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# The product of two $\sigma$-supersoluble groups

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**Abstract**: Let $\mathfrak{N}_\sigma$ denote the classes of all $\sigma$-nilpotent groups and $G^{\mathfrak{N}_\sigma}$ be the $\sigma$-nilpotent residual of $G$. We say that $G$ is $\sigma$-supersoluble if each chief factor of $G$ below $G^{\mathfrak{N}_\sigma}$ is cyclic. A subgroup $H$ of $G$ is said to be completely $c$-permutable with a subgroup $T$ of $G$ if there exists an element $x\in\langle H,T\rangle$ such that $HT^x=T^xH$. The structure of finite group which is the product of two $\sigma$-supersoluble subgroups was studied by means of the completely $c$-permutability of subgroups.

**Key words**: finite groups; $\sigma$-nilpotent groups; $\sigma$-supersoluble groups; $\sigma$-subnormal; completely $c$-permutable

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# 两个$\sigma$-超可解子群的积

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**摘要**:设$\mathfrak{N}_\sigma$是指所有$\sigma$-幂零群所有构成的群类,并记$G^{\mathfrak{N}_\sigma}$是$G$的$\sigma$-幂零群上根.我们称群$G$是$\sigma$-超可解的,如果$G$的含于$G^{\mathfrak{N}_\sigma}$的主因子是循环的.群$G$的子群$H$称为与子群$T$是完全$c$-置换的,如果存在元素$x\in\langle H,T\rangle$满足$HT^x=T^xH$.利用子群的完全$c$-置换性研究两个$\sigma$-超可解群的积所构成的有限群的结构.

**关键词**:有限群;$\sigma$-幂零群;$\sigma$-超可解群;$\sigma$-次正规;完全$c$-置换

## 0 Introduction

Throughout this paper, all groups are finite and $G$ always denotes a finite group. If $n$ is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing $n$; as usual, $\pi(G)=\pi(|G|)$, the set of all primes dividing the order of $G$.

In what follows, $\sigma=\{\sigma_i \mid i\in I\}$ is some partition of all primes $\mathbb{P}$, that is, $\mathbb{P}=\bigcup_{i\in I}\sigma_i$ and $\sigma_i\cap\sigma_j=\emptyset$ for all $i\neq j$. We write

$$\sigma(G)=\{\sigma_i \mid \sigma_i\cap\pi(G)\neq\emptyset\}.$$

Following Refs.[1-2], the group $G$ is said to

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be $\sigma$-primary if $|\sigma(G)|\leqslant 1$. A chief factor $H/K$ of $G$ is said to be $\sigma$-central in $G$ if $(H/K)\rtimes(G/C_G(H/K))$ is $\sigma$-primary. Recall also that $G$ is $\sigma$-soluble if every chief factor of $G$ is $\sigma$-primary; $\sigma$-nilpotent if every chief factor of $G$ is $\sigma$-central. We use $\mathfrak{S}_\sigma$ and $\mathfrak{N}_\sigma$ to denote the classes of all $\sigma$-soluble groups and $\sigma$-nilpotent groups, respectively. $G^{\mathfrak{N}_\sigma}$ denotes the $\sigma$-nilpotent residual of $G$, that is, the intersection of all normal subgroups $N$ of $G$ with $\sigma$-nilpotent quotient $G/N$.

Moreover, a set $\mathscr{H}$ of subgroups of $G$ is said to be a complete Hall $\sigma$-set of $G$ if every non-identity member of $\mathscr{H}$ is a Hall $\sigma_i$-subgroup of $G$ for some $\sigma_i$ and $\mathscr{H}$ contains exactly one Hall $\sigma_i$-subgroup for every $\sigma_i\in\sigma(G)$. Let $\mathscr{H}=\{H_1,\cdots,H_t\}$ be a complete Hall $\sigma$-set of $G$. $\mathscr{H}$ is said to be a $\sigma$-basis of $G$ if $H_iH_j=H_jH_i$ for all $i,j$. $G$ is said to be $\sigma$-full if $G$ possesses a complete Hall $\sigma$-set; a $\sigma$-full group of Sylow type if every subgroup of $G$ is a $D_{\sigma_i}$-group for all $\sigma_i\in\sigma(G)$. Recently, Guo et al.[3] introduce the definition of $\sigma$-supersoluble group: the group $G$ is said to be $\sigma$-supersoluble if every chief factor of $G$ below $G^{\mathfrak{N}_\sigma}$ is cyclic. They also give some important results about $\sigma$-supersoluble groups. In this paper, we use $\mathscr{U}_\sigma$ to denote the class of all $\sigma$-supersoluble groups.

A subgroup $H$ of $G$ is said to be completely $c$-permutable with $T$ in $G$[4] if there exists $x\in\langle H,T\rangle$ such that $HT^x=T^xH$, where $\langle H,T\rangle$ is the subgroup of $G$ generated by $H$ and $T$. By using the concept of completely $c$-permutable, some conditions under which the product $G=AB$ of two supersoluble subgroups $A$ and $B$ is still supersoluble[5]. Therefore, similar to the above discussion, by applying completely $c$-permutablity, we may study the product $G=AB$ which $A$ and $B$ are two $\sigma$-supersoluble subgroups. In this paper, we determine the structure of the above group. Some new criterions of $\sigma$-supersoluble groups will be given.

We prove here the following results in this line researches.

**Theorem 0.1** Suppose that $G$ has a complete Hall $\sigma$-set $\mathscr{H}=\{H_1, H_2, \cdots, H_t\}$ such that $H_i$ is supersoluble whenever $H_i\cap G^{\mathfrak{N}_\sigma}\neq 1$. Let $G=AB$, where $A$ and $B$ are normal $\sigma$-supersoluble subgroups of $G$. If $G'$ is $\sigma$-nilpotent or $(|G:A|,|G:B|)=1$, then $G$ is $\sigma$-supersoluble.

**Theorem 0.2** Suppose that $G$ has a complete Hall $\sigma$-set $\mathscr{H}=\{H_1, H_2, \cdots, H_t\}$ such that $H_i$ is supersoluble for $i=1,2,\cdots,t$. Let $G=AB$, where $A$ and $B$ are $\sigma$-subnormal subgroups of $G$. If $A$ and $B$ are $\sigma$-supersoluble and every $\sigma$-subnormal subgroup of $A$ is completely $c$-permutable with every subgroup of $B$ in $G$, then $G$ is $\sigma$-supersoluble.

**Theorem 0.3** Suppose that $G$ has a complete Hall $\sigma$-set $\mathscr{H}=\{H_1, H_2, \cdots, H_t\}$ such that $H_i$ is supersoluble for $i=1,2,\cdots,t$. Let $G=AB$, where $A$ and $B$ are $\sigma$-subnormal subgroups of $G$. If $A$ and $B$ are $\sigma$-supersoluble and every primary cyclic subgroup of $A$ is completely $c$-permutable with primary cyclic subgroup of $B$ in $G$, then $G$ is $\sigma$-supersoluble.

All unexplained terminologies and notations are standard. The reader is referred to Refs.[6-8] if necessary.

## 1 Preliminaries

**Lemma 1.1**[1,Lemma 2.1] The class $\mathfrak{S}_\sigma$ and $\mathfrak{N}_\sigma$ are closed under taking direct products, homomorphic images and subgroups. Moreover, any extension of a $\sigma$-soluble group by a $\sigma$-soluble group is a $\sigma$-soluble group as well.

**Lemma 1.2**[3,Lemma 1.3] The class $\mathscr{U}_\sigma$ is a hereditary formation.

**Lemma 1.3**[9,Theorem A] If $G$ is $\sigma$-soluble and $G$ has a Hall $\Pi$-subgroup $E$ for any $\Pi$, then every $\Pi$-subgroup of $G$ is contained in some conjugate of $E$ and permutes with some Sylow $p$-subgroup of $G$ for all primes $p$.

**Lemma 1.4**[10,Lemma 5] Let $H, K$ and $N$ be pairwise permutable subgroups of $G$, and suppose that $H$ is a Hall subgroup of $G$. Then $N\cap HK=(N\cap H)(N\cap K)$.

**Lemma 1.5**[1,Lemma 2.6;11,Lemma 2.1] Let $A$, $K$ be subgroups of $G$ and $N$ be a normal subgroup of $G$. Suppose that $A$ is $\sigma$-subnormal in $G$.

① If $K \leqslant A$ and $A$ is $\sigma$-nilpotent, then $K$ is $\sigma$-subnormal in $G$.

② If $H \neq 1$ is a Hall $\Pi$-subgroup of $G$ and $A$ is not a $\Pi'$-group, then $A \cap H \neq 1$ is a Hall $\pi$-subgroup of $A$.

③ $AN/N$ is $\sigma$-subnormal in $G/N$.

④ If $N \leqslant K$ and $K/N$ is $\sigma$-subnormal in $G/N$, then $K$ is $\sigma$-subnormal in $G$.

⑤ If $G$ is $\sigma$-group and $A$ is $\sigma$-nilpotent, then $A \leqslant F_\sigma(G)$.

⑥ If $G$ is $\pi$-full and $A$ is a $\Pi$-group, then $A \leqslant O_\pi(G)$.

**Lemma 1.6**[3, Lemma 1.4]　$G$ is $\sigma$-supersoluble if and only if the following assertions hold:

① $G^{\mathfrak{N}_\sigma}$ is nilpotent;

② $G'$ is $\sigma$-nilpotent;

③ $[G^{\mathfrak{U}}, G^{\mathfrak{N}_\sigma}] = 1$ and $G^{\mathfrak{U}} \cap G^{\mathfrak{N}_\sigma} \leqslant \Phi(G) \cap Z_{\mathfrak{U}}(G)$.

**Lemma 1.7**[3, Lemma 2.9]　Let $G$ be a $\sigma$-supersoluble group and $N$ be a normal subgroup of $G$.

① $G/N$ is $\sigma$-supersoluble.

② If for some $\sigma_i \in \sigma(G)$ we have that $\sigma_i \cap \pi(G) \subseteq p$, then $G$ is $p$-supersoluble.

**Lemma 1.8**[3, Lemma 2.10]　Let $A = G/O_{p'}(G)$. Then $G$ is $p$-supersoluble if and only if $A/O_p(A)$ is an abelian group of exponent dividing $p-1$, $p$ is the largest prime dividing $|A|$ and $F(A) = O_p(A)$ is a normal Sylow subgroup of $A$.

**Lemma 1.9**[12, Theorem 2.16]　Let $G$ be a $p$-supersoluble group. Then the derived subgroup $G'$ of $G$ is $p$-nilpotent. In particular, if $O_{p'}(G) = 1$, then $G$ is supersoluble and has a unique Sylow $p$-subgroup.

**Lemma 1.10**[1, Corollary 2.4]　If $A, B$ are normal $\sigma$-nilpotent subgroups of $G$, then $AB$ is $\sigma$-nilpotent.

**Lemma 1.11**[5, Lemma 2.8]　If $T/N$ is a primary cyclic subgroup of $AN/N$, then $T = \langle a \rangle N$ for some $a \in A$ of prime power order.

**Lemma 1.12**　Let $G$ be a $\sigma$-supersoluble group and $\mathscr{H} = \{H_1, H_2, \cdots, H_t\}$ be a complete Hall $\sigma$-set of $G$ such that every $H_i$ is supersoluble for $i = 1, 2, \cdots, t$. Suppose that $p$ is the largest prime divisor of $|G|$ and $P$ is the Sylow $p$-subgroup of $G$. Then $P$ is normal in $G$ and so $G$ satisfies the Sylow tower property, that is, $G$ is soluble.

**Proof**　Without loss of generality, we assume that $p \in \pi(H_1)$. If $G^{\mathfrak{N}_\sigma} = 1$, then $G$ is $\sigma$-nilpotent. Hence $H_i$ is normal in $G$ for every $i$. Since $H_1$ is supersoluble, $P$ is normal in $H_1$ and so $P$ is normal in $G$. Now assume that $G^{\mathfrak{N}_\sigma} \neq 1$ and let $N$ be a minimal normal subgroup of $G$ contained in $G^{\mathfrak{N}_\sigma}$. Then $|N| = q$, where $q$ is a prime divisor of $|G|$. It is clear that $G/N$ satisfies the hypothesis of the lemma by induction on $|G|$. Hence $PN/N$ is normal in $G/N$ and so $PN$ is normal in $G$. If $p = q$, clearly, $P$ is normal in $G$. Assume that $p \neq q$. Then $PN = P \times N$ by Ref. [8, Chapter Ⅳ, Theorem 2.8]. It follows that $P$ is normal in $G$. Let $p > p_1 > p_2 > \cdots p_n$ be the all distinct prime divisor of $|G|$. Now we consider $G/P$, then $p_1$ is the largest prime divisor of $|G/P|$. Clearly, $G/P$ is $\sigma$-supersoluble by Lemma 1.7 and

$$\widetilde{\mathscr{H}} = \{H_1P/P, H_2P/P, \cdots, H_tP/P\}$$

is a complete Hall $\sigma$-set of $G/P$ such that every $H_iP/P$ is supersoluble for $i = 1, 2, \cdots, t$. Hence by induction, $PP_1$ is normal in $G$, where $P_1$ is the Sylow $p_1$-subgroup of $G$. The rest can be deduced by analogy that there exists Sylow subgroups $P_2$, $P_3, \cdots, P_n$ such that $PP_1P_2P_3 \cdots P_k$ is normal in $G$, where $k = 2, \cdots, n$. This shows that $G$ satisfies the Sylow tower property and so $G$ is soluble.

Recall that $G$ is called a $CLT_\sigma$-group[3] if $G$ has a complete Hall $\sigma$-set $\mathscr{H} = \{H_1, H_2, \cdots, H_t\}$ such that for all $A_i \leqslant H_i$, $G$ has a subgroup of order $|A_1| \cdots |A_t|$.

**Lemma 1.13**[3, Theorem 1.12]　Let $D = G^{\mathfrak{N}_\sigma}$. Suppose that $G$ has a complete Hall $\sigma$-set $\mathscr{H} = \{H_1, H_2, \cdots, H_t\}$ such that $H_i$ is supersoluble whenever $H_i \cap D \neq 1$. Then $G$ is $\sigma$-supersoluble if and only if every section of $G$ is a $CLT_\sigma$-group.

## 2　Proofs of Theorems 0.1, 0.2 and 0.3

**Proof of Theorem 0.1**　Assume that this is false and let $G$ be a counterexample with minimal $|G|$. We now proceed via the following steps.

① $G$ has a unique minimal normal subgroup $N$ such that $G/N$ is $\sigma$-supersolule.

Let $N$ be a minimal normal subgroup of $G$. Obviously, $\widetilde{\mathscr{H}}=\{H_1N/N, H_2N/N, \cdots, H_tN/N\}$ is a complete Hall $\sigma$-set of $G/N$. By Lemma 1.7, we have that $AN/N$ and $BN/N$ are $\sigma$-supersoluble. Assume that $H_iN/N \cap G^{\mathfrak{N}_\sigma}N/N \neq 1$ for some $i$. Because $H_iN \cap G^{\mathfrak{N}_\sigma}N = (H_i \cap G^{\mathfrak{N}_\sigma})N$ by Lemma 1.4, so $H_i \cap G^{\mathfrak{N}_\sigma} \neq 1$. Then by hypothesis, $H_i$ is supersoluble and so $H_iN/N$ is supersoluble. If $G'$ is $\sigma$-nilpotent or $(|G:A|, |G:B|)=1$, then clearly, $G'N/N$ is $\sigma$-nilpotent or $(|G/N:AN/N|, |G/N:BN/N|)=1$. Hence $G/N$ satisfies that hypothesis of the theorem. The choice of $G$ shows that $G/N$ is $\sigma$-supersoluble. It follows from Lemma 1.2 that $N$ is the unique minimal normal subgroup of $G$.

② $N \nsubseteq \Phi(G)$.

Assume that $N \leqslant \Phi(G)$. Then $N$ is an abelian $p$-group, say $p \in \pi(H_1)$. It implies that by ① that $O_p(G)=F(G)$. By ① and Lemma 1.6, we have that $G^{\mathfrak{N}_\sigma}/N$ is nilpotent, and thereby $G^{\mathfrak{N}_\sigma}$ is nilpotent. This follows that $G^{\mathfrak{N}_\sigma} \leqslant F(G) \leqslant H_1$. Since $G/G^{\mathfrak{N}_\sigma}$ is $\sigma$-nilpotent, $H_1/G^{\mathfrak{N}_\sigma}$ is normal in $G/G^{\mathfrak{N}_\sigma}$ and so $H_1$ is normal in $G$. By the hypothesis of theorem, we know that $H_1$ is supersoluble. Then by ①, we have that $p$ is the largest prime divisor of $H_1$. Let $P$ be the Sylow $p$-subgroup of $G$. Then it is easy to see that $P$ is normal in $G$. Now let $V$ be a complement to $P$ in $H_1$ and $U$ be a complement to $P$ in $G$ such that $V \leqslant U$. Since $G^{\mathfrak{N}_\sigma} \leqslant O_p(G) \leqslant P$, $U \cong G/P \cong (G/G^{\mathfrak{N}_\sigma})/(P/G^{\mathfrak{N}_\sigma})$ is $\sigma$-nilpotent. Hence

$$U = V \times H_2 \times H_3 \times \cdots \times H_t.$$

Let $S_i = PH_i$, where $i \in \{2, 3, \cdots, t\}$. First we show that $G \neq PH_i$ for every $i$. If not, assume that for some $i$, we have $G=PH_i$. Then in this case, $H_1 = P$, that is, $\mathscr{H}=\{P, H_i\}$. It follows that $\widetilde{\mathscr{H}}=\{P/N, H_iN/N\}$ is a complete Hall $\sigma$-set of $G/N$. Since $G/N$ is $\sigma$-supersoluble, $G/N$ is $p$-supersoluble by Lemma 1.7, which implies that $G$ is $p$-supersoluble and so $G$ is supersoluble by Lemma 1.9 because $O_{p'}(G)=1$. This is a contradiction. Hence $G \neq PH_i$ for every $i$.

By Lemma 1.4, we know that

$$PH_i = (P \cap A)(P \cap B)(H_i \cap A)(H_i \cap B) = (P \cap A)(H_i \cap A)(P \cap B)(H_i \cap B) = (PH_i \cap A)(PH_i \cap B),$$

that is, $S_i = (S_i \cap A)(S_i \cap B)$. Clearly, $S_i' \leqslant G'$ is $\sigma$-nilpotent or $(|S_i:S_i \cap A|, |S_i:S_i \cap B|)=1$, so $S_i$ satisfies the hypothesis of the theorem. The choice of $G$ implies that $S_i$ is $\sigma$-supersoluble. Then by Lemma 1.7, we have that $S_i$ is $p$-supersolule. Since

$$[O_p(G), O_{p'}(S_i)] = 1,$$

$$O_{p'}(S_i) \leqslant C_G(O_p(G)) \leqslant O_p(G),$$

which forces that $O_{p'}(S_i)=1$. Hence $H_i$ is an abelian group of exponent dividing $p-1$ by Lemma 1.8. Similarly, $V$ is an abelian group of exponent dividing $p-1$. Therefore $U$ is an abelian group of exponent dividing $p-1$, which implies that $G$ is supersoluble, a contradiction. Hence $N \nsubseteq \Phi(G)$.

③ $N=F(G)=C_G(N)=O_p(G)$ and $H_1$ is supersoluble, say $p \in \pi(H_1)$.

Since $A$ and $B$ are $\sigma$-supersoluble, $A^{\mathfrak{N}_\sigma}$ and $B^{\mathfrak{N}_\sigma}$ are nilpotent by Lemma 1.6. If $A^{\mathfrak{N}_\sigma}=B^{\mathfrak{N}_\sigma}=1$, then $A$ and $B$ are $\sigma$-nilpotent. By Lemma 1.10, we have that $G$ is $\sigma$-nilpotent, a contradiction. Therefore $A^{\mathfrak{N}_\sigma} \neq 1$ or $B^{\mathfrak{N}_\sigma} \neq 1$. Without loss of generalization, we assume that $A^{\mathfrak{N}_\sigma} \neq 1$. Then by ①, $N \leqslant A^{\mathfrak{N}_\sigma}$ and so $N$ is an elementary abelian $p$-group, say $p \in \pi(H_1)$. Then by ① and ②, it is easy to see that $N=F(G)=C_G(N)=O_p(G)$. Obviously, $N \leqslant G^{\mathfrak{N}_\sigma}$ and thereby $H_1 \cap G^{\mathfrak{N}_\sigma} \neq 1$. Then by hypothesis of the theorem, $H_1$ is supersoluble.

④ If $H$ is a normal $\sigma$-supersoluble subgroup of $G$, then $H$ is supersoluble.

First assume that $H^{\mathfrak{N}_\sigma}=1$. Then $H$ is $\sigma$-nilpotent and so $H \cap H_i$ is normal in $H$ for every $i$. Clearly, $N \leqslant H$ by ①. Hence when $i=2, \cdots, t$, $H \cap H_i \leqslant C_G(N)=N$ by ③, which implies that $H \cap H_i = 1$. So $H \leqslant H_1$ is supersoluble by ③.

Now suppose that $H^{\mathfrak{N}_\sigma} \neq 1$. Since $H$ is $\sigma$-supersoluble, $[H^{\mathfrak{U}}, H^{\mathfrak{N}_\sigma}]=1$ and $H^{\mathfrak{N}_\sigma}$ is nilpotent by Lemma 1.6. Clearly, that $N=F(H)$. Hence $N \leqslant H^{\mathfrak{N}_\sigma} \leqslant F(H)=N$ and so $N=H^{\mathfrak{N}_\sigma}$. This implies from ③ that $H^{\mathfrak{U}} \leqslant C_G(N)=N$. Hence $H/N$ is supersoluble. Clearly, $N=N_1 \times N_2 \times \cdots \times N_s$, where $N_i$ is a minimal normal subgroup of $H$ for every $i \in \{1, 2, \cdots, s\}$. Since $H$ is $\sigma$-supersoluble and $N_i \leqslant N=H^{\mathfrak{N}_\sigma}$, $|N_i|=p$. It derives that $H$ is supersoluble.

⑤ The final contradiction.

By ④, we know that $A$ and $B$ are supersoluble. If $(|G:A|, |G:B|)=1$, then by Ref. [7, Chapter 1, Corollary 4.7], $G$ is supersoluble, a contradiction. Hence assume that $G'$ is $\sigma$-nilpotent. Clearly, $A'$ and $B'$ are nilpotent and $N=F(A)=F(B)$. If $A'=1$, clearly, $A/N$ is abelian. Now assume that $A' \neq 1$. Then $A'=N$, which follows that $A/N$ is abelian. By a similar discussion, we always have that $B/N$ is abelian. It implies that $G/N$ is nilpotent. Hence

$$G/N=H_1/N \times H_2N/N \times \cdots \times H_tN/N.$$

Since $G'$ is $\sigma$-nilpotent, $G' \leqslant F_\sigma(G)=O_{\sigma_1}(G) \leqslant H_1$ by ①. This implies that $H_1$ is normal in $G$ and $G/H_1$ is abelian. By a similar discussion as above, we have that $H_1/N$ is abelian. Because $H_i \cong H_iH_1/H_1$ is abelian, so

$$G/N=H_1/N \times H_2N/N \times \cdots \times H_tN/N$$

is abelian. Therefore $G'=N$ is nilpotent. By Ref. [7, Chapter 1, Corollary 4.6], $G$ is supersoluble. The contradiction completes the proof the theorem.

**Proof of Theorem 0.2** Assume that this is false and let $G=AB$ be a counterexample of minimal order. Without loss of generality, we may assume that for any proper $\sigma$-subnormal subgroup $A_1$ of $A$ and any proper subgroup $B_1$ of $B$, we have that $G \neq A_1B$ and $G \neq AB_1$. We prove the theorem via the following steps:

① $G$ has a unique minimal normal subgroup $N$ such that $G/N$ is $\sigma$-supersoluble and so $N$ is non-cyclic.

Let $N$ be a minimal normal subgroup of $G$. Clearly, $\widetilde{\mathscr{H}}=\{H_1N/N, H_2N/N, \cdots, H_tN/N\}$ be a complete Hall $\sigma$-set of $G/N$ such that every $H_iN/N$ is supersoluble for $i=1, 2, \cdots, t$. By Lemmas 1.5③ and 1.7, we have that $AN/N$ and $BN/N$ are $\sigma$-subnormal subgroup of $G/N$ and they are $\sigma$-supersoluble. Now let $H/N$ be a $\sigma$-subnormal subgroup of $AN/N$ and $T/N$ be a subgroup of $BN/N$. Then by Lemma1.5④, $H$ is a $\sigma$-subnormal subgroup of $AN$ and so $H \cap A$ is a $\sigma$-subnormal subgroup of $A$. Hence by the hypothesis of the theorem, there exists $x \in \langle H, T\rangle$ such that $(H \cap A)(T \cap B)^x=(T \cap B)^x(H \cap A)$. It follows that

$$\begin{aligned}&(H/N)(T/N)^{xN}=\\&(H \cap AN)/N((T \cap BN)/N)^{xN}=\\&((H \cap A)N/N)((T \cap B)N/N)^{xN}=\\&(H \cap A)(T \cap B)^xN/N=\\&(T \cap B)^x(H \cap A)N/N=(T/N)^{xN}(H/N),\end{aligned}$$

where $xN \in \langle H, T\rangle/N$. This shows that $G/N$ satisfies that hypothesis of the theorem. Hence $G/N$ is $\sigma$-supersoluble. It implies from Lemma 1.2 that $N$ is the unique minimal normal subgroup of $G$. If $N$ is cyclic, then clearly, $G$ is $\sigma$-supersoluble. The contradiction shows that $N$ is non-cyclic.

② $G$ is $\sigma$-soluble and $N$ is an abelian $p$-group, where $p \in \pi(H_1)$. Moreover, $O_{\sigma_1'}(G)=1$ and so $F_\sigma(G)=O_{\sigma_1}(G) \leqslant H_1$.

If $F_\sigma(G)=1$, then by Lemmas 1.5⑤ and 1.6, $A^{\mathfrak{N}_\sigma} \leqslant F_\sigma(G)=1$. Hence $A$ is $\sigma$-nilpotent and so $A \leqslant F_\sigma(G)=1$. Then $G=B$ is $\sigma$-soluble. Hence we assume that $F_\sigma(G) \neq 1$. Then by ①, $N \leqslant F_\sigma(G)$. It follows from ① and Lemma 1.1 that $G$ is $\sigma$-soluble. Hence $N \leqslant H_i$ for some $i$, without loss of generality, we may say $i=1$. Because $H_1$ is supersoluble, so $N$ is an abelian $p$-group, where $p \in \pi(H_1)$. It follows from ① that $O_{\sigma_1'}(G)=O_{p'}(G)=1$ and so $F_\sigma(G)=O_{\sigma_1}(G)$.

③ Every proper subgroup of $G$ containing $A$ or $B$ is $\sigma$-supersoluble.

Let $K$ be a proper subgroup of $G$ containing $A$ or $B$. Then $K=A(K \cap B)$ or $K=(K \cap A)B$. By Lemma 1.1, $K$ is $\sigma$-soluble, so by Lemma 1.3, $K$

has a complete Hall $\sigma$-set $\mathscr{H}_k=\{K_1, K_2, \cdots, K_s\}$ such that every $K_i \leqslant H_i^x$ is supersoluble for some $x \in G$. Hence $K$ satisfies the hypothesis of the theorem. The choice of $G$ implies that $K$ is $\sigma$-supersoluble.

④ If $K$ is a $\sigma$-supersoluble subgroup of $G$, then $K$ is soluble.

By a similar discussion as in ③, $K$ has a complete Hall $\sigma$-set $\mathscr{H}_k=\{K_1, K_2, \cdots, K_s\}$ such that every $K_i$ is supersoluble. Hence by Lemma 1.12, $K$ is soluble.

⑤ $N \nsubseteq \Phi(G)$ and so

$$N=F(G)=C_G(N)=O_p(G).$$

Assume that $N \leqslant \Phi(G)$. By ① and Lemma 1.6, we have that $G^{\mathfrak{N}_\sigma}/N$ is nilpotent and so $G^{\mathfrak{N}_\sigma}$ is nilpotent. It is easy to see that $F(G) \leqslant H_1$. This follows that $G^{\mathfrak{N}_\sigma} \leqslant F(G) \leqslant H_1$. Since $G/G^{\mathfrak{N}_\sigma}$ is $\sigma$-nilpotent, $H_1/G^{\mathfrak{N}_\sigma}$ is normal in $G/G^{\mathfrak{N}_\sigma}$ and so $H_1$ is normal in $G$. Because $H_1$ is supersoluble, so by ①, it is obvious that $p$ is the largest prime divisor of $|H_1|$. Let $P$ be the Sylow $p$-subgroup of $G$. Then $P$ is normal in $G$ and so $G$ is $p$-soluble. Now let $V$ be a complement to $P$ in $H_1$ and $U$ be a complement to $P$ in $G$ such that $V \leqslant U$. Since $G^{\mathfrak{N}_\sigma} \leqslant O_p(G) \leqslant P$, $U \cong G/P$ is $\sigma$-nilpotent. Hence $U=V \times H_2 \times H_3 \times \cdots \times H_t$. Moreover, since $G'/N$ is $\sigma$-nilpotent by ① and Lemma 1.6, $G'$ is $\sigma$-nilpotent by Lemma 1.1. Hence $G' \leqslant F_\sigma(G)=O_{\sigma_1}(G) \leqslant H_1$ by ②. It implies that

$$H_i \cong H_i H_1/H_1 \leqslant G/H_1$$

is abelian, where $i=2, 3, \cdots, t$. Therefore $(H_i \cap A)(H_i \cap B)$ is a group. By Lemma 1.5②, we have that $H_i \cap A$ and $H_i \cap B$ are Hall $\sigma_i$-subgroups of $A$ and $B$, respectively. Since

$$|G:(H_i \cap A)(H_i \cap B)|=$$
$$|AB:(H_i \cap A)(H_i \cap B)|$$

divides $|A:H_i \cap A||B:H_i \cap B|$, $|G:(H_i \cap A)(H_i \cap B)|$ is a $\sigma_i'$-number and so $(H_i \cap A)(H_i \cap B)$ is a Hall $\sigma_i$-subgroup of $G$. Hence

$$H_i=(H_i \cap A)(H_i \cap B).$$

Now, let $S_i=PH_i$, where $i \neq 1$. By using a same argument as in Step ② of Theorem 0.1, $G \neq S_i$ for every $i$. We show that $S_i$ satisfies the hypothesis of the theorem. By Lemma 1.3, $S_i$ has a complete Hall $\sigma$-set such that every member of the set is supersoluble. Moreover, it is obvious that

$$(PH_i \cap A)(PH_i \cap B)=$$
$$(H_i \cap A)(P \cap A)(P \cap B)(H_i \cap B)=$$
$$(H_i \cap A)(H_i \cap B)P=H_iP,$$

that is, $S_i=(S_i \cap A)(S_i \cap B)$. Since $A^{\mathfrak{N}_\sigma} \leqslant G^{\mathfrak{N}_\sigma} \leqslant P$, $A/(P \cap A)$ is $\sigma$-nilpotent and thereby $(H_i \cap A)(P \cap A)/(P \cap A)$ is $\sigma$-subnormal in $A/(P \cap A)$. This implies by Lemma 1.5 that $S_i \cap A=PH_i \cap A=(H_i \cap A)(P \cap A)$ is $\sigma$-subnormal in $A$. Let $L$ be any $\sigma$-subnormal subgroup of $S_i \cap A$, then $L$ is a $\sigma$-subnormal subgroup of $A$. Hence by hypothesis, $L$ completely $c$-permutes with every subgroup of $B$. Therefore $S_i$ satisfies the hypothesis of the theorem. The choice of $G$ implies that $S_i$ is $\sigma$-supersoluble. It follows from Lemma 1.7 that $S_i$ is $p$-supersoluble. Because $G$ is $p$-soluble, so $C_G(O_p(G)) \leqslant O_p(G)$. Since $[O_p(G), O_{p'}(S_i)]=1$, $O_{p'}(S_i)=1$. Hence $H_i$ is an abelian group of exponent dividing $p-1$ by Lemma 1.8. Similarly, $V$ is an abelian group of exponent dividing $p-1$. Therefore $U$ is an abelian group of exponent dividing $p-1$, which implies that $G$ is supersoluble, a contradiction. Hence $N \nsubseteq \Phi(G)$. Then by ① and ②, it is easy to see that

$$N=F(G)=C_G(N)=O_p(G).$$

⑥ $p$ is the largest prime of $|G|$ and $N$ is the Sylow $p$-subgroup of $G$. Moreover, $AN/N$ and $BN/N$ are $\sigma$-nilpotent.

Let $q$ be the largest prime divisor of $|G|$ with $p \neq q$ and $Q$ be the Sylow $q$-subgroup of $G$. By Lemma 1.13, $A$ exists a maximal subgroup $A_1$ such that $|A:A_1|=r$ and $B$ exists a maximal subgroup $B_1$ such that $|B:B_1|=s$, where $r$ is the least prime divisor of $|A|$ and $s$ is the least prime divisor of $|B|$. Then $A_1$ is normal in $A$ and $B_1$ is normal in $B$. By the hypothesis of the theorem, we have that $T_1=A_1B$ and $T_2=AB_1$ are two subgroups of $G$. And $|G:T_1|=|AB:A_1B|=r$ and $|G:T_2|=|AB:AB_1|=s$, so $T_1$ and $T_2$ are two maximal subgroups of $G$. By ③, we have that

$T_1$ and $T_2$ are $\sigma$-supersoluble. We show that $N \leqslant T_1 \cap T_2$. Assume that $N \nsubseteq T_1$ or $N \nsubseteq T_2$. Without loss of generality, we may assume that $N \nsubseteq T_1$. Then by ⑤, it is clear that $G = N \rtimes T_1$. It follows that $|N| = |G : T_1| = r = p$, which contradicts ①. Hence $N \leqslant T_1 \cap T_2$. If $r = q$, then the order of $A$ is power of prime $q$ because $q$ be the largest prime divisor of $|A|$. Clearly, $s \neq q$. Hence $Q \leqslant T_2$ because $|G : T_2| = s$. Since $T_2$ is $\sigma$-supersoluble, $Q$ is normal in $T_2$ by Lemma 1.12 and so $Q \leqslant C_G(N) = N$ by ⑤, a contradiction. Hence $r \neq q$. Then $Q \leqslant T_1$ and so $Q \leqslant C_G(N) = N$ by ⑤ too. This contradiction shows that $p$ is the largest prime divisor of $|G|$. Since $G/N$ is $\sigma$-supersoluble by ①, $P/N$ is normal in $G/N$ by Lemma 1.12 and so $P = N$ by ⑤.

Moreover, since

$$[O_{p'}(T_1), N] = [O_{p'}(T_2), N] = 1$$

and

$$C_G(N) = N,\ O_{p'}(T_2) = O_{p'}(T_1) = 1.$$

It follows from ⑤ that

$$F(T_1) = O_p(T_1) = F(T_2) = O_p(T_2) = N.$$

Since $T_1$ and $T_2$ are $\sigma$-supersoluble, by Lemma 1.6, $T_1^{\mathfrak{N}_\sigma}$ and $T_2^{\mathfrak{N}_\sigma}$ are nilpotent and so $T_1^{\mathfrak{N}_\sigma}$, $T_2^{\mathfrak{N}_\sigma} \leqslant N$. This implies that $T_1/N$ and $T_2/N$ are $\sigma$-nilpotent and thereby $AN/N$ and $BN/N$ are $\sigma$-nilpotent.

⑦ Final contradiction.

Since $A/A \cap N \cong AN/N$ and $B/B \cap N \cong BN/N$ are $\sigma$-nilpotent by ⑥, $A^{\mathfrak{N}_\sigma}$, $B^{\mathfrak{N}_\sigma} \leqslant N$. Assume that $A^{\mathfrak{N}_\sigma} \neq 1$. Let $N_1$ be a minimal normal subgroup of $A$ such that $N_1 \leqslant A^{\mathfrak{N}_\sigma}$. Then $|N_1| = p$. Let $q$ be any prime divisor of $|B|$ such that $p \neq q$ and $B_q$ be a Sylow $q$-subgroup of $B$. Then by hypothesis, there exists an element $x \in \langle N_1, B_q \rangle$ such that $N_1 B_q^x = B_q^x N_1$. This follows that $N_1 = N \cap N_1 B_q^x$ is normal in $N_1 B_q^x$, that is, $B_q^x \leqslant N_G(N_1)$. It implies that $B_q^b \leqslant N_G(N_1)$ for some $b \in B$. Clearly, $B_q^b$ is a Sylow $q$-subgroup of $B$ and $B \cap N \leqslant N_G(N_1)$. Hence $B \leqslant N_G(N_1)$ and so $N_1$ is normal in $G$. This implies that $|N| = p$, which contradicts to ①. Hence $A^{\mathfrak{N}_\sigma} = 1$. Then $A$ is $\sigma$-nilpotent and so $A \leqslant F_\sigma(G) = O_{\sigma_1}(G) \leqslant H_1$ by Lemma 1.5 and ②. Similarly, it is easy to derive that $B^{\mathfrak{N}_\sigma} \neq 1$. Therefore we suppose that $N_2$ is a minimal normal subgroup of $B$ such that $N_2 \leqslant B^{\mathfrak{N}_\sigma}$. Then $|N_2| = p$. By the hypothesis of theorem, we know that $AN_2^x = N_2^x A$ for some $x \in G$. Let $x = ab$, where $a \in A$, $b \in B$. Then it is obvious that $AN_2 = N_2 A$. If $A \cap N = 1$, then $N_2 = AN_2 \cap N$ is normal in $AN_2$, that is, $A \leqslant N_G(N_2)$. This follows that $N_2$ is normal in $G$, which is impossible from the above discussion. Therefore we assume that $A \cap N \neq 1$ and let $R$ be a minimal normal subgroup of $A$ such that $R \leqslant A \cap N$. Then $|R| = p$ because $A$ is supersoluble. By a similar argument as above, we can still derive that $R$ is normal in $G$ and so $|N| = p$. The final contradiction completes the proof of the theorem.

**Proof of Theorem 0.3** Assume that this is false and let $G$ be a counterexample with minimal $|G|$.

① $G$ has a unique minimal normal subgroup $N$ such that $G/N$ is $\sigma$-supersoluble and so $N$ is non-cyclic.

Let $N$ be a minimal normal subgroup of $G$. Let $T/N$ and $L/N$ be primary cyclic subgroups of $AN/N$ and $BN/N$, respectively. Then by Lemma 1.11, there exists elements $a \in A$ and $b \in B$ with prime power order such that $T = \langle a \rangle N$ and $L = \langle b \rangle N$. Hence $\langle a \rangle \langle b \rangle^x = \langle b \rangle^x \langle a \rangle$ for some $x \in \langle \langle a \rangle, \langle b \rangle \rangle$ by hypothesis of the theorem. It follows that

$$(\langle a \rangle N/N)(\langle b \rangle N/N)^{xN} = (\langle a \rangle \langle b \rangle^x)N/N = (\langle b \rangle^x \langle a \rangle)N/N = (\langle b \rangle N/N)^{xN}(\langle a \rangle N/N),$$

where $xN \in \langle \langle a \rangle N/N, \langle b \rangle N/N \rangle$. By a similar proof as in Step ① of Theorem 0.2, we have that $G/N$ satisfies that hypothesis of the theorem. Hence $G/N$ is $\sigma$-supersoluble, $N$ is the unique minimal normal subgroup of $G$ and $N$ is non-cyclic.

② $N$ is an abelian $p$-group, say $p \in \pi(H_1)$, $O_{\sigma_1'}(G) = 1$ and so $F_\sigma(G) = O_{\sigma_1}(G) \leqslant H_1$. Moreover, every $\sigma$-supersoluble subgroup of $G$ is soluble.

See Steps ② and ④ of Theorem 0.2.

③ $N \nsubseteq \Phi(G)$ and so

$$N = F(G) = C_G(N) = O_p(G).$$

See Step ⑤ of Theorem 0.2.

④ $p$ is the largest prime of $|G|$ and $N = P$ is the Sylow $p$-subgroup of $G$.

Assume that $q$ is the largest prime divisor of $|G|$ with $p \neq q$ and $Q$ is the Sylow $q$-subgroup of $G$. If $Q \leqslant H_1$, then $Q \leqslant C_G(N) = N$, a contradiction. Hence $q \in \pi(H_1)$. Clearly, $q$ is the largest prime divisor of $|A|$ or $|B|$. without loss of generality, we may assume that $q$ is the largest prime divisor of $|A|$. Let $A_q$ be the Sylow $q$-subgroup of $A$. Then by Lemma 1.12, $A_q$ is normal in $A$ and so $A_q$ is $\sigma$-subnormal in $G$. It follows from ② that $A_q \leqslant O_{\sigma_1'}(G) = 1$. This contradiction shows that $p$ is the largest prime of $|G|$. Let $P$ be the Sylow $p$-subgroup of $G$. Since $G/N$ is $\sigma$-supersoluble by ①, $P/N$ is normal in $G/N$ by Lemma 1.12 and so $P = N$ by ③.

⑤ Final contradiction.

First, we show that $N \nsubseteq A$ and $N \nsubseteq B$. Without loss of generality, we may assume that $N \leqslant A$. Then $O_{p'}(A) \leqslant C_G(N) = N$ by ③, so $O_{p'}(A) = 1$. It follows that $O_p(A) = F(A) = N$. Hence $A^{\mathfrak{N}_\sigma} \leqslant N$ by Lemma 1.6. Let $N_1$ be a minimal normal subgroup of $A$ such that $N_1 \leqslant N$. First, we assume that $A^{\mathfrak{N}_\sigma} = 1$, then $A$ is $\sigma$-nilpotent. By ② and Lemma 1.5, $A \leqslant H_1$ and so $A$ is supersoluble. So $|N_1| = p$. Now assume that $A^{\mathfrak{N}_\sigma} \neq 1$, we also let $N_1 \leqslant A^{\mathfrak{N}_\sigma}$. Then $|N_1| = p$ too. Let $b$ be an arbitrary element of $B$ of prime power order. If the order of $b$ is $p^\alpha$, then $\langle b \rangle \leqslant N \leqslant N_G(N_1)$ because $N$ is abelian. Now suppose that $b$ is a $p'$-element. Then by hypothesis, $N_1^x \langle b \rangle = \langle b \rangle N_1^x$ for $x \in \langle \langle b \rangle, N \rangle = \langle b \rangle N$. Hence $N_1^x = P^x \cap N_1^x \langle b \rangle$ and so $\langle b \rangle \leqslant N_G(N_1^x)$. Denote that $x = b'n$, where $b' \in \langle b \rangle$, $n \in N$. Then $\langle b \rangle \leqslant N_G(N_1^{b'}) = N_G^{b'}(N_1)$ and so $\langle b \rangle = \langle b \rangle^{b'} \leqslant N_G(N_1)$. Since $B$ is generated by all its elements of prime power order, $B \leqslant N_G(N_1)$. It follows that $N_1$ is normal in $G$ and thereby $|N| = p$, which contradicts ①. Therefore $N \nsubseteq A$ and $N \nsubseteq B$.

Clearly, $A_p = N \cap A$ is a normal Sylow $p$-subgroup of $A$. If $A_p = 1$, then $A$ is a $p'$-group and so $N \leqslant B$, a contradiction. Hence $A_p \neq 1$. It is clear that $A_p = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_s \rangle$, where every $\langle a_i \rangle$ is a cyclic group of order $p$ contained in $N$. We use a similar claim as above, let $b$ be an arbitrary element of $B$ of prime power order. If the order of $b$ is $p^\alpha$, then $\langle b \rangle \leqslant N \leqslant N_G(\langle a_i \rangle)$. Assume that $b$ is a $p'$-element. Then by hypothesis of the theorem, there exists some element $x \in \langle \langle a_i \rangle, \langle b \rangle \rangle \leqslant \langle b \rangle N$ such that $\langle a_i \rangle \langle b \rangle^x = \langle b \rangle^x \langle a_i \rangle$ for every $i$. Let $x = b_1 p$, where $b_1 \in \langle b \rangle$ and $p \in N$. Then we have

$$\langle a_i \rangle \langle b \rangle^{b_1 p} = \langle b \rangle^{b_1 p} \langle a_i \rangle$$

and so $\langle a_i \rangle \langle b \rangle = \langle b \rangle \langle a_i \rangle$ because $\langle a_i \rangle$ is normal in $P$. It follows that $\langle a_i \rangle = P \cap \langle a_i \rangle \langle b \rangle$ and so $\langle b \rangle \leqslant N_G(\langle a_i \rangle)$. This shows that $B \leqslant N_G(\langle a_i \rangle)$ for every $i$ and thereby $B \leqslant N_G(A_p)$. Hence $A_p$ is normal in $G$, which implies that $A_p = N$ or $A_p = 1$. These two cases are impossible. This completes the proof of the theorem.

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