第48卷第11期 中国科学技术大学学报 Vol. 48, No. 11
2018年11月 JOURNAL OF UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA Nov. 2018

文章编号:0253-2778(2018)11-0885-05

On a class of locally dually flat (α,β)-metrics

HUA Yiping[1], SONG Weidong[2]

(1. *College of Mathematics and Computer Science, Chizhou University, Chizhou* 247000, *China*;
2. *School of Mathematics and Statistics, Anhui Normal University, Wuhu* 241003, *China*)

Abstract: Locally dually flat weak Landsberg (α,β)-metrics in the form of $F=\alpha\phi(\frac{\beta}{\alpha})$ were studied, where α is a Riemannian metric and β is a 1-form. And locally dually flat (α,β)-metrics $F=\alpha\phi(\frac{\beta}{\alpha})$ with relatively isotropic mean Landsberg curvature were characterized, where $\phi=\phi(s)$ is a polynomial in s.

Key words: locally dually flat; (α,β)-metrics; weak Landsberg; mean Landsberg curvature; Minkowskian

CLC number: O186.1 **Document code**: A doi:10.3969/j.issn.0253-2778.2018.11.003

2010 Mathematics Subject Classification: Primary 53B40; Secondary 53C60

Citation: HUA Yiping, SONG Weidong. On a class of locally dually flat (α,β)-metrics[J]. Journal of University of Science and Technology of China, 2018, 48(11): 885-889.
华义平,宋卫东. 一类局部对偶平坦的(α,β)度量[J]. 中国科学技术大学学报,2018,48(11):885-889.

一类局部对偶平坦的(α,β)度量

华义平[1],宋卫东[2]

(1.池州学院数学与计算机学院,安徽池州 247000; 2.安徽师范大学数学与统计学院,安徽芜湖 241003)

摘要:刻画了定义在$n(n\geqslant 3)$维流形M上的局部对偶平坦的弱Landsberg的(α,β)度量$F=\alpha\phi(\frac{\beta}{\alpha})$,其中$\alpha=\sqrt{a_{ij}(x)y^iy^j}$是一个黎曼度量,$\beta=b_i(x)y^i$是一个1形式.还刻画了定义在$n(n\geqslant 3)$维流形上局部对偶平坦且具有相对迷向平均Landsberg曲率的(α,β)度量$F=\alpha\phi(\frac{\beta}{\alpha})$,其中$\phi(s)$是关于$s$的多项式.

关键词:局部对偶平坦;(α,β)度量;弱Landsberg;平均Landsberg曲率;闵可夫斯基

0 Introduction

Locally dually flat Finsler metrics are studied in Finsler information geometry and naturally arise from the investigation of the so-called flat information structure[1]. A Finsler metric $F=F(x,y)$ on a manifold M is called locally dually flat if at every point there is a coordinate system

Received: 2017-06-24; **Revised**: 2017-07-22

Foundation item: Supported by the NSF of Chizhou University (2014ZRZ011).

Biography: HUA Yiping (corresponding author), female, born in 1982, master/lecturer. Research field: Differential geometry. E-mail: huayiping@yeah.net

(x^i) in which the geodesic spray coefficients are in the following form

$$G^i = -\frac{1}{2} g^{ij} H_{y^j} \tag{1}$$

where $H = H(x, y)$ is a C^∞ scalar function on $TM \backslash \{0\}$ satisfying $H(x, \lambda y) = \lambda^3 H(x, y)$ for all $\lambda > 0$. Such a coordinate system is called an adapted coordinate system.

The first example of non-Riemannian dually flat metrics is the Funk metric given as follows[1]

$$F = \frac{\sqrt{|y|^2 - (|x|^2 |y|^2 - \langle x, y \rangle^2)}}{1 - |x|^2} \pm \frac{\langle x, y \rangle}{1 - |x|^2}.$$

This metric is defined on the unit ball $B^n \subset R^n$. Recently, there are many results on the locally dually flat Finsler metrics. Ref. [2] characterized locally dually flat and locally projectively flat Finsler metric and found some equations that characterized locally dually flat Randers metrics. Ref. [3] studied and gave a characterization of locally dually flat (α, β)-metrics on an n-dimensional manifold M $(n \geqslant 3)$.

In Finsler geometry, there are several important non-Riemannian quantities, one of them being the distortion τ is a primary quantity. The vertical derivative of τ on tangent spaces gives rise to the mean Cartan torsion $I = \tau_{y^m} \mathrm{d}x^m$, and the horizontal derivative of I along geodesics is called the mean Landsberg curvature $J = I_{;k} y^k$, thus J / I is regarded as the relative growth rate of the mean Cartan torsion along geodesic. We call a Finsler metric F is of relatively isotropic mean Landsberg curvature if F satisfies $J + cFI = 0$, where $c = c(x)$ is a scalar function on the Finsler manifold. In particular, when $c = 0$, Finsler metrics with $J = 0$ are called weak Landsberg metrics. Refs. [4, 8] studied and characterized (α, β)-metrics with relatively isotropic mean Landsberg curvature, and obtained some useful results.

In this paper, we first study and characterize locally dually flat weak Landsberg (α, β)-metrics which are not Riemannian, and obtain the following theorem:

Theorem 0.1 Let $F = \alpha\phi(\frac{\beta}{\alpha})$ be an (α, β)-metric on an n-dimensional manifold M $(n \geqslant 3)$. Suppose F is not Riemannian and ϕ satisfies one of the following:

① $\phi(s)$ is a polynomial of s with $\phi'(0) = 0$;

② $\phi(s)$ is an analytic function with $\phi'(0) = \phi''(0) = 0$;

③ $\phi'(0) \neq 0$,

$$s(k_2 - k_3 s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi') \neq 0;$$

where k_1, k_2 and k_3 are constants. Then, if F is a locally dually flat weak Landsberg metric, F must be locally Minkowskian.

The third condition in Theorem 0.1 is to make locally dually flat (α, β)-metric satisfy Eqs. (15)～(17), and these are very important for the proof of Theorem 0.1. In other words, if a locally dually flat (α, β)-metric can not satisfy any condition of ①～③ (for example Randers metric[2]), then Eqs. (15)～(17) can not be held. Moreover, there are many functions $\phi(s)$ satisfying the third condition, for example $\phi(s) = \mathrm{e}^s$, $\phi(s) = \frac{1}{1-s} + \epsilon s (\epsilon \neq -1)$, ect.

Further, we study locally dually flat (α, β)-metrics with relatively isotropic mean Landsberg curvature. We have the following theorem.

Theorem 0.2 Let $F = \alpha\phi(\frac{\beta}{\alpha})$ be an (α, β)-metric on an n-dimensional manifold M $(n \geqslant 3)$. $\phi(s) = \sum_{i=0}^{n} a_i s^i$ is a polynomial in s satisfying $a_1 = 0$ or $a_i \neq 0 (0 \leqslant i \leqslant n)$, If F is locally dually flat with relatively isotropic mean Landsberg curvature, then it must be locally Minkowskian.

1 Preliminary

Let M be an n-dimensional smooth manifold. We denote by TM the tangent bundle of M and by $(x, y) = (x^i, y^i)$ the local coordinates on the tangent bundle TM. A Finsler manifold (M, F) is a smooth manifold equipped with a function F: $TM\backslash\{0\} \to [0,\infty)$, which has the following properties:

①Regularity: F is smooth in $TM\backslash\{0\}$;

②Positively homogeneity: $F(x,\lambda y)=\lambda F(x,y)$, for $\lambda > 0$;

③ Strong convexity: the Hessian matrix of F^2, $(g_{ij}(x,y)=\frac{1}{2}[F^2]_{y^iy^j})$ is positive definite on $TM\backslash\{0\}$. We call F and the tersor g_{ij} the Finsler metric and the fundamental tensor of M respectively.

In Finsler geometry, (α,β)-metric is a class of important Finsler metric. By definition, an (α,β)-metric is expressed as the following form,

$$F=\alpha\phi(s),\ s=\frac{\beta}{\alpha} \tag{2}$$

where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta=b_i(x)y^i$ is a 1-form, $\phi=\phi(s)$ is a C^∞ positive function on an open interval $(-b_0,b_0)$ satisfying

$$\phi(s)-s\phi'(s)+(b^2-s^2)\phi''(s)>0,\ |s|\leqslant b<b_0 \tag{3}$$

where $b=\|\beta_x\|_\alpha$. It is known that $F=\alpha\phi(s)$ is a Finsler metric if and only if $\|\beta_x\|_\alpha<b_0$ for any $x\in M$[5]. In particular, if $\phi(s)=1+s$, then (α,β)-metric is a Randers metric. Let $G^i(x,y)$ and $G^i_\alpha(x,y)$ denote the spray coefficients of F and α, respectively. To express formulate for the spray coefficients G^i of F in term of α and β, we need to introduce some notations. Let $b_{i|j}$ be a covariant derivative of b_i with respect to α. Denote

$$r_{ij}=\frac{1}{2}(b_{i|j}+b_{j|i}),\ s_{ij}=\frac{1}{2}(b_{i|j}-b_{j|i}) \tag{4}$$

$$s^i_j=a^{ih}s_{hj},\ s_j=b^is_{ij},\ r_j=b^ir_{ij} \tag{5}$$

$$r_0=r_jy^j,\ s_0=s_jy^j,\ r_{00}=r_{ij}y^iy^j \tag{6}$$

In fact, we can express the spray coefficients G^i as follows[5]:

$$G^i=G^i_\alpha+\alpha Qs^i_0+\Theta(-2\alpha Qs_0+r_{00})\frac{y^i}{\alpha}+\Psi(-2\alpha Qs_0+r_{00})b^i \tag{7}$$

where

$$Q=\frac{\phi'}{\phi-s\phi'} \tag{8}$$

$$\Theta=\frac{\phi'(\phi-s\phi')}{2\phi[(\phi-s\phi')+(b^2-s^2)\phi'']}-s\Psi \tag{9}$$

$$\Psi=\frac{\phi''}{2[(\phi-s\phi)+(b^2-s^2)\phi'']} \tag{10}$$

here $b^i=a^{ij}b_j$ and $b^2=b_ib^i$.

Let

$$\left.\begin{aligned}&\Delta=1+sQ+(b^2-s^2)Q',\\ &\Psi_1=\sqrt{b^2-s^2}\,\Delta^{\frac{1}{2}}\Big[\frac{\sqrt{b^2-s^2}\,\phi}{\Delta^{\frac{3}{2}}}\Big]'\end{aligned}\right\} \tag{11}$$

$$\Phi=-(n\Delta+1+sQ)(Q-sQ')-(b^2-s^2)(1+sQ)Q'',\ h_j=b_j-\alpha^{-1}sy_j \tag{12}$$

The mean Landsberg curvature of the (α,β)-metric $F=\alpha\phi(\frac{\beta}{\alpha})$ is given by[6]

$$\begin{aligned}J_j=\frac{-1}{2\alpha^4\Delta}\{&\frac{2\alpha^2}{b^2-s^2}[\frac{\Phi}{\Delta}+(n+1)(Q-sQ')]\cdot(r_0+s_0)h_j+\frac{\alpha}{b^2-s^2}[\Psi_1+s\frac{\Phi}{\Delta}](r_{00}-2\alpha Qs_0)h_j+\\&\alpha[-\alpha Q's_0h_j+\alpha Q(\alpha^2s_j-y_js_0)+\alpha^2\Delta s_{j0}+\alpha^2(r_{j0}-2\alpha Qs_j)-(r_{00}-2\alpha Qs_0)y_j]\frac{\Phi}{\Delta}\}\end{aligned} \tag{13}$$

2 Some lemmas

To prove Theorems 0.1 and 0.2, we need some lemmas.

Lemma 2.1 If $b^2Q+s=0$, then $\phi(s)=k\sqrt{b^2-s^2}$, in this case F is Riemannian.

Lemma 2.2[7] If $\phi=\phi(s)$ satisfies $\Psi_1=0$, then F is Riemannian.

Let $J=J_ib^i$, by Eq. (13) we get

Lemma 2.3[6] For an (α,β)-metric $F=\alpha\phi(\frac{\beta}{\alpha})$, the quantity J is given by

$$J=-\frac{1}{2\alpha^2\Delta}\{\Psi_1(r_{00}-2\alpha Qs_0)+\alpha\Psi_2(r_0+s_0)\} \tag{14}$$

where $\Psi_2=2(n+1)(Q-sQ')+\frac{3\Phi}{\Delta}$.

Lemma 2.4[3] Let $F=\alpha\phi(\frac{\beta}{\alpha})$ be an (α,β)-metric on an n-dimensional manifold ($n\geqslant 3$). Suppose F is not Riemannian and ϕ satisfies one of the following:

(H_1) $\phi(s)$ is a polynomial of s with $\phi'(0)=0$,

(H_2) $\phi(s)$ is an analytic function with $\phi'(0)=\phi''(0)=0$,

(H_3) $\phi'(0)\neq 0$,

$$s(k_2-k_3s^2)(\phi\phi'-s\phi'^2-s\phi\phi'')-(\phi'^2+\phi\phi'')+k_1\phi(\phi-s\phi')\neq 0;$$

where k_1,k_2,k_3 are constants. Then F is locally dually flat on M if and only if α and β satisfy

$$s_{l0}=\frac{1}{3}(\beta\theta_l-\theta b_l) \quad (15)$$

$$r_{00}=\frac{2}{3}[\theta\beta-(\theta_l b^l)\alpha^2] \quad (16)$$

$$G^l_\alpha=\frac{1}{3}(2\theta y^l+\theta^l\alpha^2) \quad (17)$$

where $\theta=\theta_i(x)y^i$ is a 1-form on M and $\theta^l=a^{lk}\theta_k$.

In fact, there are many functions $\phi(s)$ which are solutions of assumption (H_3), let $\phi(s)=\sum_{i=0}^{n}a_is^i$ with constants $a_i\neq 0(0\leqslant i\leqslant n)$. It is not difficult to verify these $\phi(s)$ satisfy condition (H_3)[3]. Thus, the (α,β)-metrics defined by $\phi(s)=\sum_{i=0}^{n}a_is^i$ with constants $a_i\neq 0(0\leqslant i\leqslant n)$ are locally dually flat if and only if α and β satisfy Eqs. (15)~(17).

We assume that $\phi(s)=\sum_{i=0}^{n}a_is^i$ is a polynomial in s, where $a_1=0$ or $a_i\neq 0(0\leqslant i\leqslant n)$, if F is locally dually flat. By (H_1) and (H_3), Eqs. (15)~(17) hold.

Lemma 2.5[8] Let $F=\alpha\phi(s)$, $s=\frac{\beta}{\alpha}$ be an (α,β)-metric with relatively isotropic mean Landsberg curvature on a manifold of dimension n ($n\geqslant 3$). Suppose that the following conditions: ① $\phi(s)$ is a polynomial in s; ② F is not of Randers type. Then F is a Berwald metric.

3 Proof of theorems

Proof of Theorem 0.1 Assume that $F=\alpha\phi(\frac{\beta}{\alpha})$ is a (α,β)-metric satisfying the condition of Theorem 0.1, then Eqs. (15)~(17) hold. Thus

$$\left.\begin{aligned} s_{ij}&=\frac{1}{3}(b_j\theta_i-b_i\theta_j),\\ r_{ij}&=\frac{1}{3}(\theta_ib_j+\theta_jb_i-2\mu a_{ij})\end{aligned}\right\} \quad (18)$$

where $\mu(x)=\theta_lb^l$.

Contracting (18) with b^i yields

$$s_j=\frac{1}{3}(\mu b_j-b^2\theta_j),\ r_j=\frac{1}{3}(b^2\theta_j-\mu b_j) \quad (19)$$

That is

$$s_0=\frac{1}{3}(\mu\beta-b^2\theta),\ r_0=\frac{1}{3}(b^2\theta-\mu\beta) \quad (20)$$

So

$$r_0+s_0=0 \quad (21)$$

Assume that F is a weak Landsberg metric, then it satisfies $J=0$. Putting (21) into (14), we get $\Psi_1=0$ or $r_{00}-2\alpha Q s_0=0$. If $\Psi_1=0$, by Lemma 2.2, F is Riemannian, this is impossible. Thus

$$r_{00}-2\alpha Q s_0=0 \quad (22)$$

In order to overcome the difficulty in computation, we take an orthonormal basis at any point x with respect to α such that

$$\alpha=\sqrt{\sum_{i=1}^{n}(y^i)^2},\ \beta=by^1 \quad (23)$$

where $b=\|\beta_x\|_\alpha$. Then we have the following coordinate transformation in T_xM[8], $\varphi:(s,u^A)\to(y^i)$

$$y^1=\frac{s}{\sqrt{b^2-s^2}}\bar{\alpha},\ y^A=u^A,\ 2\leqslant A\leqslant n \quad (24)$$

where $\bar{\alpha}=\sqrt{\sum_{A=2}^{n}(u^A)^2}$, we have

$$\alpha=\frac{b}{\sqrt{b^2-s^2}}\bar{\alpha},\ \beta=\frac{bs}{\sqrt{b^2-s^2}}\bar{\alpha} \quad (25)$$

In this coordinate system, we have the following expressions:

$$\left.\begin{aligned} r_{00}&=\frac{2}{3}\left[\frac{b\theta_1s^2}{b^2-s^2}\bar{\alpha}^2+\frac{bs\bar{\theta}_0}{\sqrt{b^2-s^2}}\bar{\alpha}-\frac{\mu b^2}{b^2-s^2}\bar{\alpha}^2\right],\\ s_0&=\frac{1}{3}\left[\frac{\mu bs}{\sqrt{b^2-s^2}}\bar{\alpha}-\frac{b^2\theta_1s}{\sqrt{b^2-s^2}}\bar{\alpha}-b^2\bar{\theta}_0\right]\end{aligned}\right\} \quad (26)$$

It follows from Eq. (22) that

$$\frac{2b}{3(b^2-s^2)}(\theta_1s^2-\mu b-\mu bsQ+b^2\theta_1Qs)\bar{\alpha}^2+\frac{2b\bar{\theta}_0}{3\sqrt{b^2-s^2}}(s+b^2Q)\bar{\alpha}=0 \quad (27)$$

Note $b \neq 0$ then

$$\left.\begin{array}{l}\theta_1 s^2 - \mu b - \mu bsQ + b^2\theta_1 Qs = 0,\\(s + b^2Q)\bar{\theta}_0 = 0\end{array}\right\} \quad (28)$$

Letting $s = 0$ and using $b \neq 0$ in the first equation of (28), we have $\mu = 0$. Then (28) is equivalent to

$$\theta_1(b^2Q + s) = 0,\ (s + b^2Q)\bar{\theta}_0 = 0 \quad (29)$$

Since F is non-Riemannian, by Lemma 2.1, $b^2Q + s \neq 0$, then (29) are reduced to

$$\theta_1 = 0,\ \theta_A = 0 \quad (29)$$

Plugging (30) into(17)~(18) yields

$$r_{ij} = 0,\ s_{ij} = 0,\ G^i_\alpha = 0 \quad (31)$$

We know that α is flat and β is parallel with respect to α, which implies that F is locally Minkowskian.

Proof of Theorem 0.2 We assume that F is a locally dually flat (α, β)-metric with relatively isotropic mean Landsberg curvature, $\phi(s) = \sum_{i=0}^{n} a_i s^i$ is a polynomial on s satisfying $a_1 = 0$ or $a_i \neq 0 (0 \leqslant i \leqslant n)$, and by Lemma 2.4, (15)~(17) hold. In addition, noting $\phi(s)$ is a polynomial of degree $n (n \geqslant 3)$, so F is not of Randers type, and by Lemma 2.5, F is a Berwald metric, which implies that $r_{ij} = 0$, $s_{ij} = 0$. Then (15) and (16) are reduced to

$$\beta\theta_l - \theta b_l = 0,\ \theta\beta - \mu\alpha^2 = 0 \quad (32)$$

Contracting the first equation of (32) with b^l yields $\theta = \frac{\mu}{b^2}\beta$. Plugging this into the second equation of (32), we get $\frac{\mu}{b^2}\beta^2 - \mu\alpha^2 = 0$, thus deriving $\mu = 0$, $\theta = 0$. Plugging this into (17), we get $G^i_\alpha = 0$. Now α is flat and β is parallel with respect to α, which implies F is locally Minkowskian.

Acknowledgements We would like to thank the anonymous reviewers for their kind comments and valuable suggestions.

References

[1] SHEN Z. Riemann-Finsler geometry with application to information geometry[J]. Chin Ann Math, 2006, 27(1): 73-94.

[2] CHENG X, SHEN Z, ZHOU Y. On locally dually flat Finsler metrics [J]. International Journal of Mathematics, 2010, 21(11): 1531-1543.

[3] XIA Q. On locally dually flat (α, β)-metrics[J]. Differential Geometry and Its Application, 2011, 29 (2): 233-243.

[4] ZOU Y, CHENG X. The generalized unicorn problem on (α,β)-metric[J]. J Math Anal Appl, 2014, 414: 574-589.

[5] CHERN S S, SHEN Z. Riemannian-Finsler Geometry [M]. Singapore: World Scientific Publisher, 2005.

[6] LI B, SHEN Z. On a class of weak Landsberg metrics [J]. Science in China Series A, 2007, 50: 75-85.

[7] CHENG X, LI H, ZOU Y. On conformally flat (α, β)-metrics with relatively isotropic mean Landsberg curvature [J]. Publ Math Debrecen, 2014, 85: 131-144.

[8] CHENG X, ZOU Y. The generalized unicorn problem in Finsler geometry[J]. Differ Geom Dyn syst, 2015, 17: 38-48.

[9] CHENG X, SHEN Z. Randers metrics with special curvature properties[J]. Osaka J Math, 2003, 40: 87-101.