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On total {k}-domatic number of Cartesian and direct product of graphs

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Abstract: For a positive integer k, the total {k}-dominating function ($T\{k\}DF$) of a graph G without isolated vertices is a function f from the vertex set $V(G)$ to the set $\{0,1,2,\cdots,k\}$ such that for each vertex $v\in V(G)$, the sum of the values of all its neighbors assigned by f is at least k. A set $\{f_1,f_2,\cdots,f_d\}$ of pairwise different $T\{k\}DF$s of G with the property that $\sum_{i=1}^{d} f_i(v)\leqslant k$ for each $v\in V(G)$, is called a total {k}-dominating family ($T\{k\}D$ family) of G. The total {k}-domatic number of a graph G, denoted by $d_t^{\{k\}}(G)$, is the maximum number of functions in $T\{k\}D$ family. In 2013, Aram et al. proposed a problem that whether or not $d_t^{\{k\}}(C_m\Box C_n)=3$ when $4\nmid nmk$, and $d_t^{\{k\}}(C_m\Box C_n)=4$ when $4\mid nmk$. It was shown that $d_t^{\{k\}}(C_m\Box C_n)=3$ if $4\nmid nmk$ and $k\geqslant 2$ or $4\mid nmk$ and $2\nmid nk$, which partially answered the above problem. In addition, the total {k}-domatic number of the direct product of a cycle and a path, two paths, and two cycles was studied, respectively.

Key words: total {k}-domatic number; Cartesian product; direct product

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笛卡尔乘积和直积图的全{k}控制划分数

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摘要:给定正整数k,不含孤立点的图G的全{k}控制函数($T\{k\}DF$)是从顶点集$V(G)$到$\{0,1,2,\cdots,k\}$的映射f使得对任意的$v\in V(G)$,与v相邻的点在f下的赋值之和至少为k.若元素两两不同的全{k}控制函数集合$\{f_1,f_2,\cdots,f_d\}$满足$\sum_{i=1}^{d} f_i(v)\leqslant k$对任意$v\in V(G)$,则称该集合为$G$的全{$k$}控制族($T\{k\}D$族).含有函数最多的$G$的全{$k$}控制族的函数数量成为全{$k$}控制划分数,记为$d_t^{\{k\}}(G)$.2013年,Aram等提出

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了以下问题：是否当 $4\nmid nmk$ 时 $d_t^{\{k\}}(C_m\Box C_n)=3$，当 $4\nmid nmk$ 时 $d_t^{\{k\}}(C_m\Box C_n)=4$.这里证明了当 $4\nmid nmk$ 且$k\geqslant 2$ 或 $4|nmk$ 且 $2\nmid nk$ 时 $d_t^{\{k\}}(C_m\Box C_n)=3$.该结论部分回答了上述问题. 更进一步，确定了路和圈、路和路、圈和圈的全$\{k\}$控制划分数.

关键词：全$\{k\}$控制划分数；笛卡尔乘积；直积

0 Introduction

For terminology and notation on graph theory not given here, we refer the reader to Ref.[1]. Throughout this paper, the graphs we talk about are simple graphs with no isolated vertex. Let $G=(V,E)$ be a graph with vertex set $V=V(G)$ and edge set $E=E(G)$. For a vertex $v\in V(G)$, the open neighborhood $N_G(v)$ is the set $N_G(v)=\{u\in V(G)\mid uv\in E(G)\}$ and the degree of v, denoted by $d_G(v)$, is the cardinality of $N_G(v)$. For a set $D\subseteq V(G)$, the open neighborhood $N_G(D)$ is defined to be $\bigcup_{u\in D}N_G(u)$ and the closed neighborhood is $N_G[D]=N_G(D)\cup D$. The minimum and maximum degree of a graph G are denoted by δ and Δ, respectively. Write P_n and C_n for a path and a cycle on n vertices, respectively. For a real number x, write $\lfloor x\rfloor$ for the greatest integer not greater than x, and $\lceil x\rceil$ for the smallest integer not less than x.

For graphs G and H, the Cartesian product $G\Box H$ is a graph with vertex set $V(G\Box H)=V(G)\times V(H)$ and two vertices (u,v) and (u',v') are adjacent if and only if $u=u'$ and $vv'\in E(H)$ or $v=v'$ and $uu'\in E(G)$. The Cartesian product of a cycle C_m and a path P_n is called a cylinder and the Cartesian product of two cycles is called a torus. The direct product $G\times H$ is the graph defined by $V(G\times H)=V(G)\times V(H)$ and two vertices (u,v) and (u',v') are adjacent if and only if $uu'\in E(G)$ and $vv'\in E(H)$. Throughout this paper, we assume that $V(G)=\{0,1,2,\cdots,n-1\}$ for any graph G of order n. Then $V(G\Box H)=V(G\times H)=\{(i,j)\mid i\in V(G),j\in V(H)\}$. For convenience, we assume that $V(G\Box H)=V(G\times H)=\{x_{i,j}\mid i\in V(G),j\in V(H)\}$, where $x_{i,j}=(i,j)$.

A subset S of vertices of G without isolated vertices is a total dominating set if $N_G(S)=V$. The total domination number $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G. A total domatic partition is a partition of V into total dominating sets, and the total domatic number $d_t(G)$ is the largest number of sets in a total domatic partition. The total domatic number was introduced in Ref.[2].

Let k be a positive integer, a total $\{k\}$-dominating function ($T\{k\}DF$) of a graph G without isolated vertices is a function f from the vertex set $V(G)$ to the set $\{0,1,2,\cdots,k\}$ such that for any vertex $v\in V(G)$, $\sum_{u\in N_G(v)}f(u)\geqslant k$. The weight of a $T\{k\}DF$ f is the value $\omega(f)=\sum_{v\in V(G)}f(v)$. The total $\{k\}$-domination number of G, denoted by $\gamma_t^{\{k\}}(G)$, is the minimum weight of a $T\{k\}DF$ of G. Note that $\gamma_t^{\{k\}}(G)$ is the classical total domination number $\gamma_t(G)$ when $k=1$. The total $\{k\}$-domination number was introduced in Ref.[3]. A set $\{f_1,f_2,\cdots,f_d\}$ of pairwise different $T\{k\}DF$s of G with the property that $\sum_{i=1}^{d}f_i(v)\leqslant k$ for each $v\in V(G)$, is called a total $\{k\}$-dominating family ($T\{k\}D$ family) of G. The total $\{k\}$-domatic number of a graph G, denoted by $d_t^{\{k\}}(G)$, is the maximum number of functions in a $T\{k\}D$ family. The total $\{k\}$-domatic number is well-defined and $d_t^{\{k\}}(G)\geqslant 1$ for all graphs G without isolated vertices, since the set consisting of the function $f:V(G)\mapsto\{0,1,2,\cdots,k\}$ defined by $f(v)=k$ for each $v\in V(G)$, forms a $T\{k\}D$ family on G. The total $\{k\}$-domatic number was introduced in Ref.[4] and has also been studied in Ref.[5]. Aram et al.[6] presented bounds for the total k-domatic number, and studied the total k-domatic number of $C_m\Box P_n$ and $C_m\Box C_n$. In addition, they proposed the following problem on

$d_t^{\{k\}}(C_m\square C_n)$.

Problem 0.1[6] Prove or disprove: Let $G=C_m\square C_n$ be a torus of order nm. if $4\nmid nmk$ then $d_t^{\{k\}}(G)=3$, and $d_t^{\{k\}}(G)=4$ otherwise.

The main result of this paper is as follows.

Theorem 0.1

$$d_t^{\{k\}}(C_m\square C_n)\begin{cases}=3, \text{ if } 4\nmid nmk \text{ and } k\geqslant 2;\\ =3, \text{ if } 4\mid nmk \text{ but } 2\nmid nk;\\ =4, \text{ if } 4\mid m \text{ and } 4\mid n;\\ \leqslant 4, \text{ otherwise.}\end{cases}$$

It can be seen that Theorem 0.1 partially answered Problem 0.1. In addition, we also study the total $\{k\}$-domatic number of the direct product of a cycle and a path, two paths, and two cycles, respectively.

1 The proof of Theorem 0.1

The following lemmas will be used in our proofs.

Lemma 1.1[5] Let G be a graph without isolated vertices and $\delta=\delta(G)$. If $\delta\mid k$, then $d_t^{\{k\}}(G)\geqslant\delta-1$, and if $\delta\nmid k$, then $d_t^{\{k\}}(G)\geqslant\lfloor k/\lceil\frac{k}{\delta}\rceil\rfloor$.

Lemma 1.2[4] For every graph G without isolated vertices,

$$d_t^{\{k\}}(G)\leqslant\delta(G).$$

Moreover, if $d_t^{\{k\}}(G)=\delta(G)$, then for each function of any $T\{k\}D$ family $\{f_1,f_2,\cdots,f_d\}$ and for all vertices v of degree $\delta(G)$, $\sum_{u\in N_G(v)}f_i(u)=k$ and $\sum_{i=1}^{d}f_i(u)=k$ for every $u\in N_G(v)$.

Lemma 1.3[4] if $G=C_m\square C_n$ such that $4\nmid nmk$, then $d_t^{\{k\}}(G)\leqslant 3$.

Proposition 1.1 Let $m\geqslant 1$ and $n\geqslant 1$. if $G=C_{4m}\square C_{4n}$, then $d_t^{\{k\}}(G)=4$.

Proof According to Lemma 1.2, we have $d_t^{\{k\}}(G)\leqslant\delta(G)\leqslant 4$. Define $f_s: V(G)\mapsto\{0,1,\cdots,k\}$, $s=1,2,3,4$ as follows:

$$f_1(x_{i,j})=\begin{cases}k, \text{ if } i\equiv 0,1 \pmod 4 \text{ and } j\equiv 0 \pmod 4;\\ k, \text{ if } i\equiv 2,3 \pmod 4 \text{ and } j\equiv 2 \pmod 4;\\ 0, \text{ otherwise;}\end{cases}$$

$$f_2(x_{i,j})=\begin{cases}k, \text{ if } i\equiv 0,1 \pmod 4 \text{ and } j\equiv 2 \pmod 4;\\ k, \text{ if } i\equiv 2,3 \pmod 4 \text{ and } j\equiv 0 \pmod 4;\\ 0, \text{ otherwise;}\end{cases}$$

$$f_3(x_{i,j})=\begin{cases}k, \text{ if } i\equiv 0,1 \pmod 4 \text{ and } j\equiv 1 \pmod 4;\\ k, \text{ if } i\equiv 2,3 \pmod 4 \text{ and } j\equiv 3 \pmod 4;\\ 0, \text{ otherwise;}\end{cases}$$

$$f_4(x_{i,j})=\begin{cases}k, \text{ if } i\equiv 0,1 \pmod 4 \text{ and } j\equiv 3 \pmod 4;\\ k, \text{ if } i\equiv 2,3 \pmod 4 \text{ and } j\equiv 1 \pmod 4;\\ 0, \text{ otherwise.}\end{cases}$$

It is easy to check that $\{f_1,f_2,f_3,f_4\}$ is a $T\{k\}D$ family on G. Hence, $d_t^{\{k\}}(G)\geqslant 4$, and thus $d_t^{\{k\}}(G)=4$. This completes the proof.

Proposition 1.2 if $k\geqslant 2$, $4\nmid nmk$ and $G=C_m\square C_n$, then $d_t^{\{k\}}(G)=3$.

Proof It suffices to show $d_t^{\{k\}}(G)\geqslant 3$ by Lemma 1.3. In view of $4\nmid nmk$, we have $4\nmid k$, which is equivalent to $\delta(G)\nmid k$. By Lemma 1.1, we conclude that $d_t^{\{k\}}(G)\geqslant\lfloor k/\lceil\frac{k}{\delta(G)}\rceil\rfloor$. Let $k=4s+t$ where $1\leqslant t\leqslant 3$ since $4\nmid k$. Therefore,

$$d_t^{\{k\}}(G)\geqslant\lfloor(4s+t)/\lceil\frac{4s+t}{4}\rceil\rfloor=$$

$$\lfloor(4s+t)/(s+\lceil\frac{t}{4}\rceil)\rfloor=\lfloor\frac{4s+t}{s+1}\rfloor=4+\lfloor\frac{4-t}{s+1}\rfloor.$$

If $s\geqslant 2$, in view of $1\leqslant t\leqslant 3$, then $s+1\geqslant 4-t$, and thus $d_t^{\{k\}}(G)\geqslant 3$ by the above inequality. It implies that $d_t^{\{k\}}(G)\geqslant 3$ for $k\geqslant 9$ and $4\nmid nmk$.

If $s<2$, in view of $4\nmid k$, then $k=2,3,5,6,7$. It is routine to check that $d_t^{\{k\}}(G)\geqslant\lfloor k/\lceil\frac{k}{\delta(G)}\rceil\rfloor=3$ when $k=3,6,7$. In order to complete the proof, it is necessary to show that $d_t^{\{k\}}(G)\geqslant 3$ when $k=2,5$.

When $k=2$, then $2\nmid m$ and $2\nmid n$. Define functions f, g and h from $V(G)$ to $\{0,1,\cdots,k\}$ as follows:

$$f(x_{i,j})=\begin{cases}1, i\equiv 0 \pmod 2 \text{ and } 0\leqslant j\leqslant n-1;\\ 0, \text{ otherwise;}\end{cases}$$

$g(x_{i,j})=$

$$\begin{cases}1, i\equiv 1 \pmod 2, i=m-1 \text{ and } 0\leqslant j\leqslant n-1;\\ 0, \text{ otherwise;}\end{cases}$$

$$h(x_{i,j})=\begin{cases}1, & 0\leqslant i\leqslant m-2 \text{ and } 0\leqslant j\leqslant n-1;\\ 0, & \text{otherwise.}\end{cases}$$

Clearly, $\{f,g,h\}$ is a $T\{k\}D$ family on G. Hence, $d_t^{\{k\}}(G)\geqslant 3$.

When $k=5$, then $4\nmid mn$.

If $2\mid mn$, assume that $2\mid m$. Define functions f,g and h from $V(G)$ to $\{0,1,\cdots,k\}$ as follows:

$$f(x_{i,j})=\begin{cases}3, & i\equiv 0 \pmod 2 \text{ and } 0\leqslant j\leqslant n-1;\\ 0, & \text{otherwise};\end{cases}$$

$$g(x_{i,j})=\begin{cases}3, & i\equiv 1 \pmod 2,\ i=m-1 \text{ and } 0\leqslant j\leqslant n-1;\\ 0, & \text{otherwise};\end{cases}$$

$h(x_{i,j})=2,\ 0\leqslant i\leqslant m-1$ and $0\leqslant j\leqslant n-1$.

Clearly, $\{f,g,h\}$ is a $T\{k\}D$ family on G. Hence, $d_t^{\{k\}}(G)\geqslant 3$.

If $2\nmid mn$, define functions f,g and h from $V(G)$ to $\{0,1,\cdots,k\}$ as follows:

$$f(x_{i,j})=\begin{cases}2, & i\equiv 0 \pmod 2 \text{ and } 0\leqslant j\leqslant n-1;\\ 1, & \text{otherwise};\end{cases}$$

$$g(x_{i,j})=\begin{cases}1, & i\equiv 0 \pmod 2 \text{ and } 0\leqslant j\leqslant n-1;\\ 2, & \text{otherwise};\end{cases}$$

$h(x_{i,j})=2,\ 0\leqslant i\leqslant m-1$ and $0\leqslant j\leqslant n-1$.

Clearly, $\{f,g,h\}$ is a $T\{k\}D$ family on G. Hence, $d_t^{\{k\}}(G)\geqslant 3$. This completes the proof.

Proposition 1.3 If $G=C_m\square C_n$ such that $4\mid nmk$ and $2\nmid nk$, then $d_t^{\{k\}}(G)=3$.

Proof By Lemma 1.2, $d_t^{\{k\}}(G)\leqslant 4$. Suppose to the contrary that $d_t^{\{k\}}(G)=4$. Let $\mathscr{F}$ be a total $\{k\}$-dominating family and $f\in\mathscr{F}$. By Lemma 1.2, $\sum_{x\in N_G(x_{i,j})} f(x)=k$. It follows that

$$mnk=\sum_{v\in V(G)}\sum_{u\in N_G(v)} f(u)=4\sum_{u\in V(G)} f(u)=4\omega(f).$$

Let $\sum_{j=0}^{n-1} f(x_{i,j})=k_i$. Thus $k_{i-1}+2k_i+k_{i+1}=nk$ for each $i=1,2,\cdots,m-1$, namely, $k_{i-1}+k_i+k_i+k_{i+1}=nk$. Without loss of generality, assume

$$k_0+k_1\geqslant\lceil\frac{nk}{2}\rceil,\ k_1+k_2\leqslant\lfloor\frac{nk}{2}\rfloor.$$

Note that $4\mid nmk$ and $2\nmid nk$ implies that $4\mid m$. So we have

$$k_2+k_3\geqslant\lceil\frac{nk}{2}\rceil,\ k_3+k_4\leqslant\lfloor\frac{nk}{2}\rfloor,$$

$$\cdots$$

$$k_{m-2}+k_{m-1}\geqslant\lceil\frac{nk}{2}\rceil,\ k_{m-1}+k_0\leqslant\lfloor\frac{nk}{2}\rfloor,$$

which implies that $\omega(f)=\sum_{i=0}^{m-1}k_i\geqslant\frac{m}{2}\lceil\frac{nk}{2}\rceil>\frac{mnk}{4}$, a contradiction. So $d_t^{\{k\}}(G)\leqslant 3$.

Now define functions f,g and h from $V(G)$ to $\{0,1,\cdots,k\}$ as follows:

$$f(x_{i,j})=\begin{cases}k, & \text{if } i\equiv 0,1 \pmod 4 \text{ and } j\equiv 0 \pmod 4;\\ k, & \text{if } i\equiv 2,3 \pmod 4 \text{ and } j\equiv 2 \pmod 4;\\ 0, & \text{otherwise};\end{cases}$$

$$g(x_{i,j})=\begin{cases}k, & \text{if } i\equiv 2,3 \pmod 4 \text{ and } j\equiv 0 \pmod 4;\\ k, & \text{if } i\equiv 0,1 \pmod 4 \text{ and } j\equiv 2 \pmod 4;\\ 0, & \text{otherwise};\end{cases}$$

$$h(x_{i,j})=\begin{cases}k, & 0\leqslant i\leqslant m-1 \text{ and } j\equiv 1 \pmod 2;\\ 0, & \text{otherwise.}\end{cases}$$

Now $\{f,g,h\}$ is a $T\{k\}D$ family on G. Therefore $d_t^{\{k\}}(G)\geqslant 3$. The result follows.

Remark 1.1 Theorem 0.1 follows directly from Propositions 1.1～1.3.

2 Total $\{k\}$-domatic numbers of $P_m\times P_n$, $C_m\times P_n$ and $C_m\times C_n$

Proposition 2.1 $d_t^{\{k\}}(P_m\times P_n)=1$.

Proof Let $G=P_m\times P_n$. Note that $\delta(G)=1$ and $d_t^{\{k\}}(G)\geqslant 1$ for all graphs G without isolated vertices. The result follows directly from Lemma 1.2.

Proposition 2.2 if $2\nmid k$ and $4\nmid m$, then $d_t^{\{k\}}(C_m\times P_n)=1$.

Proof Let $G=C_m\times P_n$. Suppose that $d_t^{\{k\}}(G)=2(=\delta(G))$. Then $\sum_{v\in N_G(x_{i,j})} f(v)=k$ for any $f\in\mathscr{F}$, where $\mathscr{F}$ is a $T\{k\}D$ family on G. Hence,

$$\sum_{v\in N_G(x_{1,0})} f(v)=f(x_{0,1})+f(x_{2,1})=k,$$

$$\sum_{v\in N_G(x_{2,0})} f(v)=f(x_{1,1})+f(x_{3,1})=k,$$

$$\sum_{v\in N_G(x_{3,0})} f(v)=f(x_{2,1})+f(x_{4,1})=k,$$

$$\cdots$$

$$\sum_{v\in N_G(x_{m-1,0})} f(v)=f(x_{m-2,1})+f(x_{0,1})=k,$$

$$\sum_{v\in N_G(x_{0,0})} f(v)=f(x_{m-1,1})+f(x_{1,1})=k.$$

Summing up the above m equations, we have

$2(f(x_{0,1})+f(x_{1,1})+\cdots+f(x_{m-1,1}))=mk$. Since $2\nmid k$ and $4\nmid m$, we have $2|m$ and $m=2 \pmod 4$. The above equations also imply that $f(x_{0,1})=f(x_{4,1})=f(x_{8,1})=\cdots=f(x_{m-2,1})$. Since $f(x_{m-2,1})+f(x_{0,1})=k$, we have $2|k$, a contradiction.

Proposition 2.3 If $2\nmid k$ and $4|m$, then $d_t^{\{k\}}(C_m\times P_n)=2$.

Proof Let $G=C_m\times P_n$. It suffices to show $d_t^{\{k\}}(G)\geqslant 2$ by Lemma 1.2. Define functions f and g from $V(G)$ to $\{0,1,\cdots,k\}$ as follows:

$$f(x_{i,j})=\begin{cases}\frac{k+1}{2}, \text{ if } i\equiv 0,1 \pmod 4 \text{ and } j=1,n-2;\\ \frac{k-1}{2}, \text{ otherwise},\end{cases}$$

and

$$g(x_{i,j})=\begin{cases}\frac{k+1}{2}, \text{ if } i\equiv 2,3 \pmod 4 \text{ and } j=1,n-2;\\ \frac{k-1}{2}, \text{ otherwise}.\end{cases}$$

Since $\{f,g\}$ is a $T\{k\}D$ family on G, $d_t^{\{k\}}(G)\geqslant 2$. This completes the proof.

Proposition 2.4 If $2|k$ then $d_t^{\{k\}}(G)=2$.

Proof Let $G=C_m\times P_n$. It suffices to show $d_t^{\{k\}}(G)\geqslant 2$ by Lemma 1.2. Define functions f and g from $V(G)$ to $\{0,1,\cdots,k\}$ as follows:

$$f(x_{i,j})=\begin{cases}\frac{k}{2}, \text{ if } 0\leqslant i\leqslant m-1 \text{ and } j=1,n-2;\\ \frac{k-2}{2}, \text{ otherwise};\end{cases}$$

and

$$g(x_{i,j})=\frac{k}{2},\ 0\leqslant i\leqslant m-1 \text{ and } 0\leqslant j\leqslant n-1.$$

Note that $\{f,g\}$ is a $T\{k\}D$ family on G, we have $d_t^{\{k\}}(G)\geqslant 2$. This completes the proof.

From Propositions 2.2～2.4, we can get the following theorem immediately.

Theorem 2.1

$$d_t^{\{k\}}(C_m\times P_n)=\begin{cases}1, \text{ if } 2\nmid k \text{ and } 4\nmid m;\\ 2, \text{ otherwise}.\end{cases}$$

Proposition 2.5

$$d_t^{\{k\}}(C_m\times C_n)\begin{cases}=4, \text{ if } 4\mid m \text{ and } 4\mid n;\\ =4, \text{ if } 2\mid k \text{ and } 4\mid m \text{ or } 4\mid n;\\ \leqslant 3, \text{ otherwise}.\end{cases}$$

Proof Let $G=C_m\times C_n$. By Lemma 1.2, $d_t^{\{k\}}(G)\leqslant\delta(G)=4$. We proceed by considering the following three possible cases.

Case 1 $4|m$ and $4|n$.

Define $f_s: V(G)\mapsto\{0,1,\cdots,k\}$, $s=1,2,3,4$, as follows:

$$f_1(x_{i,j})=\begin{cases}\lfloor\frac{k}{2}\rfloor, \text{ if } i\equiv 0,1 \pmod 4 \text{ and } j\equiv 0,1 \pmod 4;\\ \lceil\frac{k}{2}\rceil, \text{ if } i\equiv 2,3 \pmod 4 \text{ and } j\equiv 0,1 \pmod 4;\\ 0, \text{ otherwise};\end{cases}$$

$$f_2(x_{i,j})=\begin{cases}\lceil\frac{k}{2}\rceil, \text{ if } i\equiv 0,1 \pmod 4 \text{ and } j\equiv 0,1 \pmod 4;\\ \lfloor\frac{k}{2}\rfloor, \text{ if } i\equiv 2,3 \pmod 4 \text{ and } j\equiv 0,1 \pmod 4;\\ 0, \text{ otherwise};\end{cases}$$

$$f_3(x_{i,j})=\begin{cases}\lfloor\frac{k}{2}\rfloor, \text{ if } i\equiv 0,1 \pmod 4 \text{ and } j\equiv 2,3 \pmod 4;\\ \lceil\frac{k}{2}\rceil, \text{ if } i\equiv 2,3 \pmod 4 \text{ and } j\equiv 2,3 \pmod 4;\\ 0, \text{ otherwise};\end{cases}$$

$$f_4(x_{i,j})=\begin{cases}\lceil\frac{k}{2}\rceil, \text{ if } i\equiv 0,1 \pmod 4 \text{ and } j\equiv 0,3 \pmod 4;\\ \lfloor\frac{k}{2}\rfloor, \text{ if } i\equiv 2,3 \pmod 4 \text{ and } j\equiv 0,3 \pmod 4;\\ 0, \text{ otherwise}.\end{cases}$$

Clearly, $\{f_1,f_2,f_3,f_4\}$ is a $T\{k\}D$ family on G. Hence, $d_t^{\{k\}}(G)\geqslant 4$, which follows that $d_t^{\{k\}}(G)=4$.

Case 2 $2|k$ and $4|m$ or $4|n$. Without loss of generality assume $4|n$.

Define $f_s: V(G)\mapsto\{0,1,\cdots,k\}$, $s=1,2,3,4$, as follows

$$f_1(x_{i,j})=\begin{cases}\frac{k}{2}, \text{ if } j\equiv 0,1 \pmod 4 \text{ and } 0\leqslant i\leqslant m-1;\\ 0, \text{ otherwise};\end{cases}$$

$$f_2(x_{i,j})=\begin{cases}\frac{k}{2}, \text{ if } j\equiv 1,2 \pmod 4 \text{ and } 0\leqslant i\leqslant m-1;\\ 0, \text{ otherwise};\end{cases}$$

$f_3(x_{i,j})=$

$\begin{cases}\dfrac{k}{2}, \text{ if } j\equiv 2,3 \pmod 4 \text{ and } 0\leqslant i\leqslant m-1;\\ 0, \text{ otherwise};\end{cases}$

$f_4(x_{i,j})=$

$\begin{cases}\dfrac{k}{2}, \text{ if } j\equiv 3,0 \pmod 4 \text{ and } 0\leqslant i\leqslant m-1;\\ 0, \text{ otherwise}.\end{cases}$

Clearly, $\{f_1,f_2,f_3,f_4\}$ is a $T\{k\}D$ family on G. Hence, $d_t^{\{k\}}(G)\geqslant 4$, which follows that $d_t^{\{k\}}(G)=4$.

Case 3 The others.

By Case 1, $4\nmid m$ or $4\nmid n$. Without loss of generality, assume $4\nmid m$. By Case 2, the remaining case is $4\nmid n$ or $4|n$ but $2\nmid k$.

Suppose to the contrary that $d_t^{\{k\}}(G)=4$. Let $\mathscr{F}=\{f_1,f_2,f_3,f_4\}$ be a total $\{k\}$-dominating family. We will prove that, for any $f\in\mathscr{F}$, $f(x_{i-1,j-1})=f(x_{i+1,j+1})$ and $f(x_{i-1,j+1})=f(x_{i+1,j-1})$, then we obtain $f(x_{i,j})=\dfrac{k}{4}$ for $0\leqslant i\leqslant m-1$ and $0\leqslant j\leqslant n-1$, which implies that $|\mathscr{F}|=1$, a contradiction with $d_t^{\{k\}}(G)=4$. Since G is vertex transitive, it is sufficient to show $f(x_{0,0})=f(x_{2,2})$, $f(x_{0,2})=f(x_{2,0})$ and $f(x_{0,0})=\dfrac{k}{4}$.

By Lemma 1.2, $\sum\limits_{x\in N_G(x_{i,j})}f(x)=k$. Hence

$$f(x_{i-1,j-1})+f(x_{i-1,j+1})+ f(x_{i+1,j-1})+f(x_{i+1,j+1})=k,$$

where the sums in the subscripts are modular m and n, respectively. For $i=1,2,\cdots,m-1,0$, we have

$$f(x_{0,j-1})+f(x_{0,j+1})+f(x_{2,j-1})+f(x_{2,j+1})=k,$$
$$f(x_{1,j-1})+f(x_{1,j+1})+f(x_{3,j-1})+f(x_{3,j+1})=k,$$
$$f(x_{2,j-1})+f(x_{2,j+1})+f(x_{4,j-1})+f(x_{4,j+1})=k,$$
$$\cdots$$
$$f(x_{m-2,j-1})+f(x_{m-2,j+1})+f(x_{0,j-1})+f(x_{0,j+1})=k,$$
$$f(x_{m-1,j-1})+f(x_{m-1,j+1})+f(x_{1,j-1})+f(x_{1,j+1})=k.$$

Hence we have

$$f(x_{0,j-1})+f(x_{0,j+1})=f(x_{4,j-1})+ f(x_{4,j+1})=\cdots=f(x_{4s,j-1})+f(x_{4s,j+1}),$$

where s can be any nonnegative integer. Since $4\nmid m$, $4s \pmod m$ can be any positive integer from $\{1,2,\cdots,m-1\}$. Therefore,

$$f(x_{0,j-1})+f(x_{0,j+1})=f(x_{1,j-1})+f(x_{1,j+1})= f(x_{2,j-1})+f(x_{2,j+1})=\cdots= f(x_{m-1,j-1})+f(x_{m-1,j+1})=\frac{k}{2}.$$

This implies that $2|k$ and hence we must have $4\nmid n$ (otherwise, we have $4\mid n$ and $2\nmid k$, a contradiction). By symmetry of m and n, we have

$$f(x_{i-1,0})+f(x_{i+1,0})=f(x_{i-1,1})+f(x_{i+1,1})= f(x_{i-1,2})+f(x_{i+1,2})=\cdots= f(x_{i-1,n-1})+f(x_{i+1,n-1})=\frac{k}{2}.$$

Consider $i=1$ and $j=1$, we have $f(x_{2,0})=f(x_{0,2})$ and $f(x_{0,0})=f(x_{2,2})$. By symmetry, we can obtain $f(x_{i-1,j-1})=f(x_{i+1,j+1})$ and $f(x_{i-1,j+1})=f(x_{i+1,j-1})$, for $0\leqslant i\leqslant m-1$ and $0\leqslant j\leqslant n-1$. Therefore

$$f(x_{0,0})=f(x_{2,2})=f(x_{4,0})=f(x_{6,2})= f(x_{8,0})=\cdots=f(x_{4s,0})=f(x_{4s+2,2}).$$

Again by $4\nmid m$, we have that $4s \pmod m$ can be any positive integer in $\{1,2,\cdots,m-1\}$, and hence

$$f(x_{0,0})=f(x_{1,0})=f(x_{2,0})=\cdots=f(x_{m-1,0}).$$

Since $f(x_{0,0})+f(x_{2,0})=\dfrac{k}{2}$, we have $f(x_{0,0})=\dfrac{k}{4}$. This completes the proof.

Proposition 2.6 If $k\geqslant 2$, and $4\nmid m$ or $4\nmid n$, then $d_t^{\{k\}}(C_m\times C_n)\geqslant 3$.

Proof Let $G=C_m\times C_n$. Without loss of generality, assume $4\nmid m$. if $4|k$, that is $\delta(G)|k$, then $d_t^{\{k\}}(G)\geqslant\delta(G)-1=3$ by Lemma 1.1.

Assume $4\nmid k$. By Lemma 1.1, $d_t^{\{k\}}(G)\geqslant \lfloor k/\lceil\frac{k}{4}\rceil\rfloor$. Let $k=4s+t$ where $1\leqslant t\leqslant 3$. Moreover,

$$d_t^{\{k\}}(G)\geqslant\lfloor(4s+t)/\lceil\frac{4s+t}{4}\rceil\rfloor=4+\lfloor\frac{4-t}{s+1}\rfloor.$$

If $s\geqslant 2$, then $s+1\geqslant 4-t$, and thus $d_t^{\{k\}}(G)\geqslant 3$ for $k\geqslant 9$. Hence assume $s<2$. Since $4\nmid k$, we have $k=2,3,5,6,7$. It is obvious that $d_t^{\{k\}}(G)\geqslant\lfloor k/\lceil\frac{k}{\delta(G)}\rceil\rfloor\geqslant 3$ when $k=3,6,7$ by Lemma 1.1. In order to complete the proof, it suffices to show that $d_t^{\{k\}}(G)\geqslant 3$ when $k=2,5$.

When $m\equiv 1 \pmod 4$, define functions f,g

and h from $V(G)$ to $\{0,1,\cdots,k\}$ as follows:

$$f(x_{i,j})=\begin{cases}\lfloor \frac{k}{2} \rfloor, \text{ if } i\equiv 0,1 \pmod 4 \text{ and } 0\leqslant j\leqslant n-1;\\ \lfloor \frac{k}{2} \rfloor-1, \text{ otherwise};\end{cases}$$

$$g(x_{i,j})=\begin{cases}\lfloor \frac{k}{2} \rfloor, \text{ if } i\equiv 2,3 \pmod 4, \\ \qquad i=m-1 \text{ and } 0\leqslant j\leqslant n-1;\\ \lfloor \frac{k}{2} \rfloor-1, \text{ otherwise};\end{cases}$$

and

$$h(x_{i,j})=\begin{cases}\lfloor \frac{k}{2} \rfloor-1, \text{ if } i=m-1 \text{ and } 0\leqslant j\leqslant n-1;\\ \lfloor \frac{k}{2} \rfloor, \text{ otherwise}.\end{cases}$$

Note that $\{f,g,h\}$ is a $T\{k\}D$ family on G, hence $d_t^{\{k\}}(G)\geqslant 3$.

When $m\equiv 2 \pmod 4$, define functions f,g and h form $V(G)$ to $\{0,1,\cdots,k\}$ as follows:

$$f(x_{i,j})=\begin{cases}\lfloor \frac{k}{2} \rfloor, \text{ if } i\equiv 0,1 \pmod 4 \text{ and } 0\leqslant j\leqslant n-1;\\ \lfloor \frac{k}{2} \rfloor-1, \text{ otherwise};\end{cases}$$

$$g(x_{i,j})=\begin{cases}\lfloor \frac{k}{2} \rfloor, \text{ if } i\equiv 2,3 \pmod 4, \\ \qquad i=m-2,m-1 \text{ and } 0\leqslant j\leqslant n-1;\\ \lfloor \frac{k}{2} \rfloor-1, \text{ otherwise};\end{cases}$$

and

$$h(x_{i,j})=\begin{cases}\lfloor \frac{k}{2} \rfloor-1, \text{ if } i=m-2,m-1 \text{ and } 0\leqslant j\leqslant n-1;\\ \lfloor \frac{k}{2} \rfloor, \text{ otherwise}.\end{cases}$$

Since $\{f,g,h\}$ is a $T\{k\}D$ family on G, $d_t^{\{k\}}(G)\geqslant 3$.

When $m\equiv 3 \pmod 4$, define functions f,g and h from $V(G)$ to $\{0,1,\cdots,k\}$ as follows:

$$f(x_{i,j})=\begin{cases}\lfloor \frac{k}{2} \rfloor, \text{ if } i\equiv 0,1 \pmod 4 \text{ and } 0\leqslant j\leqslant n-1;\\ \lfloor \frac{k}{2} \rfloor-1, \text{ otherwise};\end{cases}$$

$$g(x_{i,j})=\begin{cases}\lfloor \frac{k}{2} \rfloor, \text{ if } i\equiv 2,3 \pmod 4, \\ \qquad i=0 \text{ and } 0\leqslant j\leqslant n-1;\\ \lfloor \frac{k}{2} \rfloor-1, \text{ otherwise};\end{cases}$$

and

$$h(x_{i,j})=\begin{cases}\lfloor \frac{k}{2} \rfloor-1, \text{ if } i=0 \text{ and } 0\leqslant j\leqslant n-1;\\ \lfloor \frac{k}{2} \rfloor, \text{ otherwise}.\end{cases}$$

Because $\{f,g,h\}$ is a $T\{k\}D$ family on G, $d_t^{\{k\}}(G)\geqslant 3$.

This completes the proof.

From Propositions 2.5 and 2.6, we can get the following theorem immediately.

Theorem 2.2 Let $k\geqslant 2$. We have

$$d_t^{\{k\}}(C_m\times C_n)=\begin{cases}4, \text{ if } 4\mid m \text{ and } 4\mid n \text{ (equality also holds for } k=1);\\ 4, \text{ if } 2\mid k \text{ and } 4\mid m \text{ or } 4\mid n;\\ 3, \text{ otherwise}.\end{cases}$$

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