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Ore-type condition for loose Hamilton cycles in 3-uniform hypergraphs

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Abstract: A classic result of Ore in 1960 states that if the degree sum of any two independent vertices in an n-vertex graph is at least n, then the graph is Hamiltonian. Here a similar problem for 3-uniform hypergraph was studied and an approximate result was obtained.

Key words: Ore-type condition; hypergraph; Hamilton cycle; degree

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3一致超图中线性哈密顿圈的Ore条件

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摘要:1960年,Ore证明了如下结果:如果一个n点图的任意两个独立点的度和不小于n,那么它包含一个哈密顿圈.将这个结果推广到3一致超图上,并得到一个近似最优的结果.

关键词:Ore条件;超图;哈密顿圈;度

0 Introduction

Given $k \geqslant 2$, a k-uniform hypergraph (k-graph for short) consists of a vertex set V and an edge set $E \subseteq \binom{V}{k}$. For $1 \leqslant \ell \leqslant k-1$, a k-graph is called an ℓ-cycle if its vertices can be ordered cyclically so that each of its edges consists of k consecutive vertices and every two consecutive edges share exactly ℓ vertices. A $(k-1)$-cycle is often called a tight cycle while a 1-cycle is often called a loose cycle. We say that a k-graph contains a Hamilton ℓ-cycle if it contains an ℓ-cycle as a spanning subhypergraph.

For a set $A \subseteq V(H)$, let $H[A]$ be the subgraph induced by A. For two vertex sets S, $R \subseteq V(H)$ with $|S| < k$, let $N_H(S,R) = \{T : T \subseteq R$ such that $S \cup T \in E(H)\}$ and $\deg_H(S,R) = |N_H(S,R)|$. By the definitions here, $\deg_H(S) = \deg_H(S, V(H))$. If $S = \{v\}$, write $\deg_H(v,R)$ for $\deg_H(\{v\},R)$. The minimum d-degree $\delta_d(H)$ of H is the minimum of $\deg_H(S)$ over all d-element vertex sets S in H. The subscript will be omitted if the underlying hypergraph is clear from the context.

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Ore[1] showed that a graph G with $\sigma_2(G)\geqslant n$ is Hamiltonian, here $\sigma_2(G)=\min\{d(u)+d(v): u,v$ are independent$\}$. Tang and Yan[2] generalized this theorem to k-graphs.

Definition 0.1 Let H be a k-graph, $S,T\in\binom{V}{k-1}$, we say S, T are independent if there exists no such edge e that $S\cup T\subset e$. Set

$$\sigma_2(H)=\min\{\deg(S)+\deg(T): S,T\in\binom{V(H)}{k-1}, S \text{ and } T \text{ are independent}\}.$$

Theorem 0.1[2] For every $k\geqslant 3$, $\gamma>0$, there exists n_0 such that every $n\geqslant n_0$ vertices k-graph H with $\sigma_2(H)\geqslant(1+\gamma)n$ contains a tight Hamilton cycle.

We study a similar problem by considering loose Hamilton cycles in 3-graphs.

Theorem 0.2[3] There is an integer n_0 such that every 3-graph H on $n\geqslant n_0$ vertices with n even and $\delta_2(H)\geqslant n/4$ contains a loose Hamilton cycle.

Theorem 0.3[3] Let $\gamma>0$, then there exists n_0 such that every $n\geqslant n_0$ vertices 3-graph H with n even and $\sigma_2(H)\geqslant(1/2+\gamma)n$ contains a loose Hamilton cycle.

This bound is approximately best possible since it was shown in Ref. [3] that there exists a 3-graph H on n vertices with $\delta_2(H)\geqslant(n-2)/4$, which contains no loose Hamilton cycle.

To prove Theorem 0.3, we will use the so-called absorbing method.

We say that a 3-graph H is a (loose) path if its vertices can be ordered as $v_1, v_2,\cdots,v_{2m+1}$ so that $E(H)=\{\{v_{2i+1},v_{2i+2},v_{2i+3}\}: i=0,\cdots,m-1\}$ with endpoints $\{v_1,v_{2m+1}\}$.

A path P with endpoints v_1 and v_2 is said to absorb $U\subseteq V\backslash V(P)$ if there is a path Q in H with endpoints v_1 and v_2 and such that $V(Q)=V(P)\cup U$.

1 Proof of Theorem 0.3

To prove Theorem 0.3, we need the following lemma.

Lemma 1.1 Given $\gamma>0$, let H be an n vertices 3-graph with $\sigma_2(H)\geqslant\left(\frac{1}{2}+\gamma\right)n$ and with n sufficiently large, then for any $K\in\binom{V(H)}{2}$ with $\deg(K)<n/4$ and $0<\alpha<1/40000$, there exists a path P in H such that

(Ⅰ) $|V(P)|\leqslant\alpha n$.

(Ⅱ) $K\subset V(P)$, $\{u,v\}\cap K=\emptyset$, where u,v are the endpoints of P.

(Ⅲ) Any vertex set $U\subset V\backslash V(P)$ with $|U|\leqslant\alpha^2 n$, $|U|\in 2\mathbb{N}$ can be absorbed by P.

The following lemma provides a collection of paths covering nearly all the vertices in hypergraph.

Lemma 1.2(Path cover lemma[4]) For every $\gamma,\varepsilon>0$, there exist n_0, p such that every 3-graph $H=(V,E)$ on $n>n_0$ vertices with $\delta_2(H)\geqslant(1/4+\gamma)n$ the following holds. There is a family of disjoint paths $P_1,\cdots,P_q (q\leqslant p)$, which covers all but at most εn vertices of H.

The following lemma is used to connect the paths into one path.

Lemma 1.3(Reservoir Lemma[4]) For every $d,\varepsilon>0$, there exists n_0 such that every 3-graph $H=(V,E)$ on $n>n_0$ vertices with $\delta_2(H)\geqslant dn$ the following holds. There is a set R of size at most εn such that for all 2-set $S\in\binom{V}{2}$, we have $\deg(S,R)\geqslant d\varepsilon n/2$.

Proof of Theorem 0.3 If $\deg(S)\geqslant n/4$ for all $S\in\binom{V(H)}{2}$, then by Theorem 0.2 H contains a loose Hamilton cycle. So we can fix some $K\in\binom{V(H)}{2}$ with $\deg(K)<n/4$.

Let P be the path guaranteed by lemma 1.1 (applied with $\alpha<\gamma/4$), let u,v be the endpoints of P. Let $V'=(V\backslash V(P))\cup\{u,v\}$ and let $H'=H[V']$ be the induced subhypergraph of H on V'. Then

$$\delta_2(H')>\left(\frac{1}{4}+\gamma\right)n-\alpha n>\left(\frac{1}{4}+\frac{3\gamma}{4}\right)n.$$

Due to Lemma 1.3 with $d=1/4,\varepsilon=\alpha^2/2$, we can choose a set $R\subset V'\backslash\{u,v\}$ of size at most $\alpha^2 n/2$ such that for every $S\in\binom{V'}{2}$,

$$\deg(S,R)\geqslant\frac{\alpha^2 n}{16}-2.$$

Set $V''=V\setminus(V(P)\cup R)$ and let $H''=H[V'']$ be the induced subhypergraph of H on V'', then

$$\delta_2(H'')>\left(\frac{1}{4}+\gamma\right)n-\alpha n-\frac{\alpha^2 n}{2}>\left(\frac{1}{4}+\frac{\gamma}{2}\right)n.$$

Lemma 1.2 applied to H'' with $\varepsilon=\alpha^2/2$ yields a family of disjoint paths $P_1,\cdots,P_q$, which covers all but at most $(\alpha^2 n)/2$ vertices of H''. Use T to denote the set of the uncovered vertices in V''.

Let $P_0:=P$, and let P_i^b,P_i^e be the endpoints of $P_i(i=0,1,\cdots,q)$. For sufficiently large n,

$$\deg(\{P_i^e,P^b_{(i+1)(\mathrm{mod}(q+1))}\},R)\geqslant\frac{\alpha^2 n}{16}-2>q+1.$$

Therefore for each $i\in\{0,1,\cdots,q\}$, we can choose a vertex $x_i\in R\setminus(\bigcup_{0\leqslant j<i}x_j)$ such that $P_i^e x_i P^b_{(i+1)(\mathrm{mod}\,q+1)}$ is an unused edge. Hence, we can connect all these paths to form a loose cycle $\mathfrak{C}$.

Let $U=V\setminus V(\mathfrak{C})$ be the set of vertices not covered by $\mathfrak{C}$, then $U\subseteq R\cup T$ and

$$|U|\leqslant\frac{\alpha^2 n}{2}+\frac{\alpha^2 n}{2}=\alpha^2 n.$$

Since $\mathfrak{C}$ is a loose cycle and $n\in 2\mathbb{N}$, we have $|U|\in 2\mathbb{N}$. So P absorbs U to obtain a loose Hamilton cycle of H.

2 Proof of Lemma 1.1

Proof of Lemma 1.1 For $S\in\binom{V}{2}$, a 5-set $T=\{u_1,\cdots,u_5\}$ is said to absorb S if there is a path in $H[T]$ on five vertices and a path in $H[S\cup T]$ on seven vertices both with endpoints u_1 and u_5.

Claim 2.1 For any 2-set $S\in\binom{V}{2}$ with $S=K$ or $S\cap K=\varnothing$, there are at least $\binom{n-2}{5}/500$ 5-set $T\in\binom{V\setminus K}{5}$ that absorb S.

Proof Set $S=\{v_1,v_2\}$.

Case 2.1 $S\cap K=\varnothing$.

For any 2-set $S'\in\binom{V\setminus K}{2}$, since S' and K are independent, we have $\deg(S')>(1/4+\gamma)n$.

Set $V'=V\setminus K$, we can select $T=\{u_1,u_2,u_3,u_4,u_5\}$ as follows. Let $u_1\in V'\setminus S$ be an arbitrary vertex, then u_1 has $n-4$ choices; select $u_2\in(N(u_1,v_1)\setminus\{v_2\})\cap V'$, then u_2 has $(1/4+\gamma)n-3$ choices; select $u_3\in(N(u_1,u_2)\setminus\{v_1,v_2\})\cap V'$, then u_3 has $(1/4+\gamma)n-4$ choices; select $u_4\in(N(u_2,v_2)\setminus\{v_1,u_1,u_3\})\cap V'$, then u_4 has $(1/4+\gamma)n-5$ choices; select $u_5\in(N(u_3,u_4)\setminus\{v_1,v_2,u_1,u_2\})\cap V'$, then u_5 has $(1/4+\gamma)n-6$ choices. There are at least

$$(n-4)\left(\left(\frac{1}{4}+\gamma\right)n-3\right)\left(\left(\frac{1}{4}+\gamma\right)n-4\right)\cdot\left(\left(\frac{1}{4}+\gamma\right)n-5\right)\left(\left(\frac{1}{4}+\gamma\right)n-6\right)/5!\ \geqslant\binom{n-2}{5}/500$$

choices for T. We have $u_1u_2u_3,u_3u_4u_5\in E(H)$ and $u_1v_1u_2,u_2v_2u_4,u_4u_3u_5\in E(H)$, so $T=\{u_1,u_2,u_3,u_4,u_5\}$ is indeed a 5-set which absorb S.

Case 2.2 $S=K$.

We select $T=\{u_1,u_2,u_3,u_4,u_5\}$ as follows. Firstly we select u_2 from $V\setminus(N(S)\cup S)$, then $\{u_2,v_1\},\{u_2,v_2\}$ are both independent with S, so $\deg(u_2,v_1),\deg(u_2,v_2)>(1/4+\gamma)n$. Select $u_1\in N(u_2,v_1)\setminus\{v_2\}$, we have $(1/4+\gamma)n$ choices; select $u_3\in N(u_2,u_1)\setminus\{v_1,v_2\}$, we have $(1/4+\gamma)n-2$ choices; select $u_4\in N(u_2,v_2)\setminus\{v_1,u_1,u_3\}$, we have $(1/4+\gamma)n-2$ choices; select $u_5\in N(u_3,u_4)\setminus\{v_1,v_2,u_1,u_2\}$, we have $(1/4+\gamma)n-4$ choices, so there are at least

$$\left(n-2-\frac{n}{4}\right)\left(\frac{1}{4}+\gamma\right)n\left(\left(\frac{1}{4}+\gamma\right)n-2\right)\cdot\left(\left(\frac{1}{4}+\gamma\right)n-2\right)\left(\left(\frac{1}{4}+\gamma\right)n-4\right)\backslash 5!\ \geqslant\binom{n-2}{5}/500$$

choices for T. We have $u_1u_2u_3,u_3u_4u_5\in E(H)$ and $u_1v_1u_2,u_2v_2u_4,u_4u_3u_5\in E(H)$, So $T=\{u_1,u_2,u_3,u_4,u_5\}$ is indeed a 5-set which absorb S.

From now on, whenever we mention a 5-set absorb a pair $v_1,v_2\in V$, we always assume that $\{v_1,v_2\}=K$ or $\{v_1,v_2\}\cap K=\varnothing$.

Given two distinct vertices $v_1,v_2\in V$, let $A(v_1,v_2)$ be the family of 5-sets $T\in\binom{V\setminus K}{5}$ that absorb $\{v_1,v_2\}$. By Claim 2.1,

$$|A(v_1,v_2)|\geqslant\binom{n-2}{5}/500.$$

Let $\mathfrak{F}$ be a family obtained by selecting every 5-set from $\binom{V\setminus K}{5}$ independently with probability

$$p=\frac{\alpha n}{20\binom{n-2}{5}}.$$

Then $|\mathfrak{F}|$ satisfies binomial distribution $B\left(\binom{n-2}{5},p\right)$, so its expectation is

$$\mathbb{E}[|\mathfrak{F}|]=\binom{n-2}{5}p=\frac{\alpha n}{20},$$

the variation of $|\mathfrak{F}|$ is

$$\mathrm{Var}[|\mathfrak{F}|]=\binom{n-2}{5}p(1-p).$$

By the Chebyshev's inequality,

$$\mathbb{P}(|\mathfrak{F}|-\mathbb{E}[|\mathfrak{F}|]\geqslant\mathbb{E}[|\mathfrak{F}|])\leqslant \frac{\mathrm{Var}[|\mathfrak{F}|]}{\mathbb{E}^2[|\mathfrak{F}|]}=\frac{1-p}{\binom{n-2}{5}p}=o(1),$$

which implies that

$$\mathbb{P}\left(|\mathfrak{F}|\geqslant\frac{\alpha n}{10}\right)=\mathbb{P}(|\mathfrak{F}|\geqslant 2\mathbb{E}[|\mathfrak{F}|])\leqslant o(1).$$

For any $v_1,v_2\in V$, let $X_{\{v_1,v_2\}}$ be the number of 5-sets in $\mathfrak{F}$ which absorb $\{v_1,v_2\}$. Then $X_{\{v_1,v_2\}}$ has binomial distribution and

$$\mathbb{E}[X_{\{v_1,v_2\}}]=\mathbb{E}[|A(v_1,v_2)\cap\mathfrak{F}|]\geqslant \frac{\binom{n-2}{5}}{500}p\geqslant\frac{\alpha n}{10000}>0,$$

by the Chernoff's bound,

$$\mathbb{P}\left(X_{\{v_1,v_2\}}\leqslant\frac{\mathbb{E}[X_{\{v_1,v_2\}}]}{2}\right)\leqslant \exp\left(-\frac{\mathbb{E}[X_{\{v_1,v_2\}}]}{8}\right)=o(1).$$

Let Y be the number of intersecting pairs of 5-sets in $\mathfrak{F}$, then

$$\mathbb{E}[Y]\leqslant 5\binom{n-2}{5}\binom{n-2}{4}p^2\leqslant\frac{\alpha^2 n}{10},$$

by the Markov's inequality, we have

$$\mathbb{P}(Y>2\mathbb{E}[Y])<\frac{\mathbb{E}[Y]}{2\,\mathbb{E}[Y]}=\frac{1}{2}.$$

Therefore, with positive probability, there exists a family $\mathfrak{F}$, such that

(Ⅰ) $|\mathfrak{F}|\leqslant\alpha n/10$.

(Ⅱ) For any $\{v_1,v_2\}$, $|A(v_1,v_2)\cap\mathfrak{F}|\geqslant \alpha n/20000\geqslant 2\alpha^2 n$.

(Ⅲ) The number of intersecting pairs of 5-set in $\mathfrak{F}$ is at most $\alpha^2 n/5$.

Let $\mathfrak{F}'$ be obtained from $\mathfrak{F}$ by deleting all intersecting sets and sets that do not absorb any $\{v_1,v_2\}$, then

(Ⅰ) $|\mathfrak{F}'|\leqslant\alpha n/10$.

(Ⅱ) For any $\{v_1,v_2\}$, $|A(v_1,v_2)\cap\mathfrak{F}'|\geqslant \alpha^2 n+1$.

For any $S\in\mathfrak{F}'$, $H[S]$ contains a path on five vertices. We connect these paths to a new path. Suppose that we have obtained a path P' by connecting some paths in $\mathfrak{F}'$, select an unused path Q, let u, v be the endpoints of P' and Q, respectively. Since

$$\deg(u,v)-2-(6|\mathfrak{F}'|-2)\geqslant \left(\frac{1}{4}+\gamma\right)n-\frac{3\alpha n}{5}>0,$$

we can select a vertex $x\in N(u,v)\setminus(K\cup P'\cup\mathfrak{F}')$, and use uxv to connect P' and Q. We can do this until we obtain a path which covers all 5-sets in $\mathfrak{F}'$, after absorbing K, we obtain a path P, which contains at most

$$|V(P)|\leqslant\frac{\alpha n}{10}\cdot 5+\left(\frac{\alpha n}{10}-1\right)+2\leqslant\alpha n$$

vertices such that $\{u,v\}\cap K=\emptyset$, here u, v are the endpoints of P.

Suppose that $U\subset V\setminus V(P)$, $|U|\leqslant\alpha^2 n$, and $|U|\in 2\mathbb{N}$, let $S_1,\cdots,S_q$ ($q\leqslant\alpha^2 n/2$) be an arbitrary partition of U into sets of size 2, since $|A(S_i)\cap\mathfrak{F}'|\geqslant\alpha^2 n+1$, we can select an unused $T_i\in\mathfrak{F}'$ for each S_i. This implies that P absorbs U. This completed the proof.

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