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A note on Frobenius functors

ZHAO Zhibing

(*School of Mathematical Sciences, Anhui University, Hefei* 230601, *China*)

Abstract: Some new characterizations of Frobenius bimodules in terms of Frobenius functors were given. It was proved that a bimodule is Frobenius if and only if it is finitely generated projective on both sides, and that the restriction of the corresponding tensor functor to the categories of finitely generated projective modules is a Frobenius functor. The characterizations allow us to give a new proof of the endomorphism ring theorem by a functorial method.

Key words: Frobeuius functors; Frobeuius extensions; the endomorphism ring theorem

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关于 Frobenius 函子的一个注记

赵志兵

(安徽大学数学科学学院,安徽合肥 230601)

摘要:利用 Frobenius 函子来刻画 Frobenius 双模.证明了一个双模是 Frobenius 的当且仅当它作为左模和右模均是有限生成投射的,并且所对应的函子限制到有限生成投射模类上是一个 Frobenius 函子.利用这种刻画,得到了关于经典的自同态环定理的一种新的利用函子方法的证明.

关键词:Frobenius 函子;Frobenius 扩张;自同态环定理

0 Introduction

The theory of Frobenius extensions of rings was first studied in Ref.[1]. One of the main results in the theory is the endomorphism ring theorem. Frobenius bimodules are natural generalizations of Frobenius extensions of rings. It is well-known that a Frobenius bimodule induces a Frobenius functor between module categories. Here, we recall that a functor is a Frobenius functor if it has a left adjoint that is also a right adjoint[2-3]. In general, Frobenius functors can be studied as well between other than pure module categories. And we observe that Frobenius functors appear in many contexts; see Refs.[3-4].

In this note, we obtain some new

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Biography: ZHAO Zhibing, male, born in 1979, PhD/lecturer. Research field: Homological algebra, representation theory of algebras. E-mail: zbzhao@ahu.edu.cn

characterizations of Frobenius bimodules in terms of Frobenius functor. In particular, we prove that a bimodule is Frobenius if and only if it is finitely generated projective on both sides, and the restriction of the corresponding tensor functor to the subcategories of finitely generated projective modules is a Frobenius functor. As an application, we give a new proof of the endomorphism ring theorem by a functorial method.

1 Frobeuius functors

In this section, we collect some lemmas on Frobenius functors. The main references are Refs. [3-4].

Definition 1.1 Let $F:\mathscr{C}\to\mathscr{D}$ be a functor. If there exists a functor $G:\mathscr{D}\to\mathscr{C}$ such that both (F,G) and (G,F) are adjoint pairs, then we call F a Frobenius functor, and we say that (F,G) is a Frobenius pair on $\mathscr{C}$ and $\mathscr{D}$.

We have the following well-known lemma.

Lemma 1.1 Let (F,G) be a Frobeuius pair on $\mathscr{C}$ and $\mathscr{D}$. Assume that $\mathscr{C}'\subseteq\mathscr{C}$ and $\mathscr{D}'\subseteq\mathscr{D}$ are full subcategories satisfying $F(\mathscr{C}')\subseteq\mathscr{D}'$ and $G(\mathscr{D}')\subseteq\mathscr{C}'$. Then the restrictions $F|_{\mathscr{C}'}:\mathscr{C}'\to\mathscr{D}'$ and $G|_{\mathscr{D}'}:\mathscr{D}'\to\mathscr{C}'$ form a Frobenius pair.

Lemma 1.2 Let $F:\mathscr{C}\to\mathscr{D}$ be a functor and $i:\mathscr{D}\to\mathscr{D}'$ a fully faithful functor. If $i\circ F$ is a Frobenius functor, so is F.

Proof Assume that $(i\circ F,G)$ is a Frobenius pair. We claim that $(F,G\circ i)$ is also a Frobenius pair. For any object $X\in\mathscr{C}$ and $Y\in\mathscr{D}$, we have natural isomorphisms

$$\mathrm{Hom}_{\mathscr{D}}(F(X),Y)\cong$$
$$\mathrm{Hom}_{\mathscr{D}'}(i\circ F(X),i(Y))\cong\mathrm{Hom}_{\mathscr{C}}(X,G\circ i(Y)),$$

where the left isomorphism uses the fully-faithfulness of i and the right uses the adjoint pair $(i\circ F,G)$. Similarly, we have

$$\mathrm{Hom}_{\mathscr{C}}(G\circ i(Y),X)\cong$$
$$\mathrm{Hom}_{\mathscr{D}'}(i(Y),i\circ F(X))\cong\mathrm{Hom}_{\mathscr{D}}(Y,F(X)).$$

The following result can be deduced from Ref. [4,1.3(5)].

Corollary 1.1 Let $F:\mathscr{C}\to\mathscr{D}$ be a functor and $\varphi:\mathscr{D}\to\mathscr{D}'$ an equivalence of categories. If F is a Frobenius functor, so is $\varphi\circ F$.

Proof Take a quasi-inverse φ^{-1} of φ. We observe that F is isomorphic to $\varphi^{-1}\circ(\varphi\circ F)$. Then we are done by Lemma 1.2.

Let R be a unital ring. We denote by Mod-R the category of right R-modules, and by proj-R the full subcategory consisting of finitely generated projective modules. We use $\mathrm{Hom}_R(-,-)$ to denote the Hom set in Mod-R.

Lemma 1.3 Let $F:$ Mod-$R\to$ Mod-S be a Frobenius functor. Then

① F is exact and preserves direct sums and direct products.

② $F(\text{proj-}R)\subseteq\text{proj-}S$.

Proof ① By Ref.[4,Proposition 1.3].

② Suppose that (F,G) is a Frobenius pair. It follows from ① that G is exact and preserves direct sums. We observe the following natural isomorphisms:

$$\mathrm{Hom}_S(F(R_R),-)\cong\mathrm{Hom}_R(R_R,G(-))\cong G.$$

Hence $\mathrm{Hom}_S(F(R_R),-)$ is exact and preserves direct sums. It follows that $F(R_R)$ is finitely generated projective as a right S-module. Then we have $F(\text{proj-}R)\subseteq\text{proj-}S$.

2 Frobenius bimodules

In this section, we give two characterizations of Frobenius bimodules.

Let R and S be two unital rings. We identify left R-modules with right R^{op}-modules. Recall from Ref.[5, Section 2.2] that an R-S-bimodule ${}_RP_S$ is called a Frobenius bimodule if both ${}_RP$ and P_S are finitely generated projective, and

$$\mathrm{Hom}_S({}_RP,{}_SS)\cong\mathrm{Hom}_{R^{\mathrm{op}}}(P_S,R_R)$$

as S-R-bimodules.

The implication ①→② of the following result is implicitly contained in Ref.[5,Proposition 2.4].

Theorem 2.1 Let ${}_RP_S$ be an R-S-bimodule. Then the following statements are equivalent.

① ${}_RP_S$ is a Frobenius bimodule.

② The tensor functor $-\otimes_RP:$ Mod-$R\to$ Mod-S is a Frobenius functor.

③ Both ${}_RP$ and P_S are finitely generated projective, and the restricted tensor functor $-\otimes_R P$: proj-R → proj-S is a Frobenius functor.

Proof ①⇒② We have the following natural isomorphisms:

$$-\otimes_R P \cong \mathrm{Hom}_R({}_R R, -)\otimes_R P \cong \mathrm{Hom}_R(\mathrm{Hom}_{R^{op}}(P_S, R_R), -) \cong \mathrm{Hom}_R(\mathrm{Hom}_S({}_R P, {}_S S), -),$$

where the second isomorphism follows from Ref. [6, Proposition 20.11] and the third uses the assumption. Then $-\otimes_S \mathrm{Hom}_S({}_R P, {}_S S)$ is the left adjoint to $-\otimes_R P$. On the other hand, $-\otimes_R P$ has the right adjoint $\mathrm{Hom}_S({}_R P, -)$. Then we are done by the following isomorphisms:

$$-\otimes_S \mathrm{Hom}_S({}_R P, {}_S S) \cong \mathrm{Hom}_S({}_R P, -\otimes_S S) \cong \mathrm{Hom}_S({}_R P, -),$$

where the left isomorphism follows from Ref.[6, Proposition 20.10].

②⇒③ By Lemma 1.3①, $-\otimes_R P$ is exact and commutes with direct products. Then ${}_R P$ is flat and finitely presented by Ref.[7, Section 12.9], that is, ${}_R P$ is finitely generated projective. By Lemma 1.3②, $-\otimes_R P$ sends R_R into proj-S. Hence P_S is finitely generated projective.

By ②, we have the Frobenius pair $(-\otimes_R P, \mathrm{Hom}_S({}_R P, -))$ on Mod-R and Mod-S. By Lemma 1.3②, $\mathrm{Hom}_S({}_R P, -)$ sends S_S into proj-R. Then the restricted functors $-\otimes_R P$ and $\mathrm{Hom}_S(P, -)$ form a Frobenius pair on proj-R and proj-S by Lemma 1.1.

③⇒① We need only to show that

$$\mathrm{Hom}_S({}_R P, {}_S S) \cong \mathrm{Hom}_{R^{op}}(P_S, R_R)$$

as S-R-bimodules. Assume that $(-\otimes_R P, G)$ is a Frobenius pair on proj-R and proj-S. Then

$$\mathrm{Hom}_S({}_R P, -) \cong \mathrm{Hom}_S(R\otimes_R P, -) \cong \mathrm{Hom}_R(R, G(-)) \cong G(-).$$

Hence we get $G(S_S) \cong \mathrm{Hom}_S({}_R P, S)$.

On the other hand, we have the following natural isomorphisms:

$${}_R P_S \cong \mathrm{Hom}_S(S, R\otimes_R P) \cong \mathrm{Hom}_R(G(S), R).$$

Therefore

$$\mathrm{Hom}_{R^{op}}(P_S, R_R) \cong \mathrm{Hom}_{R^{op}}(\mathrm{Hom}_R(G(S_S), R), R) \cong G(S_S) \cong \mathrm{Hom}_S({}_R P, {}_S S),$$

where the second isomorphism follows from the reflexivity of $G(S_S)$.

3 The endomorphism ring theorem

A ring extension R/S is a Frobenius extension if and only if R_S is finitely generated projective as a right S-module and ${}_S R_R \cong \mathrm{Hom}_S({}_R R, {}_S S)$ as S-R-bimodules. Equivalently, ${}_S R$ is finitely generated projective as a left S-module and ${}_R R_S \cong \mathrm{Hom}_{S^{op}}(R_R, S_S)$ as R-S-bimodules (see Ref.[5, Theorem 1.2]).

The following lemma improves slightly Refs. [5, Theorem 1.2] and [3, Theorem 28].

Lemma 3.1 Let R/S be a ring extension. The following statements are equivalent.

① R/S is a Frobenius extension.

② The functor $-\otimes_R R_S$: Mod-R → Mod-S is a Frobeuius functor.

③ The functor $-\otimes_S R_R$: Mod-S → Mod-R is a Frobeuius functor.

④ R_S is finitely generated projective as a right S-module and $-\otimes_R R_S$: proj-R → proj-S is a Frobeuius functor.

⑤ ${}_S R$ is finitely generated projective as a left S-module and $-\otimes_S R_R$: proj-S → proj-R is a Frobeuius functor.

Proof ①⇔②⇔③ by the definition of Frobenius extensions.

②⇔④ and ③⇔⑤ follow from Theorem 2.1.

The following endomorphism ring theorem is due to Ref.[8].

Theorem 3.1 Let ${}_R P_S$ be a Frobenius bimodule and $\varepsilon = \mathrm{End}_S(P_S)$ the endomorphism ring of P_S, then ε/R is a Frobenius extension.

Proof Let $-\otimes_R \varepsilon$ be the tensor functor from Mod-R to Mod-ε. Clearly, the restriction $-\otimes_R \varepsilon|_{\text{proj-}R}$ is a functor from proj-R to proj-ε, denoted by $-\otimes_R \varepsilon$ still.

Since P_S is finitely generated projective as a right S-module, $M\otimes_R P \in \mathrm{add}P_S$ for any $M \in$ proj-R. Hence $-\otimes_R P$ is a functor from proj-R to $\mathrm{add}P_S$, where $\mathrm{add}P_S$ is a full subcategory of Mod-S consisting of direct summands of finite direct sums of P_S. Let Inc be the including functor from $\mathrm{add}P_S$

to proj-S. Then $\mathrm{Inc} \circ (- \otimes_R P) = - \otimes_R P$ is a Frobenius functor from proj-R to proj-S by Theorem 2.1. It follows from Lemma 1.2 that $- \otimes_R P$ is a Frobenius functor from proj-R to addP_S.

It is well-known that there exists a natural equivalence φ from add P_S to proj-ε which sends $T_S \in$ add P_S to $\mathrm{Hom}_S({}_RP, T)$. We claim that $- \otimes_R \varepsilon \cong \varphi \circ (- \otimes_R P)$. In fact, for any $M \in$ proj-R, we have the following natural isomorphisms:

$$\varphi \circ (- \otimes_R P)(M) = \mathrm{Hom}_S(P, M \otimes_R P) \cong M \otimes_R \mathrm{Hom}_S(P, {}_RP) = M \otimes_R \varepsilon,$$

where the second isomorphism follows from Ref. [6, Proposition 20.10].

Therefore, $- \otimes_R \varepsilon$ is a Frobenius functor from proj-R to proj-ε by Corollary 1.1. Since ${}_RP$ and P_S are finitely generated projective,

$$ {}_R\varepsilon = \mathrm{Hom}_S(P, {}_RP)$$

is also a finitely generated projective left R-module. Hence ε/R is a Frobenius extension by Lemma 3.1. This completes the proof.

Remark 3.1 The relations of related module categories and functors in Theorem 3.1 can be given in Fig.1.

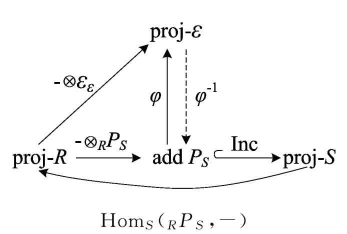

Fig.1 The relations of related module categories

And we can straightly check that $(- \otimes_R \varepsilon, \mathrm{Hom}_S({}_RP, -) \circ \mathrm{Inc} \circ \varphi^{-1})$ is a Frobenius pair on proj-R and proj-ε.

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