisely i from x $(0 \leqslant i \leqslant D)$. In addition define $\Gamma_{-1}(x) = \Gamma_{D+1}(x) := \emptyset$. We write $\Gamma(x)$ instead of $\Gamma_1(x)$ and define the local graph $\Delta(x)$ at a vertex $x \in V(\Gamma)$ as the subgraph induced on $\Gamma(x)$. Let Δ be a graph. If the local graph $\Delta(x)$ is isomorphic to Δ for any vertex x in $V(\Gamma)$, then we say Γ is locally Δ.

The complement $\overline{\Gamma}$ of a graph Γ is the graph with the same vertex as Γ, and two vertices in $\overline{\Gamma}$ are adjacent if and only if they are non-adjacent in Γ. We use sK_t to denote s copies of the complete graph K_t. A graph is coconnected if its complement is connected.

A regular graph Γ with v vertices and valency k is called strongly regular if there exist integers a, c, such that every two adjacent vertices have a common neighbors and every two non-adjacent vertices have c common neighbors. And we say Γ is a $(v, k; a, c)$-strongly regular graph.

Suppose that Γ is a regular graph with valency $k \geqslant 2$ and diameter $D \geqslant 2$, and let A_i $(0 \leqslant i \leqslant D)$ be the matrix of Γ such that the rows and columns of A_i are indexed by the vertices of Γ and the (x, y)-entry is 1 whenever x and y are at distance i and 0 otherwise. We denote the adjacency matrix of Γ as A instead of A_1. The eigenvalues of the graph Γ are the eigenvalues of A. The distance matrix of the graph Γ is $\mathscr{D} = \sum_{i=1}^{D} i A_i$, whose (x, y)-entry is $d(x, y)$, and eigenvalues of $\mathscr{D}$ are called distance eigenvalues of the graph Γ. A graph is optimistic if it has more positive than negative distance eigenvalues.

2 Strongly regular graphs with smallest eigenvalue −3

Lemma 2.1 Let Γ be a $(v, k; a, c)$-strongly regular graph. Then Γ is not coconnected if and only if $c = k$.

Proof As Γ is a $(v, k; a, c)$-strongly regular graph, we have that $\overline{\Gamma}$ is a $(v, \bar{k}; \bar{a}, \bar{c})$-strongly regular graph, with $\bar{k} = v - k - 1$, $\bar{a} = v - 2 - 2k + c$ and $\bar{c} = v - 2k + a$ (see Ref.[8, p.218]. Choose a vertex x of Γ, by counting the edges between $\Gamma(x)$ and $\Gamma_2(x)$ in two ways, we can get $(v - k - 1)c = (k - 1 - a)k$. Note that $c = k$ is equivalent to $v = 2k - a$ and $\bar{c} = v - 2k + a = 0$. It follows that Γ is non-coconnected if and only if $c = k$ by Ref.[8, Lemma 10.1.1].

Remark Besides Lemma 2.1, we have that a non-coconnected strongly regular graph is the complement of the graph sK_t for some $s, t \geqslant 2$ (see Ref.[8, Lemma 10.1.1]) and it has parameter

$(v,k;a,c)=(st,(s-1)t;(s-2)t,(s-1)t)$ and eigenvalues $(s-1)t>0>-t$ with multiplicities 1, $s(t-1), s-1$ respectively. So we only need to care about the coconnected case.

For a $(v,k;a,c)$-strongly regular graph, let $\beta=\sqrt{\Delta}$, where $\Delta=(a-c)^2+4(k-c)$.

Lemma 2.2 Let Γ be a coconnected $(v,k;a,c)$-strongly regular graph with eigenvalue $k=\theta_0>\theta_1>\theta_2$ and multiplicities $m(\theta_i)$ $(0\leqslant i\leqslant 2)$. Let $\Delta=(a-c)^2+4(k-c)$. If $\theta_2=-\alpha$ for some integer $\alpha\geqslant 2$, then the following holds:

① β is an integer satisfying $\beta\geqslant\alpha+1$;

② $\theta_1=\beta-\alpha$, $a=c+\beta-2\alpha$, $k=c+\alpha(\beta-\alpha)$ and;

③ $v=1+k+\frac{1}{c}k(k-1-a)$ and $m(\theta_1)=\frac{1}{2}\left(v-1-\frac{1}{\beta}(2k+(v-1)(a-c))\right)$.

Proof Note that Γ has eigenvalues k, $\frac{1}{2}(a-c\pm\beta)$[8, p.220]. So $\theta_1-\theta_2=\beta$, and hence $\theta_1=\beta-\alpha$. The fact $\theta_1+\theta_2=a-c$ implies that $\beta=\theta_1+\alpha$ is an integer, and $a=c+\beta-2\alpha$. Since $\sqrt{\Delta}\geqslant a-c$, we have that $\theta_1\geqslant 0$ with equality if and only if $k=c$, which results in Γ not being coconnected by Lemma 2.1. So $\beta\geqslant\alpha+1$.

Note that $c-k=\theta_1\theta_2=-\alpha(\beta-\alpha)$, we see $k=c+\alpha(\beta-\alpha)$. Since $(v-1-k)c=k(k-1-a)$, we have $v=1+k+\frac{1}{c}k(k-1-a)$. From Ref.[8, p.220] we know $m(\theta_1)=\frac{1}{2}\left(v-1-\frac{2k+(v-1)(a-c)}{\sqrt{\Delta}}\right)$.

First, we give some parameter restrictions for strongly regular graphs. The next restrictions, due to Ref. [12] are called Krein conditions.

Lemma 2.3[12] Let Γ be a coconnected $(v,k;a,c)$-strongly graph with eigenvalues $k=\theta_0>\theta_1>\theta_2$. Then the following holds:

$$\theta_1\theta_2{}^2-2\theta_1{}^2\theta_2-\theta_1{}^2-k\theta_1+k\theta_2{}^2+2k\theta_2\geqslant 0,$$
$$\theta_1{}^2\theta_2-2\theta_1\theta_2{}^2-\theta_2{}^2-k\theta_2+k\theta_1{}^2+2k\theta_1\geqslant 0.$$

The following result is the absolute bound shown in Ref.[14].

Lemma 2.4[14] Let Γ be a coconnected $(v,k;a,c)$-strongly graph with eignvalues $k=\theta_0>\theta_1>\theta_2$ and multiplicities $m(\theta_i)$. Then

$$v\leqslant\frac{1}{2}m(\theta_i)(m(\theta_i)+3),\ i=1,2.$$

Remark The above lemma gives a representation of all the parameters of Γ in some functions of α,β,c.

The following is a collection of parameter restrictions due to Ref.[2] for strongly regular graphs with smallest integeral eigenvalue $-\alpha$.

Lemma 2.5[2] Let Γ be a coconnected $(v,k;a,c)$-strongly graph with eignvalues $k>\beta-\alpha>-\alpha$ for some positive integers $\beta-1\geqslant\alpha\geqslant 2$. Then $c\leqslant\alpha^3(2\alpha-3)$ and one of the following holds

① $c=\alpha(\alpha-1)$ and
$$\beta\leqslant\frac{1}{2}(\alpha-1)(\alpha^3-\alpha^2+\alpha+2);$$

② $c=\alpha^2$ and $\beta\leqslant\frac{1}{2}(\alpha-1)(\alpha^3+\alpha+2)$;

③ $c\neq\alpha(\alpha-1),\alpha^2$ and
$$\beta\leqslant\frac{1}{2}\alpha(\alpha-1)(c+1)+\alpha-1;$$

④ $(v,k;a,c)=((\gamma+1)(\gamma(\alpha-1)+\alpha)/\alpha,\gamma\alpha;\gamma-1+(\alpha-1)^2,\alpha^2)$, where γ is a positive integer;

⑤ $(v,k;a,c)=((\gamma+1)^2,\gamma\alpha;\gamma-1+(\alpha-2)(\alpha-1),\alpha(\alpha-1))$, where γ is a positive integer.

Theorem 2.1 Let Γ be a coconnected $(v,k;a,c)$-strongly graph with eigenvalues $k>\beta-\alpha>-\alpha$. If $\alpha=3$, then one of the following holds:

① $(v,k;a,c)$ is in Tab. 1 or 2;

② $(v,k;a,c)=((\gamma+1)(\gamma(\alpha-1)+\alpha)/\alpha,\gamma\alpha;\gamma-1+(\alpha-1)^2,\alpha^2)$, where γ is a positive integer;

③ $(v,k;a,c)=((\gamma+1)^2,\gamma\alpha;\gamma-1+(\alpha-2)\cdot(\alpha-1),\alpha(\alpha-1))$, where γ is a positive integer.

Proof By Lemma 2.2, we see that the parameters $(v,k;a,c)$ and the multiplicities of eigenvalues of Γ can all be represented by the triple (α,β,c). By Lemma 2.5, we have $c\leqslant\alpha^3(2\alpha-3)=81$. Then we see that either Γ is in those two families in ② and ③, or

$$\beta\leqslant\max\Big\{\frac{1}{2}(\alpha-1)(\alpha^3-\alpha^2+\alpha+2),$$

$$\frac{1}{2}(\alpha-1)(\alpha^3+\alpha+2),$$

$$\left.\frac{1}{2}\alpha(\alpha-1)(\alpha^3(2\alpha-3)+1)+\alpha-1\right\}=248.$$

Then we check all triples (α,β,c) with $\alpha=3$, $4\leqslant\beta\leqslant 248$ and $1\leqslant c\leqslant 81$, all the parameters $(v,k;a,c)$ satisfying:

① $\frac{(v-k-1)}{c}$ and $m(\theta_1)$ are integers;

② Γ does not come from the two families in ② and ③;

③ the restrictions in Lemmas 2.3, 2.4 and 2.5; are in Tabs. 1, 2 or 3. Those graphs in Tab. 3 do not exist.

Remark There are infinitely graphs in those two families in Theorem 2.1 ② and ③ (see Ref. [13, Lemma 4.1, Lemma 4.2]).

The detailed informations (the uniqueness (Tab. 1), whether there exist known examples (Tab. 2) and non-existence (Tab. 3)) of those graphs comes from Ref. [4]. For those graphs in Tab. 3, we give the references for non-existence. For readers' convenience, we give a proof for $(v,k;a,c)=(209,16;3,1)$ or $(841,200;87,35)$.

Tab.1 Parameters feasible to Theorem 2.1 and known to exist

$(v,k;a,c)$	$\theta_0,[\theta_1]^{m(\theta_1)},[\theta_2]^{m(\theta_2)}$	extra information
(15,6;1,3)	$6,[1]^{9},[-3]^{5}$	! (uniqueness)
(16,5;0,2)	$5,[1]^{10},[-3]^{5}$	!
(26,10;3,4)	$10,[2]^{13},[-3]^{12}$	10(complete enumeration)
(36,21;12,12)	$21,[3]^{14},[-3]^{21}$	180
(40,27;18,18)	$27,[3]^{15},[-3]^{24}$	28
(45,12;3,3)	$12,[3]^{20},[-3]^{24}$	78
(50,7;0,1)	$7,[2]^{28},[-3]^{21}$	!
(50,42;35,36)	$42,[2]^{21},[-3]^{28}$	!
(56,45;36,36)	$45,[3]^{20},[-3]^{35}$	!
(64,45;32,30)	$45,[5]^{18},[-3]^{45}$	167
(77,60;47,45)	$60,[5]^{21},[-3]^{55}$	!
(81,60;45,42)	$60,[6]^{20},[-3]^{60}$	!
(100,77;60,56)	$77,[7]^{22},[-3]^{77}$	!
(105,72;51,45)	$72,[9]^{20},[-3]^{84}$	!
(112,81;60,54)	$81,[9]^{21},[-3]^{90}$	!
(120,77;52,44)	$77,[11]^{20},[-3]^{99}$	!
(125,72;45,36)	$72,[12]^{20},[-3]^{104}$	
(126,25;8,4)	$25,[7]^{35},[-3]^{90}$	
(126,60;33,24)	$60,[12]^{21},[-3]^{104}$	
(126,75;48,39)	$75,[12]^{20},[-3]^{105}$	
(162,105;72,60)	$105,[15]^{21},[-3]^{140}$	!
(175,102;65,51)	$102,[17]^{21},[-3]^{153}$	
(176,85;48,34)	$85,[17]^{22},[-3]^{153}$	
(176,105;68,54)	$105,[17]^{21},[-3]^{154}$	!
(231,30;9,3)	$30,[9]^{55},[-3]^{175}$	
(243,132;81,60)	$132,[24]^{22},[-3]^{220}$	
(253,140;87,65)	$140,[25]^{22},[-3]^{230}$	
(275,162;105,81)	$162,[27]^{22},[-3]^{252}$	!
(276,135;78,54)	$135,[27]^{23},[-3]^{252}$	

Ta.2 Parameters feasible to Theorem 2.1, but unknown whether they exist or not

$(v,k;a,c)$	$\theta_0,[\theta_1]^{m(\theta_1)},[\theta_2]^{m(\theta_2)}$
(69,20;7,5)	$20,[5]^{23},[-3]^{45}$
(75,42;25,21)	$42,[7]^{18},[-3]^{56}$
(76,35;18,14)	$35,[7]^{19},[-3]^{56}$
(85,14;3,2)	$14,[4]^{34},[-3]^{50}$
(95,54;33,27)	$54,[9]^{19},[-3]^{75}$
(96,45;24,18)	$45,[9]^{20},[-3]^{75}$
(99,42;21,15)	$42,[9]^{21},[-3]^{77}$
(105,52;29,22)	$52,[10]^{20},[-3]^{84}$
(154,81;48,36)	$81,[15]^{21},[-3]^{132}$
(162,69;36,24)	$69,[15]^{23},[-3]^{138}$
(189,60;27,15)	$60,[15]^{28},[-3]^{160}$
(196,81;42,27)	$81,[18]^{24},[-3]^{171}$
(225,96;51,33)	$96,[21]^{24},[-3]^{200}$
(232,77;36,20)	$77,[19]^{28},[-3]^{203}$
(261,84;39,21)	$84,[21]^{29},[-3]^{231}$
(288,105;52,30)	$105,[25]^{27},[-3]^{260}$
(300,117;60,36)	$117,[27]^{26},[-3]^{273}$
(351,140;73,44)	$140,[32]^{26},[-3]^{324}$
(375,102;45,21)	$102,[27]^{34},[-3]^{340}$
(405,132;63,33)	$132,[33]^{30},[-3]^{374}$
(441,88;35,13)	$88,[25]^{44},[-3]^{396}$
(476,133;60,28)	$133,[35]^{34},[-3]^{441}$
(540,147;66,30)	$147,[39]^{35},[-3]^{504}$
(550,162;75,36)	$162,[42]^{33},[-3]^{516}$
(575,112;45,16)	$112,[32]^{46},[-3]^{528}$
(703,182;81,35)	$182,[49]^{37},[-3]^{665}$
(1 344,221;88,26)	$221,[65]^{56},[-3]^{1\,287}$
(1 911,270;105,27)	$270,[81]^{65},[-3]^{1\,845}$

Tab.3 Parameters feasible to Theorem 2.1, but do not exist

$(v,k;a,c)$	$\theta_0,[\theta_1]^{m(\theta_1)},[\theta_2]^{m(\theta_2)}$	
(49,32;21,20)	$32,[4]^{16},[-3]^{32}$	Ref.[6]
(57,42;31,30)	$42,[4]^{18},[-3]^{38}$	Ref.[15]
(76,45;28,24)	$45,[7]^{18},[-3]^{57}$	Ref.[9]
(76,54;39,36)	$54,[6]^{19},[-3]^{56}$	Ref.[10]
(96,57;36,30)	$57,[9]^{19},[-3]^{76}$	Ref.[7]
(209,16;3,1)	$16,[5]^{76},[-3]^{132}$	Prop. 2.1
(841,200;87,35)	$200,[55]^{40},[-3]^{800}$	Prop. 2.1

Proposition 2.1 The $(v,k;a,c)$-strongly regular graphs with parameter (209,16;3,1) or (841,200;87,35) do not exist.

Proof In the case $(v,k;a,c)=(209,16;3,1)$. We choose a vertex $x\in V$. Since $c=1$, we have that if two vertices y,z in $\Gamma(x)$ have a common neighbor w in $\Gamma(x)$, then y and z are adjacent. It implies that the local graph is a disjoint union of $(a+1)$-cliques. We count the number of pairs (x,C) with $x\in C$, where C is a $(a+2)$-clique in Γ. Then we see $(a+2)n=v\dfrac{k}{a+1}$, where n is the number of $(a+2)$-cliques in Γ. But $(a+1)(a+2)\nmid vk$, which implies the (209,16;3,1)-strongly regular graph does not exist.

Let Γ be a (841,200;87,35)-strongly regular graph with eigenvalues $\theta_0>\theta_1>\theta_2$ and multiplicities $m(\theta_i)$ $(0\leqslant i\leqslant 2)$. We have that

$$q_{11}^{1}=\frac{m(\theta_1)}{v}\left(1+\frac{\theta_1^3}{k^2}-\frac{(\theta_1+1)^3}{(v-k-1)^2}\right)>0^{[11,\text{ p.10}]}$$

with $\theta_1=55$ and $m(\theta_1)=40$, which implies

$$841=v\leqslant\frac{1}{2}(m(\theta_1)(m(\theta_1)+1))=820^{[11,\text{ Theorem 2.6}]},$$

a contradiction.

3 Optimistic strongly regular graphs

In this section, we classify optimistic strongly regular graphs with smallest eigenvalue -3.

Let Γ be a coconnected $(v,k;a,c)$-strongly regular graph with $k=\theta_0>\theta_1>\theta_2$ and multiplicities $m(\theta_i)$ $(0\leqslant i\leqslant 2)$. Note that the distance matrix of Γ is $2J-2I-A$, whose eigenvalues are $2v-2-k>-2-\theta_2>-2-\theta_1$, with multiplicities 1, $m(\theta_2)$, $m(\theta_1)$ respectively. Thus, if the graph Γ is optimistic, then $\theta_2<-2$. And the distance matrix of $\overline{\Gamma}$ is $J-I+A$, whose eigenvalues are $v-1+k>-1+\theta_1>-1+\theta_2$, with multiplicities 1, $m(\theta_1)$, $m(\theta_2)$ respectively. Thus if the graph $\overline{\Gamma}$ is optimistic, then $\theta_1>1$.

Lemma 3.1 Let Γ be a coconnected $(v,k;a,c)$-strongly regular graph with $k=\theta_0>\theta_1>\theta_2=-2$ with multiplicities $m(\theta_i)$ $(0\leqslant i\leqslant 2)$. Then neither Γ nor $\overline{\Gamma}$ is optimistic.

Proof Note that Γ is coconnected, by Ref. [5, Theorem 3.12.4]. We see that Γ is a triangular graph $T(n)$ $(n\geqslant 5)$, a lattice graph $L_2(n)$ $(n\geqslant 3)$, one of the graphs of Petersen, Clebsch, Schläfli, Shrikhande or Chang.

Since $\theta_2=-2$, if $m(\theta_2)>m(\theta_1)\geqslant 2$, then Γ has only one positive distance eigenvalue and $m(\theta_1)\geqslant 2$ negative distance eigenvalues, and $\overline{\Gamma}$ has at most $v-m(\theta_2)$ positive eigenvalues and exactly $m(\theta_2)$ negative distance eigenvalues. Since $m(\theta_2)>m(\theta_1)$ and $m(\theta_1)+m(\theta_2)=v-1$, we see $m(\theta_2)\geqslant$

$v-m(\theta_2)$. It follows that if $m(\theta_2)>m(\theta_1)\geqslant 2$, then neither Γ or $\overline{\Gamma}$ is optimistic.

We see that $T(n)$ has parameters $\left(\frac{1}{2}(n-1)n, 2(n-2); n-2, 4\right)$ and eigenvalues $2(n-2)$, $\tau_1=n-4$, $\tau_2=-2$ with multiplicities $m(\tau_1)=n-1$, $m(\tau_2)=\frac{1}{2}n(n-3)$. As $n\geqslant 5$, we see $m(\tau_2)>m(\tau_1)\geqslant 2$. The graph $L_2(n)$ has parameters $(n^2, 2n-2, n-2, 2)$ and eigenvalues $2(n-2)$, $\tau_1=n-2$, $\tau_2=-2$ with multiplicities $m(\tau_1)=2(n-1)$, $m(\tau_2)=(n-1)^2$. When $n\geqslant 4$, we see that $m(\tau_2)>m(\tau_1)\geqslant 2$. It follows that if Γ is $T(n)$ $(n\geqslant 5)$ or $L_2(n)$ $(n\geqslant 4)$, neither Γ nor $\overline{\Gamma}$ is optimistic. If Γ is $L_2(3)$ or the Petersen graph, then $\theta_1=1$ and $\theta_2=-2$, we see that neither Γ nor $\overline{\Gamma}$ is optimistic. If Γ is one of the graphs of Clebsch, Schläfli, Shrikhande or Chang, from Ref.[13], we have $m(\theta_2)>m(\theta_1)\geqslant 2$, and hence neither Γ nor $\overline{\Gamma}$ is optimistic.

Lemma 3.2 Let Γ be a coconnected $(v, k; a, c)$-strongly regular graph with eigenvalues $k=\theta_0>\theta_1>\theta_2$ and multiplicities $m(\theta_i)$ $(0\leqslant i\leqslant 2)$. If neither Γ nor $\overline{\Gamma}$ is optimistic, then Γ is one of the following:

① the pentagon;

② a triangular graph $T(n)$ $(n\geqslant 5)$, a lattice graph $L_2(n)$ $(n\geqslant 3)$, one of the graphs of Petersen, Clebsch, Schläfli, Shrikhande or Chang;

③ the complement of one graph in ① and ②.

Proof If θ_1 and θ_2 are not integers, then $m(\theta_1)=m(\theta_2)$, Γ is a conference graph, $v\equiv 1 \bmod 4$, $\theta_1=\frac{1}{2}(-1+\sqrt{v})$ and $\theta_2=\frac{1}{2}(-1-\sqrt{v})$ by Ref.[8, Lemma 10.3.2]. Then if $v\geqslant 13$, then

$$-2-\theta_2>0,\ -1+\theta_1>0,$$

and both Γ and $\overline{\Gamma}$ are optimistic. When $v<13$, we see Γ is either the pentagon or the lattice graph $L_2(3)$ by Ref.[4]. In both cases, neither Γ nor $\overline{\Gamma}$ is optimistic.

If the eigenvalues θ_1, θ_2 are integers, we can show that $\theta_2\geqslant -2$ or $\theta_1\leqslant 1$. Suppose to the contrary that $\theta_2<-2$ or $\theta_1>1$. Then $-2-\theta_2>0$ and $-1+\theta_1<0$. As the distance eigenvalue of Γ $-2-\theta_2$ has multiplicity $m(\theta_2)$; as the distance eigenvalue of $\overline{\Gamma}$, $-1+\theta_1$ has multiplicity $m(\theta_1)$. Note that $m(\theta_1)+m(\theta_2)=v-1$. Thus at least one of the graphs Γ and $\overline{\Gamma}$ is optimistic, contradicting the hypothesis that neither Γ nor $\overline{\Gamma}$ is optimistic.

If $\theta_1=\frac{1}{2}((a-c)+\sqrt{(a-c)^2+4(k-c)})=0$, then $c=k$ and hence that Γ is not coconnected by Lemma 2.1. It is impossible!

As Γ contains a path of length 2, by Ref.[8, Theorem 9.1.1], it results that $\theta_2\leqslant -\sqrt{2}$. Thus $\theta_2=-2$ or $\theta_1=1$. Furthermore, if $\theta_1=1$, then the smallest eigenvalue of $\overline{\Gamma}$ is $-1-\theta_1=-2$. And we see that either Γ or $\overline{\Gamma}$ has smallest eigenvalue -2. The result follows from Lemma 3.1.

Remark In the case where both Γ and $\overline{\Gamma}$ are optimistic, then Γ must be a conference graph with $v\geqslant 13$.

The following lemma gives the parameter restrictions for a strongly regular graph to be optimistic.

Lemma 3.3 Let Γ be a coconnected $(v, k; a, c)$-strongly regular graph with eigenvalues $k>\theta_1>\theta_2$ and multiplicities $m(\theta_i)$ $(0\leqslant i\leqslant 2)$. The graph Γ is optimistic if and only if $a\geqslant c-\frac{2k}{v-1}$ and $a<\frac{1}{2}(c+k-4)$.

Proof The graph Γ is optimistic if and only if $m(\theta_2)\geqslant m(\theta_1)$ and $\theta_2<-2$. Note that $m(\theta_2)-m(\theta_1)=\frac{1}{\sqrt{\Delta}}(2k+(v-1)(a-c))$. Hence $m(\theta_2)\geqslant m(\theta_1)$ is equivalent to $a\geqslant c-\frac{2k}{v-1}$.

If $v=k-1$, then from $(v-k-1)c=k(k-a-1)$, it follows that $a=k-1=v-2$. Then Γ must be a complete graph. It is impossible, since the complete graph has exactly two distinct eigenvalues. Now we have $v-1>k$, and hence $a-c\geqslant -1$. Note that $\theta_2=\frac{1}{2}((a-c)-\sqrt{\Delta})$. And $a-c\geqslant -1$ and $\theta_2<-2$ implies that $a<\frac{1}{2}(c+k-4)$.

Lemma 3.4 Let Γ be a coconnected $(v, k; a, c)$-strongly regular graph and let $\alpha\geqslant 3$ be an integer. Then

① if Γ has parameter $((\gamma+1)(\gamma(\alpha-1)+\alpha)/\alpha, \gamma\alpha; \gamma-1+(\alpha-1)^2, \alpha^2)$, where γ is a positive integer, then Γ is optimistic if and only if $\gamma \geqslant 2\alpha-1$;

② if Γ has parameter $((\gamma+1)^2, \gamma\alpha; \gamma-1+(\alpha-2)(\alpha-1), \alpha(\alpha-1))$, where γ is a positive integer, then Γ is optimistic if and only if $\gamma \geqslant 2\alpha-2$.

Proof By Lemma 3.3, we only need to show $a \geqslant c-\frac{2k}{v-1}$ and $a < \frac{1}{2}(c+k-4)$. As $\frac{2k}{v-1} < 2$, we need $a \geqslant c-1$.

① For the first case, we see $2a < c+k-4$ is equivalent to $\gamma \geqslant \alpha-1$ by $\alpha \geqslant 3$. When $a \geqslant c$, we have $\gamma \geqslant 2\alpha$. When $a = c-1$, that is $\gamma = 2\alpha-1$, we have $\frac{2k}{v-1} = \frac{2\alpha}{2\alpha-1} \geqslant 1$ and $a \geqslant c-\frac{2k}{v-1}$ still holds. So we have $\gamma \geqslant 2\alpha-1$ if and only if Γ is optimistic.

② For the second case, As $\alpha \geqslant 3$, $\gamma \geqslant \alpha-2$ is equivalent to $2a < c+k-4$. And $a \geqslant c$ implies $\gamma \geqslant 2\alpha-1$. When $a = c-1$, that is $\gamma = 2\alpha-2$, we have $\frac{2k}{v-1} = 1$ and $a \geqslant c-\frac{2k}{v-1}$ still holds. It follows that $\gamma \geqslant 2\alpha-2$ if and only if Γ is optimistic.

Corollary 3.1 Let $\alpha \geqslant 3$ be an integer, then the following holds:

① There are only finitely many coconnected non-optimistic strongly regular graphs with smallest eigenvalue $-\alpha$;

② There are only finitely many coconnected non-optimistic strongly regular graphs with second largest eigenvalue $-1+\alpha$.

Proof By Theorem 0.1 and Lemma 3.4, we obtain ①. By considering their complements, we have ②.

Now we give a classification of non-coconnected (hence we also have the coconnected ones by Theorem 2.1) strongly regular graphs with smallest eigenvalue -3.

Theorem 3.1 Let Γ be a coconnected $(v,k;a,c)$-strongly regular graph with smallest eigenvalue -3. If Γ is not optimistic, then Γ has one of the following parameters:

① (16,9;4,6); ② (15,6;1,3);
③ (16,5;0,2); ④ (26,10;3,4);
⑤ (50,7;0,1).

Proof Since Γ is coconnected, we only need to deal with the graphs in Theorem 2.1. For those two families in Theorem 2.1 ② and ③, we see that the only feasible one is {16, 9; 4, 6}, by Lemma 3.4. The rest are tested by Lemma 3.3.

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