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Sufficient conditions for digraphs to be maximally connected and super-connected

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Abstract: Let D be a finite and simple digraph with vertex set $V(D)$. For a vertex $v \in V(D)$, the degree $d(v)$ of v is defined as the minimum value of its out-degree $d^+(v)$ and its in-degree $d^-(v)$. If D is a digraph with minimum degree δ and connectivity κ, then $\kappa \leqslant \delta$. A digraph is maximally connected if $\kappa = \delta$. A maximally connected digraph D is said to be super-connected if for every minimum vertex-cut S, the digraph $D - S$ is either non-strongly connected with at least one trivial strong component or trivial. Here some sufficient conditions for digraphs or bipartite digraphs with given minimum degree to be maximally connected or super-connected were presented in terms of the number of arcs, and some examples were given to demonstrate that the lower bound on these conditions are sharp.

Key words: digraph; connectivity; maximally connected; super-connected

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有向图是极大连通的和超连通的充分条件

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摘要:设 D 是顶点集为 $V(D)$ 的有限简单有向图.$V(D)$ 中的顶点 v 的度 $d(v)$ 被定义为 v 的出度 $d^+(v)$ 和入度 $d^-(v)$ 中的最小值.如果有向图 D 的最小度为 δ,连通度为 κ,则 $\kappa \leqslant \delta$.如果 $\kappa = \delta$,则称有向图是极大连通的.对极大连通的有向图 D 的每个最小点割 S,如果 $D - S$ 要么是非强连通的且至少有一个平凡的强连通分支,要么是平凡的,则称 D 是超连通的.通过弧数给出有向图或二部有向图在最小度给定时是极

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大连通的或超连通的充分条件，并举例说明这些条件中的下界是紧的.

关键词：有向图;连通度;极大连通;超连通

0 Introduction

Let D be a finite and simple digraph with the vertex set $V(D)$ and arc set $E(D)$. We define the order of D by $n=n(D)=|V(D)|$ and the size by $m=m(D)=|E(D)|$. For any vertex v of a digraph D, we denote the set of out-neighbors and in-neighbors of v by $N^+(v)=N_D^+(v)$ and $N^-(v)=N_D^-(v)$, respectively. For a vertex $v\in V(D)$, the degree of v, denoted by $d(v)=d_D(v)$, is defined as the minimum value of its out-degree $d^+(v)=d_D^+(v)=|N^+(v)|$ and its in-degree $d^-(v)=d_D^-(v)=|N^-(v)|$. The minimum out-degree and minimum in-degree of a digraph D are denoted by $\delta^+(D)$ and $\delta^-(D)$, respectively. In addition, let $\delta=\delta(D)=\min\{\delta^+(D),\delta^-(D)\}$ be the minimum degree of D. If D is a digraph and $X\subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. If X and Y are two subsets of $V(D)$, then we denote by (X,Y) the set of arcs with tail in X and head in Y. The complete digraph K_n^* is the digraph of order n such that for every pair of distinct vertices u and v there exist the arcs uv and vu. A digraph D is called a bipartite digraph if $V(D)$ can be partitioned into two disjoint subsets A and B such that each arc connects two vertices between A and B. Such a partition $\{A,B\}$ is called a bipartition of the digraph D. The complete bipartite digraph $K_{p,q}^*$ has the bipartition $\{A,B\}$ with $|A|=p$ and $|B|=q$ such that for every pair of vertices $u\in A$ and $v\in B$ there exist the arcs uv and vu.

A digraph D is strongly connected or simply strong if for every pair u,v of vertices there exists a directed path from u to v in D. Let $\kappa=\kappa(D)$ be the connectivity of D, that is, the smallest number of vertices whose deletion results in a digraph that is either non-strongly connected or trivial. It is easily shown that $\kappa(D)\leqslant\delta(D)$. Hence D is said to be maximally connected when $\kappa(D)=\delta(D)$.

A maximally connected digraph D is said to be super-connected or super-κ if for every minimum vertex-cut S, the digraph $D-S$ is either non-strongly connected with at least one trivial strong component or trivial.

Sufficient conditions for (di-) graphs to be maximally connected or super-connected were given by several authors, see for example the survey paper[1]. In 1971, Geller and Harary[2] pointed out that connectivity of graphs has been extensively investigated but connectivity of digraphs has been completed neglected. As a generalization of a result in Ref.[3], Geller and Harary[2] proved that $\kappa(D)\geqslant 2\delta(D)+2-n(D)$ for each non-complete digraph D. Extensions and other such results can be found, for example, in Refs.[4-6].

However, there seems to be no result in the literature showing that digraphs of sufficiently large size are maximally connected or super-connected. In this paper we will fill this gap.

In terms of size, we present some sufficient conditions for digraphs with given minimum degree to be maximally connected or super-connected in this paper. We give results on general and bipartite digraphs. Examples will demonstrate that the lower bounds on these conditions are sharp.

1 Maximally connected digraphs

Theorem 1.1 Let $k\geqslant 2$ be an integer, and let D be a strongly connected digraph of order n, size m, minimum degree $\delta\geqslant k$ and connectivity κ. If

$$m>n(n-1)-(\delta-k+2)(n-\delta-1),$$

then $\kappa\geqslant k$.

Proof Suppose to the contrary that $1\leqslant\kappa\leqslant k-1$. Let S be a minimum vertex-cut, and let X_1, $X_2,\cdots,X_p$ $(p\geqslant 2)$ be the vertex sets of the strong components of $D-S$. Then there exists an index $i\in\{1,2,\cdots,p\}$ such that each vertex in X_i has no out-neighbors in $D-(S\cup X_i)$. Assume, without loss of generality, that $i=1$. This implies that

$$\delta\,|X_1| \leqslant \sum_{x\in X_1} d^+(x) \leqslant |X_1|(|X_1|-1+\kappa)$$

and thus $|X_1| \geqslant \delta+1-\kappa$. Let $Y=\bigcup_{i=2}^{p} X_i$. Then there exists an index $j\in\{2,3,\cdots,p\}$ such that each vertex in X_j has no in-neighbors in $D-(S\cup X_j)$. This yields

$$\delta\,|X_j| \leqslant \sum_{x\in X_j} d^-(x) \leqslant |X_j|(|X_j|-1+\kappa)$$

and so $|X_j| \geqslant \delta+1-\kappa$. We deduce that

$$\delta+1-\kappa \leqslant |X_1| \leqslant n-\kappa-|X_j| \leqslant n-\kappa-(\delta+1-\kappa)=n-\delta-1$$

and hence

$$|X_1||Y|=|X_1|(n-\kappa-|X_1|) \geqslant \min\{(\delta+1-\kappa)(n-\kappa-(\delta+1-\kappa)), (n-\delta-1)(n-\kappa-(n-\delta-1))\} = (n-\delta-1)(\delta+1-\kappa).$$

Since there are no arcs from X_1 to Y, and since $\kappa \leqslant k-1$, it follows that

$$m \leqslant n(n-1)-|X_1||Y| \leqslant n(n-1)-(n-\delta-1)(\delta+1-\kappa) \leqslant n(n-1)-(n-\delta-1)(\delta+2-k),$$

a contradiction to the hypothesis. Therefore $\kappa \geqslant k$.

The special case $k=\delta$ in Theorem 1.1 leads to the following corollary.

Corollary 1.1 Let D be a strongly connected digraph of order n, size m, minimum degree $\delta \geqslant 2$ and connectivity κ. If

$$m > n(n-1)-2(n-\delta-1),$$

then D is maximally connected.

The next family of digraphs shows that Theorem 1.1 as well as Corollary 1.1 are best possible in the sense that

$$m=n(n-1)-(\delta-k+2)(n-\delta-1)$$

does not guarantee that $\kappa \geqslant k$.

Example 1.1 Let $\delta \geqslant k \geqslant 2$ and $n \geqslant 2\delta-k+3$ be integers. Define H as the disjoint union of the complete digraphs $K^*_{\delta-k+2}$, $K^*_{n-\delta-1}$ and K^*_{k-1} by adding all possible arcs between $K^*_{n-\delta-1}$ and K^*_{k-1} and all possible arcs between K^*_{k-1} and $K^*_{\delta-k+2}$ as well as all possible arcs from $K^*_{n-\delta-1}$ to $K^*_{\delta-k+2}$. Then $\delta(H)=\delta$ and

$$m=n(n-1)-(\delta-k+2)(n-\delta-1),$$

but obviously, $\kappa(H)=k-1$.

Theorem 1.2 Let $k \geqslant 2$ be an integer, and let D be a strongly connected bipartite digraph of order n, size m, minimum degree $\delta \geqslant k$ and connectivity κ. If

$$m > n\delta+\left\lfloor \frac{1}{2}\left(n-2\delta+\frac{k-1}{2}\right)^2 \right\rfloor,$$

then $\kappa \geqslant k$.

Proof Suppose to the contrary that $1 \leqslant \kappa \leqslant k-1$. Let S be a minimum vertex-cut. Then there exists at least one strong component D_1 of $D-S$ such that each vertex in D_1 has no out-neighbors in $D-(S\cup V(D_1))$. Let $X=V(D_1)$ and $Y=V(D)-(S\cup X)$, then $(X,Y)=\varnothing$. Let $\{U_1,U_2\}$ be the bipartition of $V(D)$, and $X_i=X\cap U_i$, $S_i=S\cap U_i$, $Y_i=Y\cap U_i$ for each $i=1,2$ (See Fig.1). Let $|X_i|=x_i$, $|S_i|=\kappa_i$, $|Y_i|=y_i$ for each $i=1,2$, then $n=|X|+|Y|+\kappa$, $|X|=x_1+x_2$, $|Y|=y_1+y_2$, and $\kappa=\kappa_1+\kappa_2$. If $v_1\in X_1$ and $v_2\in X_2$, then $\delta \leqslant d^+(v_1) \leqslant x_2+\kappa_2$ and $\delta \leqslant d^+(v_2) \leqslant x_1+\kappa_1$, which yields $x_1+\kappa_1 \geqslant \delta$, $x_2+\kappa_2 \geqslant \delta$ and $|X|=x_1+x_2 \geqslant 2\delta-\kappa$. If $u_1\in Y_1$ and $u_2\in Y_2$, then $\delta \leqslant d^-(u_1) \leqslant y_2+\kappa_2$ and $\delta \leqslant d^-(u_2) \leqslant y_1+\kappa_1$, which yields $y_1+\kappa_1 \geqslant \delta$, $y_2+\kappa_2 \geqslant \delta$ and $|Y|=y_1+y_2 \geqslant 2\delta-\kappa$. Thus

$$2\delta-\kappa \leqslant |X|, |Y| \leqslant n-2\delta.$$

	X	S	Y
U_1	X_1	S_1	Y_1
U_2	X_2	S_2	Y_2

Fig.1 The digraph D in Theorem 1.2

Since $(X,Y)=\varnothing$, it follows that

$$m \leqslant 2(|X_1|+|Y_1|+|S_1|)(|X_2|+|Y_2|+|S_2|)-|X_1||Y_2|-|X_2||Y_1| = 2(x_1+y_1+\kappa_1)(x_2+y_2+\kappa_2)-x_1y_2-x_2y_1.$$

Set

$$f(x_1,x_2,y_1,y_2) = 2(x_1+y_1+\kappa_1)(x_2+y_2+\kappa_2)-x_1y_2-x_2y_1.$$

Let $x'_1=\delta-\kappa_1$ and $x'_2=\delta-\kappa_2$, then $x_1 \geqslant x'_1$ and $x_2 \geqslant x'_2$. Let $y'_1=y_1+x_1-x'_1$ and $y'_2=y_2+x_2-x'_2$, then $y'_1 \geqslant y_1$ and $y'_2 \geqslant y_2$. Thus we have $x_1+y_1+\kappa_1 = x'_1+y'_1+\kappa_1$ and $x_2+y_2+\kappa_2=x'_2+y'_2+\kappa_2$, and so

$$f(x_1,x_2,y_1,y_2)-f(x'_1,x'_2,y'_1,y'_2)=$$

$$x'_1y'_2+x'_2y'_1-x_1y_2-x_2y_1=$$
$$(\delta-\kappa_1)(y_2+x_2+\kappa_2-\delta)-x_1y_2+$$
$$(\delta-\kappa_2)(y_1+x_1+\kappa_1-\delta)-x_2y_1=$$
$$-(x_1+\kappa_1-\delta)(y_2+\kappa_2-\delta)-$$
$$(x_2+\kappa_2-\delta)(y_1+\kappa_1-\delta)\leqslant 0.$$

Since $y'_1+y'_2=n-2\delta$ and $\frac{1}{2}pq\leqslant\frac{1}{8}(p+q)^2$ for any real numbers p and q, we deduce that

$$f(x'_1,x'_2,y'_1,y'_2)=$$
$$f(\delta-\kappa_1,\delta-\kappa_2,y'_1,y'_2)=$$
$$2(\delta+y'_1)(\delta+y'_2)-$$
$$(\delta-\kappa_1)y'_2-(\delta-\kappa_2)y'_1=$$
$$2\delta^2+\delta(y'_1+y'_2)+$$
$$\frac{1}{2}(2y'_1+\kappa_1)(2y'_2+\kappa_2)-\frac{1}{2}\kappa_1\kappa_2=$$
$$2\delta^2+\delta(n-2\delta)+$$
$$\frac{1}{2}(2y'_1+\kappa_1)(2y'_2+\kappa_2)-\frac{1}{2}\kappa_1\kappa_2\leqslant$$
$$n\delta+\lfloor\frac{1}{2}(2y'_1+\kappa_1)(2y'_2+\kappa_2)\rfloor\leqslant$$
$$n\delta+\lfloor\frac{1}{8}(2y'_1+2y'_2+\kappa)^2\rfloor\leqslant$$
$$n\delta+\lfloor\frac{1}{8}(2n-4\delta+k-1)^2\rfloor=$$
$$n\delta+\lfloor\frac{1}{2}\left(n-2\delta+\frac{k-1}{2}\right)^2\rfloor.$$

Thus we have

$$m\leqslant f(x_1,x_2,y_1,y_2)\leqslant f(x'_1,x'_2,y'_1,y'_2)\leqslant$$
$$n\delta+\lfloor\frac{1}{2}\left(n-2\delta+\frac{k-1}{2}\right)^2\rfloor,$$

a contradiction to the hypothesis. Therefore $\kappa\geqslant k$.

The special case $k=\delta$ in Theorem 1.2 leads to the following corollary.

Corollary 1.2 Let D be a strongly connected bipartite digraph of order n, size m, minimum degree $\delta\geqslant 2$ and connectivity κ. If

$$m>n\delta+\lfloor\frac{1}{2}\left(n-\frac{3\delta+1}{2}\right)^2\rfloor,$$

then D is maximally connected.

The next family of digraphs shows that Theorem 1.2 as well as Corollary 1.2 are best possible in the sense that

$$m=n\delta+\lfloor\frac{1}{2}\left(n-2\delta+\frac{k-1}{2}\right)^2\rfloor$$

does not guarantee that $\kappa\geqslant k$.

Example 1.2 Let $\delta\geqslant k\geqslant 2$ and $n\geqslant 4\delta-\frac{1}{2}(k-1)$ be integers with $2n+k-1\equiv(\mathrm{mod}\ 4)$. Set

$$a=\frac{2n+k-1}{4}-\delta,\ b=\frac{2n-k+1}{4}-\delta,$$

then $a\geqslant\delta$, $b>\delta-k+1$. Let $\{X_1\cup Y_1,\ X_2\cup S\cup Y_2\}$ be the bipartition of the complete bipartite digraphs $K^*_{\delta+a,\delta+b}$, where $|X_1|=\delta$, $|X_2|=\delta-k+1$, $|Y_1|=a$, $|Y_2|=b$ and $|S|=k-1$. Define H as the digraph obtained from $K^*_{\delta+a,\delta+b}$ by deleting the arcs from X_1 to Y_2 and the arcs from X_2 to Y_1. Then $\delta(H)=\delta$ and S is a minimum vertex-cut of H, and

$$m=2(\delta+a)(\delta+b)-b\delta-a(\delta-k+1)=$$
$$(2\delta+a+b)\delta+a(2b+k-1)=$$
$$n\delta+a(2b+k-1)=$$
$$n\delta+\frac{1}{2}\left(n-2\delta+\frac{k-1}{2}\right)^2,$$

but obviously, $\kappa(H)=k-1$.

2 Super-connected digraphs

In this section, we prove similar results to Corollary 1.1 and Corollary 1.2 for super-connected digraphs.

Theorem 2.1 Let D be a strongly connected digraph of order n, size m, minimum degree $\delta\geqslant 2$ and connectivity κ. If

$$m\geqslant n(n-1)-2(n-\delta-2),$$

then D is super-connected.

Proof Using Corollary 1.1, we observe that $\kappa=\delta$. If $\delta\geqslant n-3$, then $\kappa=\delta\geqslant n-3$ and thus D is super-κ. Therefore assume in the following that $\delta\leqslant n-4$. Suppose to the contrary that D is not super-κ. Let S be a minimum vertex-cut, and let $X_1,X_2,\cdots,X_p$ ($p\geqslant 2$) be the vertex sets of the strong components of $D-S$. By the assumption we note that $|X_i|\geqslant 2$ for $i\in\{1,2,\cdots,p\}$. There exists an index $i\in\{1,2,\cdots,p\}$, say $i=1$, such that each vertex in X_1 has no out-neighbors in $D-(S\cup X_1)$. Let $Y=\bigcup_{i=2}^{p}X_i$. Since $p\geqslant 2$, we deduce that $2\leqslant|X_i|\leqslant n-\delta-2$ and hence

$$|X_1||Y|=|X_1|(n-\delta-|X_1|)\geqslant$$

$$\min\{2(n-\delta-2),\ (n-\delta-2)(n-\delta-(n-\delta-2))\} = 2(n-\delta-2).$$

Since there are no arcs from X_1 to Y, it follows that

$$m \leqslant n(n-1) - |X_1||Y| \leqslant n(n-1) - 2(n-\delta-2).$$

If $m = n(n-1) - 2(n-\delta-2)$, then all the inequalities in the proof must be equalities. This implies that $p=2$, $|X_1|=2$, $|Y|=n-\delta-2$, $D[X_1]=K_2^*$, $D[S]=K_\delta^*$, $D[Y]=K_{n-\delta-2}^*$, and D is the complete digraph without all the arcs from X_1 to Y. However, $\delta(D) \geqslant \delta+1$, a contradiction. Therefore

$$m \leqslant n(n-1) - 2(n-\delta-2) - 1,$$

a contradiction to the hypothesis. Thus D is super-connected.

The next family of digraphs shows that Theorem 2.1 is best possible in the sense that

$$m = n(n-1) - 2(n-\delta-2) - 1$$

does not guarantee that D is super-κ.

Example 2.1 Let $\delta \geqslant 2$ and $n \geqslant \delta+4$ be integers. Define H as the disjoint union of the complete digraphs K_2^*, $K_{n-\delta-2}^*$ and K_δ^* by adding all possible arcs between $K_{n-\delta-2}^*$ and K_δ^* and all possible arcs between K_δ^* and K_2^* with one exception as well as all possible arcs from $K_{n-\delta-2}^*$ to K_2^*. Then $\delta(H)=\delta$ and

$$m = n(n-1) - 2(n-\delta-2) - 1,$$

however, H is not super-connected.

Theorem 2.2 Let D be a strongly connected bipartite digraph of order n, size m, minimum degree $\delta \geqslant 2$ and connectivity κ. If

$$m > n\delta + \left\lfloor \frac{1}{2}\left(n - \frac{3\delta}{2}\right)^2 \right\rfloor,$$

then D is super-connected.

Proof Using Corollary 1.2, we observe that $\kappa = \delta$. Suppose to the contrary that D is not super-κ. Let S be a minimum vertex-cut. Then each strong component of $D-S$ contains at least two vertices and there exists at least one strong component D_1 of $D-S$ such that each vertex in D_1 has no out-neighbors in $D-(S \cup V(D_1))$. Let $X=V(D_1)$ and $Y=V(D)-(S\cup X)$, then $(X,Y)=\varnothing$. Let $\{U_1,U_2\}$ be the bipartition of $V(D)$, and $X_i = X \cap U_i$, $S_i = S\cap U_i$, $Y_i = Y \cap U_i$ for each $i=1,2$ (See Fig.1). Let $|X_i|=x_i$, $|S_i|=\kappa_i$, $|Y_i|=y_i$ for each $i=1,2$, then $n=|X|+|Y|+\kappa$, $|X|=x_1+x_2$, $|Y|=y_1+y_2$, and $\delta=\kappa_1+\kappa_2$. If $v_1 \in X_1$ and $v_2 \in X_2$, then $\delta \leqslant d^+(v_1) \leqslant x_2+\kappa_2$ and $\delta \leqslant d^+(v_2) \leqslant x_1+\kappa_1$, which yields $x_1+\kappa_1 \geqslant \delta$, $x_2+\kappa_2\geqslant\delta$ and $|X|=x_1+x_2 \geqslant 2\delta-\kappa=\delta$. If $u_1 \in Y_1$ and $u_2 \in Y_2$, then $\delta \leqslant d^-(u_1) \leqslant y_2+\kappa_2$ and $\delta \leqslant d^-(u_2) \leqslant y_1+\kappa_1$, which yields $y_1+\kappa_1 \geqslant \delta$, $y_2+\kappa_2 \geqslant \delta$ and $|Y|=y_1+y_2 \geqslant 2\delta-\kappa=\delta$. Thus

$$2 \leqslant \delta \leqslant |X|,\ |Y| \leqslant n-2\delta.$$

Since $(X,Y)=\varnothing$, it follows that

$$\begin{aligned} m &\leqslant 2(|X_1|+|Y_1|+|S_1|)\cdot(|X_2|+|Y_2|+|S_2|) - |X_1||Y_2| - |X_2||Y_1| = \\ &2(x_1+y_1+\kappa_1)(x_2+y_2+\kappa_2) - x_1y_2 - x_2y_1. \end{aligned}$$

Set

$$f(x_1,x_2,y_1,y_2) = 2(x_1+y_1+\kappa_1)(x_2+y_2+\kappa_2) - x_1y_2 - x_2y_1.$$

Let $x_1'=\delta-\kappa_1$ and $x_2'=\delta-\kappa_2$, then $x_1 \geqslant x_1'$ and $x_2 \geqslant x_2'$. Let $y_1'=y_1+x_1-x_1'$ and $y_2'=y_2+x_2-x_2'$, then $y_1'\geqslant y_1$ and $y_2'\geqslant y_2$. Thus, we have $x_1+y_1+k_1=x_1'+y_1'+k_1$ and $x_2+y_2+k_2=x_2'+y_2'+k_2$. With the same proceeding as in the proof of Theorem 1.2, it is easy to get

$$\begin{aligned} m &\leqslant f(x_1,x_2,y_1,y_2) \leqslant f(x_1',x_2',y_1',y_2') \leqslant \\ &n\delta + \left\lfloor \frac{1}{8}(2y_1'+2y_2'+\kappa)^2 \right\rfloor \leqslant \\ &n\delta + \left\lfloor \frac{1}{8}(2n-4\delta+\delta)^2 \right\rfloor = \\ &n\delta + \left\lfloor \frac{1}{2}\left(n-\frac{3\delta}{2}\right)^2 \right\rfloor, \end{aligned}$$

a contradiction to the hypothesis. Hence, D is super-connected.

The next two families of digraphs show that Theorem 2.2 is best possible in the sense that

$$m = n\delta + \left\lfloor \frac{1}{2}\left(n - \frac{3\delta}{2}\right)^2 \right\rfloor$$

does not guarantee that D is super-κ for each $\delta \geqslant 2$.

Example 2.2 (a) Let $2 \leqslant \delta \leqslant 6$ and $n = 3\delta + 2$ be integers. Let $\{X_1 \cup Y_1, X_2 \cup S \cup Y_2\}$ be the bipartition of the complete bipartite digraphs $K^*_{2\delta,\delta+2}$, where $|X_1| = |Y_1| = |S| = \delta$, $|X_2| = |Y_2| = 1$. Define H as the digraph obtained from $K^*_{2\delta,\delta+2}$ by deleting the arcs from X_1 to Y_2 and the arcs from X_2 to Y_1. Then $\delta(H) = \delta$ and S is a minimum vertex-cut of H, and

$$m = 4\delta^2 + 6\delta = n\delta + \left\lfloor \frac{1}{2}\left(n - \frac{3\delta}{2}\right)^2 \right\rfloor.$$

However, $H - S$ contains two copies of $K^*_{1,\delta}$, which shows that H is not super-κ.

(b) Let $\delta \geqslant 4$, $k = \left\lfloor \frac{\delta}{2} \right\rfloor$, $n = 3\delta + k$ and $1 \leqslant s \leqslant k - 1$ be integers. Let $\{X_1 \cup Y_1, X_2 \cup S \cup Y_2\}$ be the bipartition of the complete bipartite digraphs $K^*_{2\delta,\delta+k}$, where $|X_1| = |Y_1| = |S| = \delta$, $|X_2| = s$ and $|Y_2| = k - s$. Define H as the digraph obtained from $K^*_{2\delta,\delta+k}$ by deleting the arcs from X_1 to Y_2 and the arcs from X_2 to Y_1. Then $\delta(H) = \delta$, $m = 4\delta^2 + 3\delta k$ and S is a minimum vertex-cut of H. Since $\left\lfloor \frac{1}{8}(\delta - 2k)^2 \right\rfloor = 0$, we obtain

$$n\delta + \left\lfloor \frac{1}{2}\left(n - \frac{3\delta}{2}\right)^2 \right\rfloor =$$

$$3\delta^2 + \delta k + \left\lfloor \frac{1}{2}\left(3\delta + k - \frac{3\delta}{2}\right)^2 \right\rfloor =$$

$$3\delta^2 + \delta k + \left\lfloor \frac{1}{8}(3\delta + 2k)^2 \right\rfloor =$$

$$4\delta^2 + 3\delta k + \left\lfloor \frac{1}{8}(\delta - 2k)^2 \right\rfloor =$$

$$4\delta^2 + 3\delta k = m.$$

However, $H - S$ contains two non-trivial strong components $K^*_{\delta,s}$ and $K^*_{\delta,k-s}$, which shows that H is not super-κ.

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