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Lin-Lu-Yau curvature and diameter of amply regular graphs

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Abstract: By Hall's marriage theorem, we study lower bounds of the Lin-Lu-Yau curvature of amply regular graphs with girth 3 or 4 under different parameter restrictions. As a consequence, we show that each conference graph has positive Lin-Lu-Yau curvature. Our approach also provides a geometric proof of a known diameter estimates of amply regular graphs in the case of girth 4 and some special cases of girth 3.

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1 Introduction and statements of result

Ricci curvature is a fundamental concept in Riemannian geometry. Its extension to general metric measure spaces, particularly, to locally finite graphs, has attracted lots of attention[1-8]. In 2009, Ollivier[2,3] introduced the notion of coarse Ricci curvature of Markov chains on metric spaces including graphs. On graphs, Ollivier's Ricci curvature κ_p of an edge is defined via the Wasserstein distance between two probability measure around the two vertices of the edge, depending on an idleness parameter $p \in [0,1]$. In 2011, Lin et al.[7] modified Ollivier's notion by taking the minus of the derivative of κ_p at $p=1$. We will study this modified Ricci curvature, which will be referred to as the Lin-Lu-Yau curvature, on amply regular graphs in this paper.

Let $G=(V,E)$ be a locally finite connected simple graph. Recall that the girth of G is the length of its shortest cycle. We denote by $d(x,y)$ the length of the shortest path connecting the two vertices x and y. We call $\mu: V \to [0,1]$ a probability measure on the graph $G=(V,E)$ if $\sum_{v\in V} \mu(v)=1$.

Definition 1.1 Let $G=(V,E)$ be a locally finite graph, μ_1 and μ_2 be two probability measures on G. The Wasserstein distance $W_1(\mu_1, \mu_2)$ between μ_1 and μ_2 is defined as

$$W_1(\mu_1,\mu_2) = \inf_{\pi} \sum_{y\in V}\sum_{x\in V} d(x,y)\pi(x,y),$$

where the infimum is taken over all maps $\pi: V\times V \to [0,1]$ satisfying

$$\mu_1(x) = \sum_{y\in V}\pi(x,y),\ \mu_2(y) = \sum_{x\in V}\pi(x,y).$$

Such a map is called a transport plan.

We consider the following particular measure around a vertex $x \in V$:

$$\mu_x^p(y) = \begin{cases} p, & \text{if } y = x; \\ \dfrac{1-p}{\deg(x)}, & \text{if } y \sim x; \\ 0, & \text{otherwise}, \end{cases}$$

where $\deg(x) := \sum_{y\in V; y\sim x} 1$ is the vertex degree of x.

Definition 1.2[2,7] Let $G=(V,E)$ be a locally finite graph. For any vertice $x, y \in V$, the p-Ollivier-Ricci curvature $\kappa_p(x,y)$, $p\in[0,1]$, is defined as

$$\kappa_p(x,y) = 1 - \frac{W_1(\mu_x^p,\mu_y^p)}{d(x,y)}.$$

The Lin-Lu-Yau curvature $\kappa(x,y)$ is defined as

$$\kappa(x,y) = \lim_{p\to 1}\frac{\kappa_p(x,y)}{1-p}.$$

Notice that $\kappa_1(x,y)$ is always 0. Hence, the Lin-Lu-Yau curvature $\kappa(x,y)$ is minus the derivative of $\kappa_p(x,y)$ at $p=1$.

Bourne et al.[9] studied the relation between p-Ollivier-Ricci curvature and Lin-Lu-Yau curvature. In particular, they proved that for an edge $xy \in E$ with $\deg(x)=\deg(y)=d$

$$\kappa(x,y) = \frac{d+1}{d}\kappa_{\frac{1}{d+1}}(x,y) \qquad (1)$$

The Lin-Lu-Yau curvature has been computed or estimated on graphs with further regularity assumptions. For regular graphs (i.e., every vertex has the same

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degree), the following upper bound estimate is known.

Theorem 1.1[10] Let $G=(V,E)$ be a d-regular graph. For any edge $xy\in E$, we have

$$\kappa(x,y)\leqslant\frac{2+|\Delta_{xy}|}{d},$$

where $\Delta_{xy}:=\Gamma(x)\cap\Gamma(y)$, $\Gamma(x):=\{z\in V|\ xz\in E\}$, and $\Gamma(y):=\{z\in V|\ yz\in E\}$.

Bonini et al.[11] derived Lin-Lu-Yau curvature formulas for strongly regular graphs in terms of the graph parameters and the size of a maximal matching in the so-called core neighborhood. In fact, a more general curvature formulas for regular graphs has been shown in Reference [12, Theorem 2.6]. In particular, their result leads to exact formulas for the Lin-Lu-Yau curvature for strongly regular graphs with girth 4 and 5. For the case of girth 3, no exact formula for the Lin-Lu-Yau curvature purely in terms of graph parameter exists: The 4×4 Rook's graph and Shrikhande graph are both strongly regular with the parameter $(16,6,2,2)$; Bonini et al.[11] computed their Lin-Lu-Yau curvature to be $\kappa=\frac{2}{3}$ and $\kappa=\frac{1}{3}$ respectively.

We study the Lin-Lu-Yau curvature of amply regular graphs with girth 3 or 4 in this paper.

Definition 1.3 (Amply regular graph[15]) We call a d-regular graph with n vertices an amply regular graph with parameter (n,d,α,β) if the following holds true:

(i) Any two adjacent vertices have α common neighbors.

(ii) Any two vertices with distance 2 have β common neighbors.

We remark that if the above property (ii) holds for any two non-adjacent vertices, the amply regular graph is strongly regular. Therefore, amply regularity is a relaxation of the strongly regularity.

For amply regular graphs with girth 4 we have the following Lin-Lu-Yau curvature formula.

Theorem 1.2 Let $G=(V,E)$ be an amply regular graph with parameter (n,d,α,β) with girth 4. For any $xy\in E$, we have

$$\kappa(x,y)=\frac{2}{d}.$$

This formula has been established for the particular cases of strongly regular graphs with girth 4 and distance regular graphs with girth 4 in Refs. [11] and [13], respectively. Observe that the Lin-Lu-Yau curvature of a given edge only involves number of common neighbors of vertices with distance at most 2. Therefore, the proofs of [11] and [13] can be adopted directly to amply regular graphs with girth 4.

Our main result is the following Lin-Lu-Yau curvature formula or estimates for amply regular graphs with girth 3 (i.e., $\alpha\geqslant1$).

Theorem 1.3 Let $G=(V,E)$ be an amply regular graph with parameter (n,d,α,β).

(i) If $\alpha=1$ and $\alpha<\beta$, then we have for any $xy\in E$ that

$$\kappa(x,y)=\frac{3}{d}.$$

(ii) If $\alpha\geqslant1$ and $\alpha=\beta-1$, then we have for any $xy\in E$ that

$$\kappa(x,y)\geqslant\frac{2}{d}.$$

(iii) If $\alpha=\beta>1$, then we have for any $xy\in E$ that

$$\kappa(x,y)\geqslant\frac{2}{d}.$$

Remark 1.1 ① Consider the 9-Paley graph which is strongly regular with parameter $(9,4,1,2)$. It fulfills the parameter restrictions in Theorem 1.3(i) and (ii). We can check directly the Lin-Lu-Yau curvature of the 9-Paley graph is $\frac{3}{4}$.

② Consider the Shrikhande graph which is strongly regular with parameter $(16,6,2,2)$. It fulfills the parameter restriction in Theorem 1.3(iii). We can check directly the Lin-Lu-Yau curvature of the Shrikhande graph is $\frac{1}{3}$. Therefore, the estimate in Theorem 1.3(iii) is sharp.

③ Bonini et al.[11, Conjecture 1.7] conjectured that the Lin-Lu-Yau curvature of any strongly regular conference graphs with parameter $(4\gamma+1,2\gamma,\gamma-1,\gamma)$ with $\gamma\geqslant2$ satisfies

$$\kappa(x,y)=\frac{1}{2}+\frac{1}{2\gamma},\ \text{for all } xy\in E.$$

Theorem 1.3(ii) implies that for such conference graphs

$$\kappa(x,y)\geqslant\frac{1}{\gamma},\ \text{for all } xy\in E.$$

④ In general, it is still open whether the Lin-Lu-Yau curvature of an amply regular graph of parameter (n,d,α,β) with girth 3 (i.e., $\alpha\geqslant1$) and $\beta\geqslant2$ is always nonnegative or not.

For Ollivier's Ricci curvature, a Bonnet-Myers type diameter estimate holds true[2]: Uniformly positive curvature lower bound implies the finiteness of the diameter. This has been extended to the Lin-Lu-Yau curvature.

Theorem 1.4 (discrete Bonnet-Myers theorem[7]) Let $G=(V,E)$ be a locally finite connected graph. Suppose $\kappa(x,y)\geqslant k>0$ holds true for any $xy\in E$. Then the diameter

$$\mathrm{diam}(G)\leqslant\frac{2}{k}.$$

Discrete Bonnet-Myers theorem has recently found important applications in coding theory: It provides a

completely elementary way to derive bounds on locally correctable and some locally testable binary linear codes[14].

Applying Theorem 1.4, we have the following consequences.

Corollary 1.1 Let $G=(V,E)$ be an amply regular graph with parameter (n,d,α,β).

(i) If G has girth 4, then
$$\operatorname{diam}(G)\leqslant d.$$
(ii) If $\alpha=1$ and $\alpha<\beta$, then
$$\operatorname{diam}(G)\leqslant\frac{2d}{3}.$$
(iii) If $\alpha\geqslant 1$ and $\alpha=\beta-1$, then
$$\operatorname{diam}(G)\leqslant d.$$
(iv) If $\alpha=\beta>1$, then
$$\operatorname{diam}(G)\leqslant d.$$

The following diameter estimate for amply regular graphs is known via classical combinatorial methods, see, e.g., Ref. [15, Theorem 1.13.2].

Theorem 1.5[15] Let $G=(V,E)$ be an amply regular graph with parameter (n,d,α,β). If $\alpha\leqslant\beta\neq 1$, then
$$\operatorname{diam}(G)\leqslant d,$$
where the equality holds if and only if G is a d-hypercube graph.

Remark 1.2 Corollary 1.1 provides a geometric proof via Lin-Lu-Yau curvature for Theorem 1.5 in the case of girth 4 and some special cases of girth 3. Moreover, we improve the estimate in Theorem 1.5 under the condition of Corollary 1.1(ii).

2 Preliminaries

We first recall the important concept of matching from graph theory.

Definition 2.1[16, Section 16.1] Let $G=(V,E)$ be a locally finite simple connected graph. A set M of pairwise nonadjacent edges is called a matching. The two vertices of each edge of M are said to be matched under M, and each vertex adjacent to an edge of M is said to be covered by M. A matching M is called a perfect matching if it covers every vertex of the graph.

The following Hall's marriage theorem will be an important tool for our purpose.

Lemma 2.1[16, Theorem 16.4] Let $H=(V,E)$ be a bipartite graph with the bipartition $V=S\sqcup T$. Then H has a matching which covers every vertex in S if and only if
$$|\Gamma_T(W)|\geqslant|W|\ \text{for all}\ W\subseteq S$$
holds, where $\Gamma_T(W):=\{v\in T\mid\text{there exists } w\in W \text{ such that } vw\in E\}$.

For any $xy\in E$, the Lin-Lu-Yau curvature $\kappa(x,y)$ only depends on the subgraph induced by vertices with the distance less than or equal to 2 to x and y[6, Lemma 2.3]. For convenience, we introduce the following notation of the core neighborhood of $xy\in E$:
$$C_{xy}=\{x\}\cup\{y\}\cup\Delta_{xy}\cup N_x\cup N_y\cup P_{xy},$$
where
$$N_x=\Gamma(x)\setminus(\{y\}\cup\Gamma(y)),$$
$$N_y=\Gamma(y)\setminus(\{x\}\cup\Gamma(x)),$$
$$P_{xy}=\{z\in V\mid d(x,z)=2,\ d(y,z)=2\}.$$

3 Proof of Theorem 1.3

In this section, we prove Theorem 1.3.

Proof of Theorem 1.3(i) We consider the core neighborhood decomposition of an edge $xy\in E$, that is, $\Gamma(x)=\{y\}\cup\Delta_{xy}\cup N_x$, $\Gamma(y)=\{x\}\cup\Delta_{xy}\cup N_y$. Since $\alpha=1$, we can denote $\Delta_{xy}=\{x_0\}$, and there are no edges connecting x_0 and any vertex in N_x or N_y. We are going to show the existence of a perfect matching between N_x and N_y via applying Lemma 2.1.

Let H be the bipartite subgraph with vertex set $V_H:=N_x\sqcup N_y$ and edge set $E_H:=\{vw\in E\mid v\in N_x,\ w\in N_y\}$. Take a subset $A\subset N_x$, let $B:=\Gamma_{N_y}(A)$ be the set of neighbors of A in N_y. Observe that for any vertex $x_i\in A$, we have $d(x_i,y)=2$. Therefore, there are β common neighbors of x_i and y. Since x is a common neighbor of x_i and y, there are $\beta-1$ neighbors of x_i in N_y. Similarly, for any vertex $y_i\in B\subset N_y$, there are $\beta-1$ neighbors of y_i in N_x. Denote by
$$E(A,B):=\{xy\in E\mid x\in A,\ y\in B\}.$$
We then have
$$\sum_{v\in A}(\beta-1)=|E(A,B)|\leqslant\sum_{w\in B}(\beta-1).$$
Since $\beta>1$, we derive from above that
$$|A|\leqslant|B|=|\Gamma_{N_y}(A)|.$$

Applying Lemma 2.1, there is a perfect matching M between N_x and N_y. We construct the following transport plan building upon such a perfect matching:
$$\pi(v,w):=\begin{cases}0, & v=x, w=y;\\ \dfrac{1}{d+1}, & v=w\in\Delta_{xy}\cup\{x\}\cup\{y\};\\ \dfrac{1}{d+1}, & vw\in M;\\ 0, & \text{otherwise.}\end{cases}$$
Noticing that $|N_x|=|N_y|=d-\alpha-1$, we calculate the Wasserstein distance
$$W_1(\mu_x^{\frac{1}{d+1}},\mu_y^{\frac{1}{d+1}})\leqslant\sum_{v\in V}\sum_{w\in V}d(v,w)\pi(v,w)=\frac{|N_x|}{d+1}=\frac{d-\alpha-1}{d+1}=\frac{d-2}{d+1}.$$
Applying Eq. (1), we have the Lin-Lu-Yau curvature
$$\kappa(x,y)=\frac{d+1}{d}\kappa_{\frac{1}{d+1}}(x,y)=\frac{d+1}{d}(1-W_1(\mu_x^{\frac{1}{d+1}},\mu_y^{\frac{1}{d+1}}))\geqslant\frac{3}{d}.$$

Combining with Theorem 1.1, we obtain

$$\kappa(x,y) = \frac{3}{d}.$$

Proof of Theorem 1.3 (ii) We construct a bipartite graph H from the core neighborhood of $xy \in E$ as follows. Denote

$$\Delta_{xy} = \{z_1, \cdots, z_\alpha\}.$$

We add a new set of vertices $\Delta'_{xy} := \{z'_1, \cdots, z'_\alpha\}$ which is considered as a copy of Δ_{xy}. Let H be the bipartite graph with vertex set

$$V_H = N_x \cup \Delta_{xy} \cup \Delta'_{xy} \cup N_y$$

and edge set

$$E_H = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5,$$

where

$$E_1 = \{vw \mid v \in N_x, w \in N_y, vw \in E\},$$
$$E_2 = \{vz'_i \mid v \in N_x, z'_i \in \Delta'_{xy}, vz_i \in E\},$$
$$E_3 = \{z_i w \mid z_i \in \Delta_{xy}, w \in N_y, z_i w \in E\},$$
$$E_4 = \{z_i z'_i \mid i = 1, \cdots, \alpha\},$$
$$E_5 = \{z_i z'_j \mid z_i z_j \in E, i \neq j, 1 \leqslant i, j \leqslant \alpha\}.$$

Notice that in the above construction, edges only exist between $N_x \cup \Delta_{xy}$ and $N_y \cup \Delta'_{xy}$. We will show that H has a perfect matching by Lemma 2.1.

Take a subset $A \subset N_x \cup \Delta_{xy}$. Let $B = \Gamma_{N_y \cup \Delta'_{xy}}(A)$ be the neighbors of A in $N_y \cup \Delta'_{xy}$.

① The case $A \subset N_x$. Similarly as in the proof of Theorem 1.3 (i), there are $\beta - 1 \geqslant 1$ neighbors of any $v \in A$ in $N_y \cup \Delta'_{xy}$, and $\beta - 1 \geqslant 1$ neighbors of any $w \in B \cap N_y$ in $N_x \cup \Delta_{xy}$. Consider any vertex $z'_i \in B \cap \Delta'_{xy}$. The corresponding vertex $z_i \in \Delta_{xy}$ and x have α common neighbors in G including the vertex y. Since there is a new additional edge between z_i and z'_i in H, the vertex z'_i has exactly α neighbors in $N_x \cup \Delta_{xy}$ in H. Therefore, we derive

$$\sum_{v \in A} (\beta - 1) = |E(A,B)| \leqslant \sum_{w \in B \cap N_y} (\beta - 1) + \sum_{w \in B \cap \Delta'_{xy}} \alpha.$$

By assumption, we have $\alpha = \beta - 1$. Hence the above estimate yields

$$|A| \leqslant |B| = |\Gamma_{N_y \cup \Delta'_{xy}}(A)|.$$

② The case $A \subset \Delta_{xy}$. Consider any vertex $z_i \in A$. The vertices z_i and y has α common neighbors in G including the vertex x. Since there is a new additional edge between z_i and $z'_i \in \Delta'_{xy}$ in H, the vertex z_i has exactly α neighbors in $N_y \cup \Delta'_{xy}$ in H. Therefore, we derive

$$\sum_{v \in A} \alpha = |E(A,B)| \leqslant \sum_{w \in B \cap N_y} (\beta - 1) + \sum_{w \in B \cap \Delta'_{xy}} \alpha.$$

Using $\alpha = \beta - 1$, we obtain

$$|A| \leqslant |B| = |\Gamma_{N_y \cup \Delta'_{xy}}(A)|.$$

③ The case $A \subset N_x \cup \Delta_{xy}$. We have

$$\sum_{v \in A \cap N_x} (\beta - 1) + \sum_{v \in A \cap \Delta_{xy}} \alpha = |E(A,B)| \leqslant \sum_{w \in B \cap N_y} (\beta - 1) + \sum_{w \in B \cap \Delta'_{xy}} \alpha.$$

By $\alpha = \beta - 1$, we obtain

$$|A| \leqslant |B| = |\Gamma_{N_y \cup \Delta'_{xy}}(A)|.$$

In conclusion, we have $|A| \leqslant |B| = |\Gamma_{N_y \cup \Delta'_{xy}}(A)|$ for any subset $A \subset N_x \cup \Delta_{xy}$. Then, Lemma 2.1 implies the existence of a perfect matching of H. Therefore, there exists a transport plan between $\mu_x^{\frac{1}{d+1}}$ and $\mu_y^{\frac{1}{d+1}}$ in which the mass at any vertex in $N_x \cup \Delta_{xy}$ is moved by a distance at most 1. And the mass at $\{x\}$ and $\{y\}$ stay put. Using such a transport plan, we estimate the Wasserstein distance

$$W_1(\mu_x^{\frac{1}{d+1}}, \mu_y^{\frac{1}{d+1}}) \leqslant \sum_{v \in V} \sum_{w \in V} d(v,w)\pi(v,w) \leqslant \frac{d-1}{d+1}.$$

Applying Eq. (1), we have the Lin-Lu-Yau curvature

$$\kappa(x,y) = \frac{d+1}{d}\kappa_{\frac{1}{d+1}}(x,y) =$$
$$\frac{d+1}{d}(1 - W_1(\mu_x^{\frac{1}{d+1}}, \mu_y^{\frac{1}{d+1}})) \geqslant \frac{2}{d}.$$

Proof of Theorem 1.3 (iii) We modify the construction of the bipartite graph H in the proof of Theorem 1.3 (ii) by dropping the edge set E_4. That is, H is the graph with vertex set $V_H = N_x \cup \Delta_{xy} \cup \Delta'_{xy} \cup N_y$ and edge set $E_H = E_1 \cup E_2 \cup E_3 \cup E_5$.

Take a subset $A \subset N_x \cup \Delta_{xy}$. Let $B = \Gamma_{N_y \cup \Delta'_{xy}}(A)$ be the neighbors of A in $N_y \cup \Delta'_{xy}$. Similarly as the analysis in the proof of Theorem 1.3 (ii), we have

$$\sum_{v \in A \cap N_x} (\beta - 1) + \sum_{v \in A \cap \Delta_{xy}} (\alpha - 1) = |E(A,B)| \leqslant \sum_{w \in B \cap N_y} (\beta - 1) + \sum_{w \in B \cap \Delta'_{xy}} (\alpha - 1).$$

By assumption, we have $\alpha = \beta > 1$. Then we derive from above that

$$|A| \leqslant |B| = |\Gamma_{N_y \cup \Delta'_{xy}}(A)|.$$

Therefore, there is a perfect matching of H by Lemma 2.1. Similarly as in the proof of Theorem 1.3 (ii), we derive

$$\kappa(x,y) \geqslant \frac{2}{d}.$$

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Conflict of interest

The authors declare no conflict of interest.

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References

[1] Chung F R K, Yau S T. Logarithmic Harnack inequalities. Math. Res. Lett., 1996,3(6): 793-812.
[2] Ollivier Y. Ricci curvature of Markov chains on metric spaces. J Funct Anal, 2009, 256(3): 810-864.
[3] Ollivier Y. A survey of Ricci curvature for metric spaces and Markov chains. In: Probabilistic Approach to Geometry. Tokyo: Math. Soc. Japan, 2010: 343-381.
[4] Bauer F, Jost J, Liu S. Ollivier-Ricci curvature and the spectrum of the normalized graph Laplace operator. Math. Res. Lett., 2012, 19(6): 1185-1205.
[5] Jost J, Liu S. Ollivier's Ricci curvature, local clustering and curvature-dimansioin inequalities on graphs. Discrete Comput. Goem., 2014, 51(2): 300-322.
[6] Bhattacharya B B, Mukherjee S. Exact and asymptotic results on coarse Ricci curvature of graphs. Discrete Math., 2015, 338(1): 23-42.
[7] Lin Y, Lu L, Yau S T. Ricci curvature of graphs. Tohoku Math. J., 2011, 63: 605-627.
[8] Lin Y, Yau S T. Ricci curvature and eigenvalue estimate on locally finite graphs. Math. Res. Lett., 2010, 17(2): 343-356.
[9] Bourne D, Cushing D, Liu S, F, et al. Ollivier-Ricci idleness functions of graphs. SIAM J. Discrete Math., 2018, 32(2): 1408-1424.
[10] Cushing D, Kamtue S, Koolen J, et al. Rigidity of the Bonnet-Myers inequality for graphs with respect to Ollivier Ricci curvature. Adv. Math., 2020, 369: 107188.
[11] Bonini V, Carroll C, Dinh U, et al. Condensed Ricci curvature of complete and strongly regular graphs. Involve, 2020, 13 (4): 559-576.
[12] Münch F, Wojciechowski R. Ollivier Ricci curvature for general graph Laplacians: Heat equation, Laplacian comparison, non-explosion and diameter bounds. Adv. Math., 2019, 356: 106759.
[13] Cushing D, Kamtue S, Kangaslampi R, et al. Curvatures, graph products and Ricci flatness. J. Graph Theory, 2021, 96(4): 522-553.
[14] Iceland E, Samorodnitsky A. On coset leader graphs of structured linear codes. Discrete Comput. Geom., 2020, 63(6): 560-576.
[15] Brouwer A E, Cohen A M, Neumaier A. Distance-Regular Graphs. Berlin: Springer, 1989.
[16] Bondy J A, Murty U S R. Graph Theory. Berlin: Springer, 2008.

Amply regular 图的林-陆-丘曲率和直径

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摘要: 利用 Hall 匹配定理,研究了在不同参数限制条件下围长为 3 或 4 的 amply regular 图的林-陆-丘曲率下界估计. 作为推论,我们证明每一个会议图均有正的林-陆-丘曲率. 我们的方法在围长为 4 以及一些特殊的围长为 3 情形为 amply regular 图的一个经典直径估计提供了几何证明.

关键词: amply regular 图;完美匹配; 最优传输距离;林-陆-丘曲率