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The threshold dividends in the Cramér-Lundberg risk model with loss-carry forward tax payments

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Abstract: A classical Cramér-Lundberg risk model with a threshold dividend strategy and loss-carry forward tax payments was studied. For this model, the closed-form expressions for the expected accumulated discounted dividends until ultimate ruin was derived and the explicit solution was presented when the individual claim amount follows an exponential distribution. Finally, numerical illustrations of the expected accumulated discounted dividends until ruin and the optimal threshold were given.

Key words: Cramér-Lundberg risk model; expected discounted dividends; threshold dividend strategy; loss-carry forward taxes

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带负载赋税的 Cramér-Lundberg 风险模型的门槛分红

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摘要:研究了一类经典 Cramér-Lundberg 风险模型,其在安全负载体系下进行赋税,且按门槛策略进行分红. 针对此模型,推导了破产前的期望折现总分红的表达式,并给出单独赔付额服从指数分布下的精确解. 最后给出破产时刻之前的期望折现总分红以及最优门槛的数值模拟结果.

关键词:Cramér-Lundberg 风险模型;期望折现分红;门槛分红策略;负载赋税

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0 Introduction

Due to their practical importance, the researches for dividend strategies under all kinds of risk models have been receiving remarkable attention. De Finetti[1] suggested that an insurance company would seek to maximize the expectation of the discounted dividends before possible ruin and showed that under the assumption of a discrete process the optimal dividend strategy is a barrier strategy. By the similar idea, Asmussen and Taksar[2] modified the problem with a bounded dividend rate and showed that in the Brownian motion model the optimal dividend strategy is a threshold strategy. Using the integro-differential equation argument, Gerber and Shiu[3] considered the Cramér-Lundberg risk model and studied the expected accumulated discounted dividends. More results on this topic can be found in Refs. [3-10].

Recently, the loss-carry forward tax system (the amount of tax payments should not lead to bunkruptsy) has been investigated extensively. Albrecher and Hipp[11] discussed how tax payments affect the behavior of a compound Poisson surplus process, and established a remarkably simple relationship between the ruin probabilities of the surplus process with and without tax payments. Albrecher et al.[21] considered a general spectrally negative Lévy risk process with tax payments of a loss-carry-forward type and studied arbitrary moments of the discounted total amount of tax payments. Wang et al.[19] considered a compound Poisson risk model with taxes paid according to a loss-carry-forward system and dividends paid under a threshold strategy, and provided the analytical expression of the expected accumulated discounted dividends paid between two consecutive taxation periods. Recently, Wang and Liu[20] also considered the compound Poisson risk model in which taxes were paid according to loss-carry-forward tax payments and dividends were paid by a threshold dividend strategy, and discussed the integro-differential equation of the expected discounted penalty function. For more recent results on loss-carry forward tax payments, the reader may consult Refs. [12-18].

In this paper, we follow our research work in 2016[20] and continue to consider the classical Cramér-Lundberg risk model by including a threshold dividend strategy and loss-carry forward tax payments. The basic assumptions of our model are as follows.

(Ⅰ) The initial surplus of the insurance portfolio is $u \geqslant 0$.

(Ⅱ) The loss-carry forward taxes are paid at a fixed rate $\gamma \in [0,1)$ of the insurer's income, whenever the insurance portfolio is in a profitable situation (or, the surplus is at a running maximum): $R_{\gamma,\alpha,b}(t) = \max\{R_{\gamma,\alpha,b}(s): s \leqslant t\}$.

(Ⅲ) When the surplus reaches a barrier of constant level b, dividends are distributed at a constant rate $\alpha < c(1-\gamma)$, where $c > 0$ is the premium rate in the classical Cramér-Lundberg risk model.

Hence, the dynamics of the surplus process $\{R_{\gamma,\alpha,b}(t), t \geqslant 0\}$ thus are determined by

$$\begin{cases} \mathrm{d}R_{\gamma,\alpha,b}(t) = (c - \alpha - c\gamma \mathbf{1}_{\{R_{\gamma,\alpha,b}(t) = \max\limits_{0\leqslant s\leqslant t} R_{\gamma,\alpha,b}(s)\}}) \mathbf{1}_{\{R_{\gamma,\alpha,b}(t) \geqslant b\}} \mathrm{d}t + \\ \qquad (c - c\gamma \mathbf{1}_{\{R_{\gamma,\alpha,b}(t) = \max\limits_{0\leqslant s\leqslant t} R_{\gamma,\alpha,b}(s)\}}) \mathbf{1}_{\{R_{\gamma,\alpha,b}(t) < b\}} \mathrm{d}t - \mathrm{d}(\sum_{n=1}^{N(t)} X_n), \\ R_{\gamma,\alpha,b}(0) = u \end{cases} \tag{1}$$

where $c > 0$ the gross premium rate, α the threshold dividend rate, $\mathbf{1}_A$ the indicator function of a set A, $\{N(t), t \geqslant 0\}$ a Poisson process with intensity $\lambda > 0$ denoting the number of claims up to time t, and $\{X_n, n \geqslant 1\}$, representing the amounts of claims and being independent of $\{N(t), t \geqslant 0\}$, a sequence of independent and identically distributed nonnegative random variables with a common distribution function $F(x)$ which has a positive mean $\mu = \int_0^{\infty} \overline{F}(x)\mathrm{d}x < \infty$. Here, $\overline{F}(x) = 1 - F(x)$ is the tail function of $F(x)$.

We denote the time of ruin by $T_{\gamma,\alpha,b}$, that is

$$T_{\gamma,\alpha,b} = \inf\{t : R_{\gamma,\alpha,b}(t) < 0\} \tag{2}$$

and $T_{\gamma,\alpha,b} = \infty$ if $R_{\gamma,\alpha,b}(t) \geqslant 0$ for all $t \geqslant 0$. Clearly, $R_{\gamma,\alpha,b}(T_{\gamma,\alpha,b}-)$ and $|R_{\gamma,\alpha,b}(T_{\gamma,\alpha,b})|$ are the surplus immediately prior to ruin and the deficit at ruin.

Let

$$m_{\gamma,\alpha,b}(u) = E_u\left[\int_0^{T_{\gamma,\alpha,b}} \mathrm{e}^{-\delta s}\mathrm{d}D(s)\right] \tag{3}$$

be the expected discounted total sum of dividend payments until the time of ruin $T_{\gamma,\alpha,b}$ under the condition that the initial surplus is u, where $D(t)$ denotes the aggregate dividends paid between time 0 and time t. $\delta \geqslant 0$ can be viewed as the force of interest for the calculation of the present value of the dividends.

For convenience, we write $(m_{\gamma,\alpha,b})_1(u)$ for $0 \leqslant u < b$ and $(m_{\gamma,\alpha,b})_2(u)$ for $u \geqslant b$. Throughout the paper we drop the subscripts γ and α whenever γ and α are zero, respectively, and drop the subscript b whenever b tends to infinity. In addition, we shall assume that the safety loading factor defined by $\theta = \dfrac{c - \alpha - \lambda\mu}{\lambda\mu}$ is always positive.

The rest of the paper is organized as follows. In Section 1, using some useful preliminaries and careful calculations, the closed-form expressions for the expected accumulated discounted dividends until ultimate ruin ($(m_{\gamma,\alpha,b})_1(u)$ and $(m_{\gamma,\alpha,b})_2(u)$) are derived and the explicit solution when the individual claim amount follows an exponential distribution is presented. In Section 2, numerical illustrations of the expected accumulated discounted dividends until ruin and the optimal threshold are given.

1 Closed-form expressions for $(m_{\gamma,\alpha,b})_1(u)$ and $(m_{\gamma,\alpha,b})_2(u)$

In this section, we derive the closed-form expressions for $(m_{\gamma,\alpha,b})_1(u)$ and $(m_{\gamma,\alpha,b})_2(u)$ over the lifetime of the surplus process $\{R_{\gamma,\alpha,b}(t), t \geqslant 0\}$. Before deriving Theorem 1.1, we restate the following results that were obtained in Ref. [19].

Firstly, let

$$B^{\alpha,b}(u, u_0) := E[\mathrm{e}^{-\delta\tau_{\alpha,b}(u,u_0)}] \tag{4}$$

denote the Laplace-Stieltjes transform of the upper exit time $\tau_{\alpha,b}(u, u_0)$ which is the time until the surplus process $\{R_{\alpha,b}(t), t \geqslant 0\}$ (with premium rate c, dividend rate α and threshold b) starting with initial surplus $u < u_0$ reaching $u_0 \geqslant b$ without leading to ruin before that event. Clearly, if we let $\delta \downarrow 0$ in $B^{\alpha,b}(u, u_0)$, it reduces to the probability that the surplus process $\{R_{\alpha,b}(t), t \geqslant 0\}$ starting from initial surplus $u < u_0$ reaching $u_0 \geqslant b$ before ruin, which is denoted by $(B^{\alpha,b})_0(u, u_0)$. We write $B^{\alpha,b}(u, u_0) = B_1^{\alpha,b}(u, u_0)$, $(B^{\alpha,b})_0(u, u_0) = (B_1^{\alpha,b})_0(u, u_0)$ for $0 \leqslant u < b$ and $B^{\alpha,b}(u, u_0) = B_2^{\alpha,b}(u, u_0)$, $(B^{\alpha,b})_0(u, u_0) = (B_2^{\alpha,b})_0(u, u_0)$ for $u \geqslant b$.

Now we provide integro-differential equations for the function $B^{\alpha,b}(u, u_0)$ in the following Proposition 1.1, which will help us to derive the closed-form expressions for $(m_{\gamma,\alpha,b})_1(u)$ and $(m_{\gamma,\alpha,b})_2(u)$.

Proposition 1.1[19] The function $B^{\alpha,b}(u, u_0)$ satisfies the following integro-differential equations. When $0 \leqslant u < b$,

$$\frac{\partial}{\partial u}B_1^{\alpha,b}(u, u_0) = \frac{\lambda + \delta}{c}B_1^{\alpha,b}(u, u_0) - \frac{\lambda}{c}\int_0^u B_1^{\alpha,b}(u - x, u_0)\mathrm{d}F(x) \tag{5}$$

and when $u \geqslant b$,

$$\frac{\partial}{\partial u}B_2^{\alpha,b}(u, u_0) = \frac{\lambda + \delta}{c - \alpha}B_2^{\alpha,b}(u, u_0) - \frac{\lambda}{c - \alpha}\Big(\int_0^{u-b} B_2^{\alpha,b}(u - x, u_0)\mathrm{d}F(x) + \int_{u-b}^{u} B_1^{\alpha,b}(u - x, u_0)\mathrm{d}F(x)\Big) \tag{6}$$

Secondly, let

$$B_\gamma(u,u_0):=E[\mathrm{e}^{-\delta\tau_\gamma(u,u_0)}] \tag{7}$$

denote the Laplace-Stieltjes transform of the upper exit time $\tau_\gamma(u,u_0)$ which is the time until the surplus process $\{R_\gamma(t),t\geqslant 0\}$ (with premium rate c and tax rate γ) starting with initial surplus $u<u_0$ reaching u_0 without leading to ruin before that event.

Before presenting the main theorem, we now introduce the analytical expression of $B_\gamma(u,u_0)$ in the following Proposition 1.2, which plays an instrumental role in analyzing the expected accumulated discounted dividends until ultimate ruin, and its proof is refereed to Ref. [20].

Proposition 1.2 The resulting Laplace-Stieltjes transform of the upper exit time $\tau_\gamma(u,u_0)$ is a power of that of the upper exit time $\tau(u,u_0)$, that is

$$B_\gamma(u,u_0)=(B(u,u_0))^{\frac{1}{1-\gamma}}=\left(\frac{h(u)}{h(u_0)}\right)^{\frac{1}{1-\gamma}} \tag{8}$$

where $h(u)$ is the solution to the integro-differential equation

$$ch'(x)-(\lambda+\delta)h(x)+\lambda\int_0^x h(x-y)\mathrm{d}F(y)=0 \tag{9}$$

Using the propositions given above, we now derive the closed-form expressions for $(m_{\gamma,\alpha,b})_1(u)$ and $(m_{\gamma,\alpha,b})_2(u)$ over the lifetime of the surplus process $\{R_{\gamma,\alpha,b}(t),t\geqslant 0\}$ in the following Theorem 1.1.

Theorem 1.1 When $0\leqslant u<b$,

$$(m_{\gamma,\alpha,b})_1(u)=B_\gamma(u,b)(m_{\gamma,\alpha,b})_2(b) \tag{10}$$

and when $u\geqslant b$,

$$(m_{\gamma,\alpha,b})_2(u)=\frac{c-\alpha}{c(1-\gamma)-\alpha}(m_{\alpha,b})_2(u)-\frac{c\gamma}{c(1-\gamma)-\alpha}\exp\left\{\int_b^u M(t)\mathrm{d}t\right\}\int_u^\infty M(s)\,(m_{\alpha,b})_2(s)\exp\left\{-\int_b^s M(t)\mathrm{d}t\right\}\mathrm{d}s \tag{11}$$

where $M(t)$ is given by

$$M(t)=\frac{1}{c(1-\gamma)-\alpha}\left\{\lambda+\delta-\lambda\left(\int_0^{t-b}B_2^{\alpha,b}(t-x,t)\mathrm{d}F(x)+\int_{t-b}^t B_1^{\alpha,b}(t-x,t)\mathrm{d}F(x)\right)\right\} \tag{12}$$

Proof When $0\leqslant u<b$, no dividends will be paid unless the process $\{R_{\gamma,\alpha,b}(t),t\geqslant 0\}$ reaches the level b, and the trajectories of the process $\{R_{\gamma,\alpha,b}(t),t\geqslant 0\}$ are identical to those of the process $\{R_\gamma(t),t\geqslant 0\}$ before they arrive at b, and implementing these considerations leads to Eq. (10).

When $u\geqslant b$, by considering whether or not there is a claim during the infinitesimal time interval from 0 to $\mathrm{d}t$ and using the similar conditioning idea of Ref. [15], we have

$$\begin{aligned}(m_{\gamma,\alpha,b})_2(u)=&\alpha\mathrm{d}t+(1-\lambda\mathrm{d}t)\mathrm{e}^{-\delta\mathrm{d}t}(m_{\gamma,\alpha,b})_2(u+(c(1-\gamma)-\alpha)\mathrm{d}t)+\lambda\mathrm{d}t\,\mathrm{e}^{-\delta\mathrm{d}t}\left\{\int_0^{u+(c(1-\gamma)-\alpha)\mathrm{d}t-b}(B_2^{\alpha,b}(u+\right.\\&(c(1-\gamma)-\alpha)\mathrm{d}t-x,u+(c(1-\gamma)-\alpha)\mathrm{d}t)(m_{\gamma,\alpha,b})_2(u+(c(1-\gamma)-\alpha)\mathrm{d}t)+\\&(m_{\alpha,b})_2(u+(c(1-\gamma)-\alpha)\mathrm{d}t-x)-B_2^{\alpha,b}(u+(c(1-\gamma)-\alpha)\mathrm{d}t-x,\\&u+(c(1-\gamma)-\alpha)\mathrm{d}t)(m_{\alpha,b})_2(u+(c(1-\gamma)-\alpha)\mathrm{d}t))\mathrm{d}F(x)+\\&\int_{u+(c(1-\gamma)-\alpha)\mathrm{d}t-b}^{u+(c(1-\gamma)-\alpha)\mathrm{d}t}(B_1^{\alpha,b}(u+(c(1-\gamma)-\alpha)\mathrm{d}t-x,u+(c(1-\gamma)-\alpha)\mathrm{d}t)(m_{\gamma,\alpha,b})_2(u+(c(1-\gamma)-\alpha)\mathrm{d}t)+\\&(m_{\alpha,b})_1(u+(c(1-\gamma)-\alpha)\mathrm{d}t-x)-B_1^{\alpha,b}(u+(c(1-\gamma)-\alpha)\mathrm{d}t-x,\\&\left.u+(c(1-\gamma)-\alpha)\mathrm{d}t)(m_{\alpha,b})_2(u+(c(1-\gamma)-\alpha)\mathrm{d}t))\mathrm{d}F(x)\right\}+o(\mathrm{d}t).\end{aligned}$$

Taylor expansion and collection of terms of order $\mathrm{d}t$ yields

$$\begin{aligned}(m_{\gamma,\alpha,b})_2'(u)=&\frac{\lambda+\delta}{(c(1-\gamma)-\alpha)}\left((m_{\gamma,\alpha,b})_2(u)-\frac{\alpha}{\lambda+\delta}\right)-\frac{\lambda}{(c(1-\gamma)-\alpha)}\cdot\\&\left\{\int_0^{u-b}\left(B_2^{\alpha,b}(u-x,u)(m_{\gamma,\alpha,b})_2(u)+(m_{\alpha,b})_2(u-x)-B_2^{\alpha,b}(u-x,u)(m_{\alpha,b})_2(u)\right)\mathrm{d}F(x)+\right.\end{aligned}$$

$$\int_{u-b}^{u}(B_1^{\alpha,b}(u-x,u)(m_{\gamma,\alpha,b})_2(u)+(m_{\alpha,b})_1(u-x)-B_1^{\alpha,b}(u-x,u)(m_{\alpha,b})_2(u))\,\mathrm{d}F(x)\Big\}.$$

From the integro-differential equation for $(m_{\alpha,b})_2(u)$ in Ref. [3], that is,

$$0=\alpha+(c-\alpha)(m_{\alpha,b})_2'(u)-(\lambda+\delta)(m_{\alpha,b})_2(u)+$$
$$\lambda\Big\{\int_0^{u-b}(m_{\alpha,b})_2(u-x)\mathrm{d}F(x)+\int_{u-b}^{u}(m_{\alpha,b})_1(u-x)\mathrm{d}F(x)\Big\},$$

then we obtain

$$(m_{\gamma,\alpha,b})_2'(u)=M(u)(m_{\gamma,\alpha,b})_2(u)-M(u)(m_{\alpha,b})_2(u)+\frac{c-\alpha}{c(1-\gamma)-\alpha}(m_{\alpha,b})_2'(u) \tag{13}$$

The general solution to this ordinary differential equation of first order is given by

$$(m_{\gamma,\alpha,b})_2(u)=\Big(C-\int_b^u\Big(M(s)(m_{\alpha,b})_2(s)-\frac{c-\alpha}{c(1-\gamma)-\alpha}(m_{\alpha,b})_2'(s)\Big)\cdot\exp\Big\{-\int_b^s M(t)\mathrm{d}t\Big\}\,\mathrm{d}s\Big)\exp\Big\{\int_b^u M(t)\mathrm{d}t\Big\} \tag{14}$$

Noting that $M(t)\geqslant\frac{\delta}{(c(1-\gamma)-\alpha)}$ (>0) and the boundary condition at threshold b, one can verify that

$$C=\int_b^\infty\Big(M(s)(m_{\alpha,b})_2(s)-\frac{c-\alpha}{c(1-\gamma)-\alpha}(m_{\alpha,b})_2'(s)\Big)\exp\Big\{-\int_b^s M(t)\mathrm{d}t\Big\}\,\mathrm{d}s.$$

Therefore, we have

$$(m_{\gamma,\alpha,b})_2(u)=\exp\Big\{\int_b^u\Big(M(t)\mathrm{d}t\Big\}\int_u^\infty\Big(M(s)(m_{\alpha,b})_2(s)-\frac{c-\alpha}{c(1-\gamma)-\alpha}(m_{\alpha,b})_2'(s)\Big)\exp\Big\{-\int_b^s M(t)\mathrm{d}t\Big\}\,\mathrm{d}s.$$

In addition, noting that

$$\frac{c-\alpha}{c(1-\gamma)-\alpha}(m_{\alpha,b})_2(u)-\frac{c-\alpha}{c(1-\gamma)-\alpha}\exp\Big\{\int_b^u M(t)\mathrm{d}t\Big\}\int_u^\infty M(s)(m_{\alpha,b})_2(s)\exp\Big\{-\int_b^s M(t)\mathrm{d}t\Big\}\,\mathrm{d}s=$$
$$-\frac{c-\alpha}{c(1-\gamma)-\alpha}\exp\Big\{\int_b^u M(t)\mathrm{d}t\Big\}\int_u^\infty(m_{\alpha,b})_2'(s)\exp\Big\{-\int_b^s M(t)\mathrm{d}t\Big\}\,\mathrm{d}s \tag{15}$$

we arrive at Eq. (11). The proof of Theorem 1. 1 is completed.

Example 1. 1　Assume that the individual claim amount is exponentially distributed with parameter $\beta>0$. We are to calculate the closed-form expressions for $(m_{\gamma,\alpha,b})_1(u)$ and $(m_{\gamma,\alpha,b})_2(u)$.

It follows from (6. 15) of Ref. [3] that

$$(m_{\alpha,b})_2(u)=\frac{\alpha}{\delta}(1-\mathrm{e}^{R_2(u-b)})-\frac{\alpha R_2}{\beta\delta}\frac{(\beta+\rho)\mathrm{e}^{\rho b}-(\beta+r_2)\mathrm{e}^{r_2 b}}{(\rho-R_2)\mathrm{e}^{\rho b}-(r_2-R_2)\mathrm{e}^{r_2 b}}\mathrm{e}^{R_2(u-b)},\ u\geqslant b \tag{16}$$

From Eq. (11) we get

$$(m_{\gamma,\alpha,b})_2(u)=\frac{\alpha}{\delta}(1-\mathrm{e}^{R_2(u-b)})-\frac{\alpha R_2}{\beta\delta}\frac{(\beta+\rho)\mathrm{e}^{\rho b}-(\beta+r_2)\mathrm{e}^{r_2 b}}{(\rho-R_2)\mathrm{e}^{\rho b}-(r_2-R_2)\mathrm{e}^{r_2 b}}\mathrm{e}^{R_2(u-b)}+$$
$$\frac{c\gamma}{c(1-\gamma)-\alpha}\Big(\mathrm{e}^{R_1 b}\mathrm{e}^{(R_2-R_1)u}+q_1(b)\Big)^{\frac{c-\alpha}{c(1-\gamma)-\alpha}}\mathrm{e}^{R_2(u-b)}\Big(\frac{\alpha R_2}{\delta}+\frac{\alpha {R_2}^2}{\beta\delta}\frac{(\beta+\rho)\mathrm{e}^{\rho b}-(\beta+r_2)\mathrm{e}^{r_2 b}}{(\rho-R_2)\mathrm{e}^{\rho b}-(r_2-R_2)\mathrm{e}^{r_2 b}}\Big)\cdot$$
$$\frac{(q_1(b))^{-\frac{c-\alpha}{c(1-\gamma)-\alpha}}}{\frac{c-\alpha}{c(1-\gamma)-\alpha}R_1-R_2}\,{}_2F_1\Big(\frac{c-\alpha}{c(1-\gamma)-\alpha},\frac{\frac{c-\alpha}{c(1-\gamma)-\alpha}R_1-R_2}{R_1-R_2};\frac{\frac{c-\alpha}{c(1-\gamma)-\alpha}R_1-R_2}{R_1-R_2}+1;-\frac{\mathrm{e}^{R_1 b}}{q_1(b)}\mathrm{e}^{(R_2-R_1)u}\Big) \tag{17}$$

where ${}_2F_1(p,q;l;z)$ is the Gauss hypergeometric series, that is,

$${}_2F_1(p,q;l;z)=\frac{\Gamma(l)}{\Gamma(q)\Gamma(l-q)}\int_0^1 s^{q-1}(1-s)^{l-q-1}(1-sz)^{-p}\mathrm{d}s.$$

When $0\leqslant u<b$, by Proposition 1. 2 and Eq. (10) we have

$$(m_{\gamma,\alpha,b})_1(u)=\left(\frac{(\beta+\rho)\mathrm{e}^{\rho u}-(\beta+r_2)\mathrm{e}^{r_2 u}}{(\beta+\rho)\mathrm{e}^{\rho b}-(\beta+r_2)\mathrm{e}^{r_2 b}}\right)^{\frac{1}{1-\gamma}}\Big[-\frac{\alpha R_2}{\beta\delta}\frac{(\beta+\rho)\mathrm{e}^{\rho b}-(\beta+r_2)\mathrm{e}^{r_2 b}}{(\rho-R_2)\mathrm{e}^{\rho b}-(r_2-R_2)\mathrm{e}^{r_2 b}}+$$
$$\frac{c\gamma}{c(1-\gamma)-\alpha}\left(\mathrm{e}^{r_2 b}+q_1(b)\right)^{\frac{c-\alpha}{c(1-\gamma)-\alpha}}\left(\frac{\alpha R_2}{\delta}+\frac{\alpha R_2^{\,2}}{\beta\delta}\frac{(\beta+\rho)\mathrm{e}^{\rho b}-(\beta+r_2)\mathrm{e}^{r_2 b}}{(\rho-R_2)\mathrm{e}^{\rho b}-(r_2-R_2)\mathrm{e}^{r_2 b}}\right)\cdot$$
$$\frac{(q_1(b))^{-\frac{c-\alpha}{c(1-\gamma)-\alpha}}}{\frac{c-\alpha}{c(1-\gamma)-\alpha}R_1-R_2}\,{}_2F_1\left(\frac{c-\alpha}{c(1-\gamma)-\alpha},\frac{\frac{c-\alpha}{c(1-\gamma)-\alpha}R_1-R_2}{R_1-R_2};\frac{\frac{c-\alpha}{c(1-\gamma)-\alpha}R_1-R_2}{R_1-R_2}+1;-\frac{\mathrm{e}^{r_2 b}}{q_1(b)}\right)\Big] \tag{18}$$

2 Numerical illustrations and the optimal threshold

In this section, we give numerical illustrations of the expected accumulated discounted dividends until ruin and the optimal threshold. We consider the case of an exponential claim size distribution with parameter $\beta=2$ and choose $c=2$, $\lambda=1$, $\delta=0.04$, $\alpha=1$. Let furthermore $\gamma=0.5$. Hence we have $\rho=0.0386$, $r_2=-0.5186$, $R_1=0.1277$, $R_2=-0.2610$, and $\frac{c-\alpha}{c(1-\gamma)-\alpha}=2$. We then obtain from Eqs. (17) and (18) that

$$(m_{\gamma,\alpha,b})_1(u)=20(1-\mathrm{e}^{-0.2610(u-b)})+5.22p(b)+1.9365(\mathrm{e}^{-0.3887u}\mathrm{e}^{0.1277b}+q_1(b))^2\cdot$$
$$\mathrm{e}^{-0.2610(u-b)}(-5.22+1.3624p(b))(q_1(b))^{-2}\,{}_2F_1\left(2,1.3285;2.3285;\frac{-\mathrm{e}^{-0.2610b}}{q_1(b)}\right) \tag{19}$$

and

$$(m_{\gamma,\alpha,b})_2(u)=\left(\frac{1.0386\mathrm{e}^{0.0386u}-0.4814\mathrm{e}^{-0.5186u}}{1.0386\mathrm{e}^{0.0386b}-0.4814\mathrm{e}^{-0.5186b}}\right)^{1.4286}\cdot\Big[5.22p(b)+1.9365(\mathrm{e}^{-0.2610b}+q_1(b))^2\cdot$$
$$(-5.22+1.3624p(b))(q_1(b))^{-2}\,{}_2F_1\left(2,1.3285;2.3285;\frac{-\mathrm{e}^{-0.2610b}}{q_1(b)}\right)\Big] \tag{20}$$

where

$$q_1(b)=-\mathrm{e}^{-0.2610b}+\frac{0.3887(1.0386\mathrm{e}^{-0.2224b}-0.4814\mathrm{e}^{-0.7796b})}{0.0658\mathrm{e}^{0.0386b}-0.4776\mathrm{e}^{-0.5186b}},\ p(b)=\frac{1.0386\mathrm{e}^{0.0386b}-0.4814\mathrm{e}^{-0.5186b}}{0.2996\mathrm{e}^{0.0386b}+0.2576\mathrm{e}^{-0.5186b}}.$$

We are interested in the optimal threshold level b^* that maximizes the expectation of the discounted dividends (until possible ruin). When the initial surplus u is given, one can take $(m_{\gamma,\alpha,b})_1(u)$ and $(m_{\gamma,\alpha,b})_2(u)$ for functions of b. Noting that $(m_{\gamma,\alpha,b})_1(u)$ and $(m_{\gamma,\alpha,b})_2(u)$ are both continuous functions of b, and since the function $m_{\gamma,\alpha,b}(u)$ tends to 0 when b tends to infinity, then the trivial bounds $0\leqslant m_{\gamma,\alpha,b}(u)\leqslant\int_0^\infty \mathrm{e}^{-\delta t}\alpha\,\mathrm{d}t$ guarantee the existence of the optimal threshold level b^*. When $u=0,1,4,5,6$ and 8, the data for $(m_{\gamma,\alpha,b})_1(u)$ and $(m_{\gamma,\alpha,b})_2(u)$ (as functions of b) are given in Tab. 1. Fig. 1 depicts the expected accumulated discounted dividend payments as a function of the threshold level b.

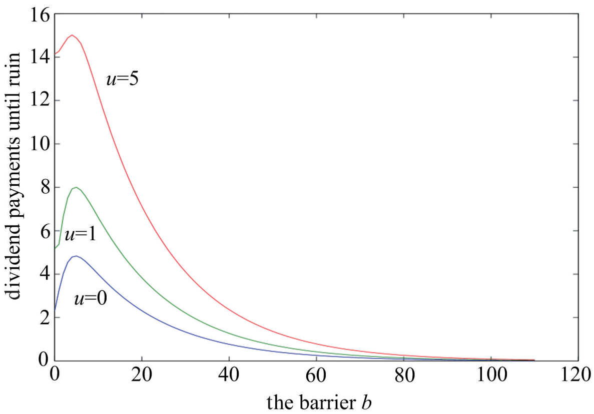

Fig. 1 The expected accumulated discounted dividend payments as a function of b

It is interesting to note that when $u<4.8$, the optimal threshold b^* remains at the level 4.8. Meanwhile, when $u>4.8$ the optimal threshold b^* increases as the initial surplus u increases.

Tab. 1　The data for $m_{\gamma,\alpha,b}(u)$ (as functions of b) for six different initial surplus level: $u=0,1,4,5,6$ and 8

b	$u=0$	$u=1$	$u=4$	$u=5$	$u=6$	$u=8$	b	$u=0$	$u=1$	$u=4$	$u=5$	$u=6$	$u=8$
0.0	2.3059	5.1696	12.4842	14.1313	15.4383	17.2658	5.2	4.8272	7.9898	13.6165	14.8519	16.0424	17.6807
0.6	2.8791	5.2518	12.5489	14.1840	15.4803	17.2917	5.8	4.7773	7.9074	13.4760	14.6986	15.8297	17.5711
1.2	3.4225	5.6649	12.7184	14.3228	15.5915	17.3605	6.4	4.6984	7.7767	13.2534	14.4557	15.5503	17.3977
1.8	3.8944	6.4459	12.9465	14.5114	15.7434	17.4551	7.0	4.5997	7.6133	12.9749	14.1520	15.2235	17.1524
2.4	4.2702	7.0680	13.1827	14.7097	15.9047	17.5565	7.6	4.488	17.4286	12.6601	13.8087	14.8543	16.8253
3.0	4.5430	7.5195	13.3793	14.8802	16.0459	17.6469	8.2	4.3688	7.2311	12.3234	13.4414	14.4592	16.4037
3.6	4.7190	7.8107	13.4948	14.9908	16.1424	17.7117	8.8	4.2452	7.0266	11.9749	13.0613	14.0503	15.9398
4.0	4.7892	7.9269	13.5093	15.0181	16.1719	17.7349	9.2	4.1618	6.8886	11.7397	12.8048	13.7743	15.6267
4.4	4.8273	7.9901	13.6170	15.0006	16.1678	17.7390	10.0	3.9950	6.6124	11.2691	12.2915	13.2221	15.0002
4.8	4.8385	8.0085	13.6484	14.9321	16.1260	17.7216	80.0	0.0846	0.140	10.2388	0.2604	0.2802	0.3178
5.0	4.8353	8.0033	13.6396	14.8770	16.0896	17.7042	b^*	4.8	4.8	4.8	4.0	4.2	4.3

Tab. 2　The data for $m_{\gamma,\alpha,b}(u)$ (as function of u) for three different threshold level: $b=10,15,20$

	$u=0$	$u=1$	$u=2$	$u=3$	$u=4$	$u=5$	$u=6$	$u=7$	$u=8$	$u=9$	$u=10$
$b=10$	3.9950	6.6124	8.5782	10.0728	11.2691	12.2915	13.2221	14.1139	15.0002	15.9029	16.8361
$b=15$	3.0486	5.0460	6.5461	7.6866	8.5995	9.3797	10.0899	10.7704	11.4468	12.1356	12.8478
$b=20$	2.3148	3.8313	4.9704	5.8363	6.5295	7.1218	7.6611	8.1778	8.6913	9.2144	9.7551

	$u=11$	$u=12$	$u=13$	$u=14$	$u=15$	$u=16$	$u=17$	$u=18$	$u=19$	$u=20$	$u=21$
$b=10$	17.6023	18.1738	18.6043	18.9306	19.1793	19.3694	19.5151	19.6269	19.7128	19.7789	19.8298
$b=15$	13.5908	14.3701	15.1901	16.0545	16.9667	17.7023	18.2505	18.6632	18.9759	19.2141	19.3962
$b=20$	10.3192	10.9110	11.5336	12.1899	12.8825	13.6137	14.3860	15.2019	16.0639	16.9748	17.7085

	$u=22$	$u=23$	$u=24$	$u=25$	$u=26$	$u=27$	$u=28$	$u=29$	$u=30$	$u=31$	$u=32$
$b=10$	19.8689	19.8990	19.9222	19.9401	19.9539	19.9645	19.9726	19.9789	19.9838	19.9875	19.9904
$b=15$	19.5357	19.6428	19.7251	19.7883	19.8370	19.8745	19.9033	19.9256	19.9427	19.9558	19.9660
$b=20$	18.2553	18.6668	18.9787	19.2163	19.3979	19.5370	19.6438	19.7258	19.7889	19.8375	19.8748

	$u=33$	$u=34$	$u=35$	$u=36$	$u=37$	$u=38$	$u=39$	$u=40$	$u=41$	$u=42$	$u=43$
$b=10$	19.9926	19.9943	19.9956	19.9966	19.9974	19.9980	19.9984	19.9988	19.9991	19.9993	19.9995
$b=15$	19.9738	19.9798	19.9845	19.9880	19.9908	19.9929	19.9945	19.9958	19.9968	19.9975	19.9981
$b=20$	19.9036	19.9258	19.9428	19.9560	19.9661	19.9739	19.9799	19.9845	19.9881	19.9908	19.9929

	$u=44$	$u=45$	$u=46$	$u=47$	$u=48$	$u=49$	$u=50$	$u=51$	$u=52$	$u=53$	$u=54$
$b=10$	19.9996	19.9997	19.9998	19.9998	19.9999	19.9999	19.9999	19.9999	19.9999	20.0000	20.0000
$b=15$	19.9985	19.9989	19.9991	19.9993	19.9995	19.9996	19.9997	19.9998	19.9998	19.9999	19.9999
$b=20$	19.9945	19.9958	19.9968	19.9975	19.9981	19.9985	19.9989	19.9991	19.9993	19.9995	19.9996

In Tab. 2, for the given threshold level $b=10, 15, 20$, we give numerical illustrations of $m_{\gamma,\alpha,b}(u)$ (as the function of the initial surplus u). It's easy to see that $m_{\gamma,\alpha,b}(u)$ is an increasing function (for fixed b) of u, this is because if the initial surplus u increases, the insurer seems easier to arrive at the threshold level and receive dividends. Hence, larger u leads to larger value of $m_{\gamma,\alpha,b}(u)$. The trends are also shown in Fig. 2, which depicts that the expected accumulated discounted dividend payments are increasing as a function of u and the trivial bound $0 \leqslant m_{\gamma,\alpha,b}(u) \leqslant 20$ holds.

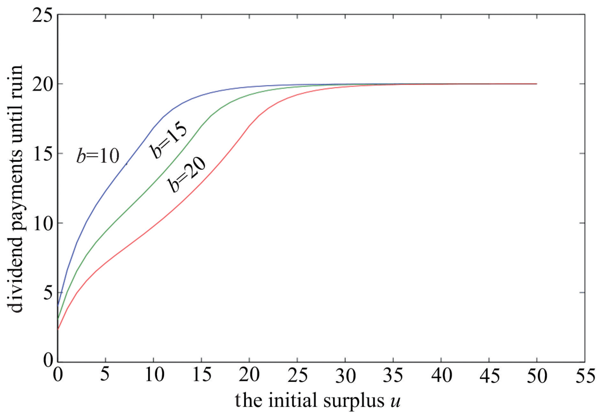

Fig. 2 The expected accumulated discounted dividend payments as a function of u

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