第48卷第11期 中国科学技术大学学报 Vol. 48, No. 11
2018年11月 JOURNAL OF UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA Nov. 2018

文章编号:0253-2778(2018)11-0898-04

Complemented Sylow subgroups of finite groups

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Abstract: The relationship between the complementability of Sylow subgroups and the structure of a group was considered, and some results about the construction of composition factors were obtained. Further, one of Heliel's results is the corollary of our results.

Key words: composition factor; complementability; Sylow subgroup; simple group

CLC number: O152.1 **Document code**: A doi:10.3969/j.issn.0253-2778.2018.11.005

2010 Mathematics Subject Classification: Primary 20D10; Secondary 20D20

Citation: BAO Hongwei, ZHANG Jia, MIAO Long. Complemented Sylow subgroups of finite groups[J]. Journal of University of Science and Technology of China, 2018,48(11):898-901.
鲍宏伟,张佳,缪龙. 有限群的可补的 Sylow 子群[J]. 中国科学技术大学学报,2018,48(11):898-901.

有限群的可补的 Sylow 子群

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摘要:研究 Sylow 子群的可补性与群结构的关系,得到了一些关于合成因子构造的结果,推广了 Heliel 的一个结果.

关键词:合成因子;可补性;Sylow 子群;单群

0 Introduction

All the groups in this paper are finite. Most of the notation are standard, as in Refs.[1-2]. Let π be a set of prime numbers, $|G|$ be the order of a group G and $\pi(G)$ denote the set of all prime divisors of $|G|$. Further, assume that H/K is a composition factor (chief factor) of G. If $p\in\pi(H/K)$, then H/K is called a pd-composition factor (chief factor) of G.

A subgroup H of a group G is said to be complemented in G if there exists a subgroup K of G such that $G=HK$ and $H\cap K=1$. In 1937, Hall[3] showed that G is solvable if and only if all Sylow subgroups of G are complemented in G. In 1982, Arad and Ward[4] proved that G is solvable

Received: 2017-07-08; **Revised**: 2018-01-05

Foundation item: Supported by the NSFC (11871062, 11501235), the Natural Science Foundation of Jiangsu Province (BK20181451), the Key Natural Science Foundation of Anhui Education Commission (KJ2017A569), the Fundamental Research Funds of China West Normal University (17E091).

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if and only if every Sylow 2-subgroup and every Sylow 3-subgroup are complemented in G. In 1999, Ballester-Bolinches and Guo[5] proved that a finite group G is supersolvable if it contains a normal group N such that G/N is supersolvable and all subgroups of N of prime order are complemented in G. In 2015, Qian and Tang[6] investigated the structure of G under the assumption that all subgroups of P with order p^m are complemented in G, where m is a given integer satisfying $1 \leqslant m \leqslant -1+\log_p|P|$ and P is a Sylow subgroup of G.

Recently, Monakhov and Kniahina[7] established that if every subgroup of P of order p is complemented in G, then P is complemented in G. But the converse is not true. For instance, suppose that $G=PSL(2,8)$, every Sylow 3-subgroup of G is complemented in G, but every subgroup of order 3 of G is not complemented in G. Naturally, we shall consider the following question which is a motivation of our research.

What is the structure of a group when we consider the complementability of Sylow subgroups of finite groups?

Further, we obtain the following results:

Theorem 0.1 Let G be a group, $p\in\pi(G)$. Suppose that every Sylow p-subgroup is complemented in G. If G is not p-solvable, then every nonabelian pd-composition factor of G is isomorphic to one of the following groups:

① A_p, $p\geqslant 5$ is a prime.

② $PSL(2,11)$ and $p=11$, M_{23} and $p=23$, M_{11} and $p=11$.

③ $PSL(n,q)$, $p^a=(q^n-1)/(q-1)$, n is a prime.

Theorem 0.2 Let G be a group, $r\in\pi(G)$, $\pi=\pi(G)\setminus\{r\}$. Suppose that every Sylow p-subgroup of G is complemented in G for each prime $p\in\pi$. Then every nonabelian composition factor of G is isomorphic to $PSL(2,7)$.

Theorem 0.3 Let G be a group, and let $r,t\in\pi(G)$, $\pi=\pi(G)\setminus\{r,t\}$. Suppose that every Sylow p-subgroup is complemented in G for each prime $p\in\pi$. Then every nonabelian composition factor of G is isomorphic to one of the following groups: ① $PSL(2,5)$, ② $PSL(2,7)$, ③ $PSL(2,8)$, ④ $PSL(3,3)$.

Theorem 0.4 Let G be a group, $r,t,v\in\pi(G)$, $\pi=\pi(G)\setminus\{r,t,v\}$. Suppose that every Sylow p-subgroup is complemented in G for each $p\in\pi$. Then every nonabelian composition factor of G is isomorphic to one of the following groups: ① A_7; ② M_{11}; ③ $PSL(3,q)$, $q=3,5,8,17$; ④ $PSL(2,q)$, $q=5,7,8,11,16,31,127$.

1 Preliminaries

For the sake of convenience, we first list here some results which will be used in the sequel.

Lemma 1.1[8, Lemma 2.1] Let G be a group and N be a normal subgroup of G. Then the following statements hold:

① If H is complemented in G, $H\leqslant M\leqslant G$, then H is also complemented in M.

② If $N\leqslant H$ and H is complemented in G, then H/N is complemented in G/N.

③ Let π be a set of primes, N a normal π'-subgroup and H a π-subgroup of G. Then H is complemented in G if and only if HN/N is complemented in G/N.

Lemma 1.2[7, Lemma 3] Suppose that a simple group G is the product of a biprimary subgroup A with a p-subgroup B for some prime p. Then G is isomorphic to one of the following groups:

① $PSL(2,5)=A_4Z_5$.

② $PSL(2,7)=S_4Z_7=(Z_7\rtimes Z_3)D_8$.

③ $SL(2,8)=(E_{2^3}\rtimes Z_7)Z_9$.

④ $PSL(3,3)=AZ_{13}$, $|A|=2^4\cdot 3^3$.

Lemma 1.3[9, Theorem 1] Let G be a nonabelian finite simple group. If there exists a prime p, a positive integer a and a subgroup H of G such that $|G:H|=p^a$, then one of the following holds:

① $G\cong A_n$, $H\cong A_{n-1}$, where $n=p^a$.

② $G\cong PSL(n,q)$, $|G:H|=(q^n-1)/(q-1)=p^a$, where n is a prime.

③ $G\cong PSL(2,11)$, $H\cong A_5$.

④ $G\cong M_{23}$, and $H\cong M_{22}$.

⑤ $G\cong M_{11}$, and $H\cong M_{10}$.

⑥ $G\cong PSU(4,2)\cong PSp_4(3)$, and $H\cong 2^4:A_5$.

Lemma 1.4[10, Lemma 3.8.2] Let A, B be subgroups of a group G and K a normal subgroup of G. If $(|G:A|,|G:B|)=1$, then

$$K=(K\cap A)(K\cap B).$$

Lemma 1.5[1, Lemma 1.2] Let U, V and W be subgroups of a group G. Then the following statements are equivalent:

① $U\cap VW=(U\cap V)(U\cap W)$.

② $UV\cap UW=U(V\cap W)$.

Lemma 1.6 Let p be a prime and U a subnormal subgroup or a quotient group of G. If every Sylow p-subgroup of G is complemented in G, then every Sylow p-subgroup of U is complemented in U.

Proof Assume that U_p is a Sylow p-subgroup of U and $U_p=P\cap U$ for some Sylow p-subgroup P of G. If U is a subnormal subgroup of G, without loss of generality, then we may assume that U is a normal subgroup of G. Further, by the hypothesis, P is complemented in G, i.e., $G=PB$ and $P\cap B=1$. Then $U=U\cap G=U\cap PB=(U\cap P)(U\cap B)=U_p(U\cap B)$ by Lemma 1.4, that is, U_p is complemented in U and every Sylow p-subgroup of U is complemented in U.

If U is a quotient group of G, then we may set $U=\overline{G}$. Since every Sylow p-subgroup of U is $\overline{P}$ for some Sylow p-subgroup P of G, every Sylow p-subgroup of U is complemented in U by Lemma 1.5.

Lemma 1.7 Suppose that a nonabelian simple group G which satisfies that $\pi(G)=4$ and a Sylow p-subgroup of G is complemented in G only for one prime divisor p belongs to $\pi(G)$. Then G is isomorphic to one of the following groups: ① A_7; ② M_{11}; ③ $PSL(3,q)$, $q=5,8,17$; ④ $PSL(2,q)$, $q=11,16,31,127$.

Proof Firstly, by Refs. [11, Theorem 1 and Theorem 2] and [9, Theorem 1], we can verify that G is isomorphic to one of the following groups: A_7, M_{11}, $PSL(3,5)$, $PSL(3,8)$, $PSL(3,17)$, $PSL(2,11)$, $PSL(2,16)$, $PSL(2,31)$.

Next, by Ref. [9, Theorem 1], we consider the group $G\cong PSL(2,2^n-1)$, $2^n-1\geqslant 19$ is a Mersenne prime. Further, $n>3$ is a prime and $n=6k\pm 1$, k is a positive integer, $G\cong PSL(2,2^{6k\pm 1}-1)$ and $G\cong PSL(2,127)$ by Ref. [11, Theorem 2].

Lemma 1.8[3] G is solvable if and only if all Sylow subgroups of G are complemented in G.

2 Proof of theorems

Proof of Theorem 0.1 Let U be a nonabelian pd-composition factor of G. Then U is a nonabelian simple group with $p\in\pi(U)$. By Lemma 1.6, U also satisfies the hypothesis. Now the required result follows by Lemma 1.3.

Proof of Theorem 0.2 By Lemma 1.6 and Lemma 1.8, we may assume that G is a nonabelian simple group. By Burnside's p^aq^b-Theorem[1], $|\pi(G)|\geqslant 3$. Then $G\cong PSL(2,7)$ since $PSL(2,7)$ is the only simple group with subgroups of two different prime power indices by Guralnick[9].

Proof of Theorem 0.3 By Lemma 1.6, Lemma 1.8 and the proof of Theorem 0.2, we may assume that G is a nonabelian simple group. If $|\pi(G)|\geqslant 4$, then $G\cong PSL(2,7)$ by Guralnick[9]. If $|\pi(G)|=3$, then the required result follows by Lemma 1.2.

Proof of Theorem 0.4 By Lemma 1.6, Lemma 1.8 and the proofs of Theorem 0.2 and Theorem 0.3, we may assume that G is a nonabelian simple group. If $|\pi(G)|\geqslant 5$, then $G\cong PSL(2,7)$ by Guralnick[9]. If $|\pi(G)|=4$, then every nonabelian composition factor of G is isomorphic to one of ①～④ in the theorem by Lemma 1.7.

3 Applications

Firstly, we introduced the concept of c-supplemented subgroups which is related to our corollaries.

Definition 3.1[12, Definition 1.1] A subgroup H of

G is called c-supplemented in G, if there exists a subgroup K of G such that $G=HK$ and $H\cap K\leqslant H_G$ where H_G is the core of H in G.

Corollary 3.1 Let G be a group, $p\in\pi(G)$. Suppose that every Sylow p-subgroup is c-supplemented in G. If G is not p-solvable, then nonabelian pd-composition factors of G are isomorphic to one of the following groups:

① A_p, $p\geqslant 5$ is a prime.

② $PSL(2,11)$ and $p=11$, M_{23} and $p=23$, M_{11} and $p=11$.

③ $PSL(n,q)$, $p^a=(q^n-1)/(q-1)$, n is a prime.

Proof By Ref. [12, Lemma 2.1(2)], $O_p(G)=1$. Further, every c-supplemented Sylow p-subgroup of G is complemented in G. Hence it is proved by Theorem 0.1.

In proofs of the following corollaries, we also only consider the condition that every c-supplemented Sylow p-subgroup of G is complemented in G the same as the proof of Corollary 3.1.

Corollary 3.2 Let G be a group, $p\in\pi(G)$. Suppose that every Sylow p-subgroup is c-supplemented in G. If G is not p-solvable, then every nonabelian pd-chief factor of G is the direct product of simple groups, each being isomorphic to one of ①～③ in Corollary 3.1.

Corollary 3.3 Let G be a group, $r\in\pi(G)$, $\pi=\pi(G)\setminus\{r\}$. Suppose that every Sylow p-subgroup is c-supplemented in G for each prime $p\in\pi$. Then every nonabelian composition factor of G is isomorphic to $PSL(2,7)$.

When $r=2$, we generalized the following result of Heliel:

Corollary 3.4[13, Theorem 1.4] Let G be a group. Then G is solvable if and only if every Sylow subgroup of odd order is c-supplemented in G.

Corollary 3.5 Let G be a group, $3\neq r\in\pi(G)$, $\pi=\pi(G)\setminus\{r\}$. Suppose that every Sylow p-subgroup is c-supplemented in G for each prime $p\in\pi$. Then G is solvable.

Corollary 3.6 Let G be a group, $r\in\pi(G)$, $\pi=\pi(G)\setminus\{r\}$. Suppose that every Sylow p-subgroup is c-supplemented in G for each prime $p\in\pi$. Then every nonabelian chief factor of G is the direct product of simple groups, each being isomorphic to $PSL(2,7)$.

Corollary 3.7 Let G be a group, $r,t\in\pi(G)$, $\pi=\pi(G)\setminus\{r,t\}$. Suppose that every Sylow p-subgroup is c-supplemented in G for each prime $p\in\pi$. Then every nonabelian composition factor of G is isomorphic to one of the following groups: ① $PSL(2,5)$, ② $PSL(2,7)$, ③ $PSL(2,8)$, ④ $PSL(3,3)$.

Corollary 3.8 Let G be a group, $r,t\in\pi(G)$, $\pi=\pi(G)\setminus\{r,t\}$. Suppose that every Sylow p-subgroup is complemented in G for each prime $p\in\pi$. Then every nonabelian chief factor of G is the direct product of simple groups, each being isomorphic to one of ①～④ in Corollary 3.7.

Acknowledgments The authors wish to thank professors Victor Monakhov, Qian Guohua, Danila Revin and the reviewers for their suggestions which have helped to improve our original version.

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