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The a_i-invariants of powers of ideals

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Abstract: Inspired by the recent work of Lu and O'Rourke, we study the a_i-invariants of (symbolic) powers of some graded ideals. When I and J are two graded ideals in two distinct polynomial rings R and S over a common field $\mathbb{K}$. We study the a_i-invariants of the powers of the fiber product via the corresponding conditions on I and J. When I_Δ is the Stanley-Reisner ideal of a k-dimensional complex Δ with $k\geqslant 2$. We investigate the a_i-invariants of the symbolic powers of I_Δ.

Key words: a_i-invariants; local cohomology; complex; symbolic power
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理想幂次的 a_i-不变量

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摘要:在 Lu 和 O'Rourke 最近的工作基础上,我们研究了分次理想及其幂次的 a_i-不变量. 设 R 和 S 是域 $\mathbb{K}$ 上的两个多项式环,$T=S\otimes_{\mathbb{K}}R$,I 和 J 分别是 R 和 S 中的分次理想. 我们利用 I 和 J 的信息研究 $a_i(T/(I+J+\mathfrak{m}\mathfrak{n})^k)$ 的性质. 再设 $k\geqslant 2$,并令 Δ 为一个 k 维复形且 I_Δ 是其 Stanley-Reisner 理想. 我们研究 $I_\Delta^{(n)}$ 的 a_i-不变量.

关键词:a_i-不变量;局部上同调;复形;形式幂

0 Introduction

Let $S=\mathbb{K}[x_1,\cdots,x_s]$ and $R=\mathbb{K}[y_1,\cdots,y_r]$ be two polynomial rings over a field $\mathbb{K}$ and $T=S\otimes_{\mathbb{K}}R$. Let $I\subseteq S$ and $J\subseteq R$ be two graded ideals. The fiber product of I and J is defined by $F=I+J+\mathfrak{m}\mathfrak{n}$, where $\mathfrak{m}$ and $\mathfrak{n}$ are the graded maximal ideals of S and R respectively. One may observe that $(S\otimes_{\mathbb{K}}R)/(I+J+\mathfrak{m}\mathfrak{n})$ can be decomposed as a direct sum of rings $\frac{S}{I}\oplus\frac{R}{J}$.

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Furthermore, if I and J are edge ideals of two vertex-disjoint graphs, then $I+J+\mathfrak{mn}$ corresponds to the edge ideal of the join of the graphs. Fiber products of ideals were studied by many authors; c. f. [1-8]. But little is known about the a_i-invariants of $T/(I+J+\mathfrak{mn})^k$ yet.

Recall that when M is a finitely generated S-module and $0\leqslant i\leqslant \dim(M)$, the a_i-invariant of M is given by

$$a_i(M):=\max\{t: H_{\mathfrak{m}}^i(M)_t\neq 0\},$$

where $H_{\mathfrak{m}}^i(M)$ is the i-th local cohomology module of M with support in $\mathfrak{m}$. Notice that $a_{\dim(M)}(M)$ is exactly the a-invariant introduced by Goto and Watanabe in Ref. [9]. It plays an important role in local duality, since $-a(M)$ is the initial degree of the canonical module of M; see, for instance, Refs. [9-10].

The a_i-invariant also has a close relation with the Castelnuovo-Mumford regularity:

$$\operatorname{reg}(M):=\max\{a_i(M)+i: 0\leqslant i\leqslant \dim(M)\}.$$

In fact, the a_i-invariant takes an important part in studying its asymptotic behaviour. For example, in Ref. [11], Herzog, Hoa and Trung proved that if J is a homogeneous ideal of R, then $\operatorname{reg}(R/J^n)$ is a linear function of the form $cn+e$ for $n\gg 0$ via investigating $a_i(R/J^n)$. Meanwhile, in Ref. [2], Hoa and Trung showed that $a_i(R/J^n)$ is also asymptotoically a linear function of n.

For any subset F of $[s]$, we set

$$\boldsymbol{x}_F:=\prod_{i\in F}x_i\in S.$$

For a positive integer s, let $=\{1,2,\cdots,s\}$. For any simplicial complex Δ on $[s]$, we use I_Δ to denote its Stanley-Reisner ideal. Precisely speaking,

$$I_\Delta:=(\boldsymbol{x}_F: F\in\mathfrak{N}(\Delta))\subseteq S,$$

where $\mathfrak{N}(\Delta)$ is the set of minimal non-faces of Δ. When G is a simple graph on $[s]$ considered as a 1-dimensional simplicial complex and G' is obtained from G by adding an isolated vertex $\{s+1\}$, we may find that $I_G{}'=(I_G,\mathfrak{m}\,x_{s+1})$. Based on this observation, in addition to other beautiful results, Lu showed the following important result in Ref. [4].

Theorem 1.1[4] Let $S=\mathbb{K}[x_1,\cdots,x_s]$ be a polynomial ring over a field $\mathbb{K}$. Assume that $m=(x_1,\cdots,x_s)$ is the graded maximal ideal of S and y is a new variable over S. $I\subseteq S$ is a monomial ideal and $J=(I,\mathfrak{m}\,y)\subseteq S[y]$.

(a) If $i\geqslant 2$, then $a_i(S[y]/J^k)=\max\{a_i(S/I^{k-t})+t: 0\leqslant t\leqslant k-1\}$.

(b) If $\sqrt{I}\neq\mathfrak{m}$, then $a_1(S[y]/J^k)=\max\{2k-2, a_1(S/I^{k-t})+t: 0\leqslant t\leqslant k-1\}$.

Notice that the ideal $(I,\mathfrak{m}\,y)$ above can also be considered as a fiber product of $I\subseteq S$ and $0\subseteq\mathbb{K}[y]$. It is then very natural to ask: what can be said towards $a_i(T/(I+J+\mathfrak{mn})^k)$ in a more general framework? We will answer this in Theorem 2.9.

Next, we turn our attention to the Stanley-Reisner ideal of simplicial complexes. Assume that Δ is a simplicial complex on $[s]$ and I_Δ is the Stanley-Reisner ideal of Δ in $S=\mathbb{K}[x_1,\cdots,x_s]$. We will deal with the its powers I_Δ^n and its symbolic powers $I_\Delta^{(n)}$. Assume P is a prime ideal of S, the P-primary component of the n-th power of P is called the n-th symbolic power of P, written as $P^{(n)}$. The symbolic powers of ideals have a nice geometric description, due to Zariski and Nagata in Ref. [12]. Recall that the n-th symbolic power of an ideal $I\subseteq S$ is defined to be

$$I^{(n)}:=\bigcap_{\mathfrak{p}\in Ass(S/I)}\mathfrak{p}^n$$

for $n\geqslant 1$. Since by Ref. [13], the ideal I_Δ has the following primary decomposition

$$I_\Delta=\bigcap_{F\in\mathcal{F}(\Delta)}P_F. \tag{1}$$

Then it follows from (1) that the n-th symbolic power of I_Δ in our situation is precisely

$$I_\Delta^{(n)}=\bigcap_{F\in\mathcal{F}(\Delta)}P_F^n \tag{2}$$

The research of related topics has continuously attracted the attention of many researchers; see for instance the recent survey[14] and the references therein.

Previous related work mainly focuses on symbolic powers of 2-dimensional square free ideals. In Refs. [1, 6], the a_i-invariants of symbolic powers of Stanley-Reisner ideals was described explicitly in this case. And in Ref. [4], the author proved that for any 1-dimensional complex Δ without isolated vertex, one has $a_2(S/I_\Delta^{(n)})=a_2(S/I_\Delta^n)$. From these phenomena, it is natural to ask whether $a_{k+1}(S/I_\Delta^{(n)})=a_{k+1}(S/I_\Delta^n)$ always holds and under what conditions will $a_{k+1}(S/I_\Delta^{(n)})$ be maximal when dim $(\Delta)=k\geqslant 2$. We will give definite answers to these two questions in Theorem 3.9 and Tehorem 3.11.

2 a_i-invariants of powers of fiber product ideal

In this section, we will always assume the following settings.

Setting 2.1 Let $S=\mathbb{K}[x_1,\cdots,x_s]$ and $R=\mathbb{K}[y_1,\cdots,y_r]$ be two polynomial rings over a common field $\mathbb{K}$ and $\mathfrak{m}$ and $\mathfrak{n}$ be the corresponding graded maximal ideals respectively. Let $I\subseteq\mathfrak{m}$ and $J\subseteq\mathfrak{n}$ be two graded ideals and $F=I+J+\mathfrak{m}\mathfrak{n}$ the {fiber product} of I and J in $T=S\otimes_{\mathbb{K}}R$. Fix a positive integer k.

The aim of this section is to describe the a_i-invariants of T/F^k via the corresponding conditions of I and J.

Let us start by recalling some pertinent facts of local cohomology and Čech complex.

Definition 2.2 Let M be an S-module M and $\mathfrak{a}$ be an S-ideal.

(a) Set

$\Gamma_{\mathfrak{a}}(M):=\{x\in M:\mathfrak{a}^t x=0\text{ for some }t\in\mathbb{N}\}$.

Let $H_{\mathfrak{a}}^i(-)$ be the i-th right derived functor of $\Gamma_{\mathfrak{a}}(-)$, namely, $H_{\mathfrak{a}}^i(M):=H^j(\Gamma_{\mathfrak{a}}(I^\cdot))$, in which $I^\cdot$ is an injective resolution of M. The module $H_{\mathfrak{a}}^i(M)$ will be called the i-th local cohomology of M with support in $\mathfrak{a}$.

(b) The module M is called $\mathfrak{a}$-torsion if $\Gamma_{\mathfrak{a}}(M)=M$, namely, if each element of M is annihilated by some power of $\mathfrak{a}$.

Next, we collect some well-known facts from Refs. [3] and [15] regarding local cohomology modules.

Lemma 2.3 Let M be an S-module and $\mathfrak{a}$ an S-ideal.

(a) Let $\{M_\gamma\}$ be a family of S-modules. Then $H_{\mathfrak{a}}^j(\oplus_\gamma M_\gamma)\cong\oplus_\gamma H_{\mathfrak{a}}^j(M_\gamma)$ for all $j\geqslant 0$.

(b) If $S\to R$ is a ring homomorphism and N is an R-module, then $H_{\mathfrak{a}}^j(N)=H_{\mathfrak{a}R}^j(N)$.

(c) Any short exact sequence of S-modules $0\to M\to N\to L\to 0$ induces a long exact sequence of local cohomology modules

$$\cdots\to H_{\mathfrak{a}}^j(M)\to H_{\mathfrak{a}}^j(N)\to H_{\mathfrak{a}}^j(L)\to H_{\mathfrak{a}}^{j+1}(M)\to\cdots.$$

(d) Assume that M is $\mathfrak{b}$-torsion for some S-ideal $\mathfrak{b}$. Then, $H_{\mathfrak{a}+\mathfrak{b}}^j(M)\cong H_{\mathfrak{a}}^j(M)$ for all $j\geqslant 0$.

(e) If M is $\mathfrak{a}$-torsion, then $H_{\mathfrak{a}}^j(M)=0$ for all $j>0$.

Our argument afterwards also depends heavily on the computation of local cohomologies in terms of Čech complexes.

Definition 2.4 For elements $m_1,\cdots,m_r$ in a commutative ring R, set $m_\sigma=\prod_{i\in\sigma}m_i$ for $\sigma\subseteq[r]$. The Čech complex $\check{C}^\cdot(m_1,\cdots,m_r)$ is the cochain complex (upper indices increasing from the copy of R sitting in cohomological degree 0)

$$0\to R\to\bigoplus_{i=1}^{r}R[m_i^{-1}]\to\cdots\to\bigoplus_{|\sigma|=k}R[m_\sigma^{-1}]\to\cdots\to R[m_{[r]}^{-1}]\to 0,$$

with the map

$$\partial_{|\sigma|}^i:R[m_\sigma^{-1}]\to R[m_{\sigma\cup\{i\}}^{-1}]$$

between the summands in $\check{C}^\cdot(m_1,\cdots,m_r)$ being sign$(i,\sigma\cup\{i\})$ times the canonical localization homomorphism.

Čech complex facilitates the computation of local cohomologies.

Lemma 2.5[5] The local cohomology of M supported on the ideal $\mathfrak{a}=(m_1,\cdots,m_r)$ in R is the cohomology of the Čech complex tensored with M:

$$H_{\mathfrak{a}}^i(M)=H^i(M\otimes\check{C}^\cdot(m_1,\cdots,m_r)).$$

The following results are also crucial for our argument in this section.

Lemma 2.6[16] Take the assumptions as in setting-section-2. Assume in addition that $I\subseteq\mathfrak{m}^2$ and $J\subseteq\mathfrak{n}^2$. Furthermore, let $H=I+\mathfrak{m}\mathfrak{n}$. For each $1\leqslant t\leqslant k$, denote $G_t=H^k+\sum_{i=1}^{t}(\mathfrak{mn})^{k-i}J^i$ and $G_0=H^k$. Then,

(a) there is an equality $F^k=H^k+\sum_{i=1}^{k}(\mathfrak{mn})^{k-i}J^i$ for each positive integer k.

(b) one has $G_{t-1}\cap(\mathfrak{m}\,\mathfrak{n})^{k-t}J^t=\mathfrak{m}^{k-t+1}\mathfrak{n}^{k-t}J^t$ for each t.

Lemma 2.7[12] Let $(S,\mathfrak{m})$ be a local ring, and let M be a finitely generated S-module. We have $H_{\mathfrak{m}}^i(M)=0$ for $i<\operatorname{depth}(M)$ and for $i>\dim(M)$.

Before presenting the main result of this section, we collect some preliminary results.

Proposition 2.8 Take the assumptions as in setting 2.1.

(a) For any integer $0\leqslant t<k$, $a_1(S/\mathfrak{m}^tI^{k-t})=a_1(S/I^{k-t})$.

(b) If $\dim(R)>2$ and $\dim(S)>2$, then $a_1\left(\frac{T}{\mathfrak{m}^k\mathfrak{n}^k}\right)=2k-2$.

(c) If $\dim(S)>2$, then $a_1(S/I)=a_2(I)$.

(d) Set $m_{\sigma\delta}:=\prod_{i\in\sigma}x_i\cdot\prod_{j\in\delta}y_j$ for $\sigma\subseteq[s]$ and $\delta\subseteq[r]$. Let $F=I+J+\mathfrak{mn}\subseteq T$. We use ∂_j to denote the differential map in $\frac{T}{F^k}\otimes\check{C}^{\cdot}(x_1,\cdots,x_s,y_1,\cdots,y_r)$ at the positions from j to $j+1$. Let $\partial_j^1\}$ and ∂_j^2 be the restriction of ∂_j on $\bigoplus_{|\sigma|=j}\frac{T}{F^k}[m_{\sigma\varnothing}^{-1}]$ and $\bigoplus_{|\delta|=j}\frac{T}{F^k}[m_{\varnothing\delta}^{-1}]$ respectively. Then $\partial_j=\partial_j^1\oplus\partial_j^2$ for each integer $j\geqslant 1$.

Proof When $0\leqslant t\leqslant k-1$, the following short exact sequence

$$0\to\frac{I^{k-t}}{I^{k-t}\mathfrak{m}^t}\to\frac{S}{I^{k-t}\mathfrak{m}^t}\to\frac{S}{I^{k-t}}\to 0$$

induces a long exact sequence

$$\cdots\to H_{\mathfrak{m}}^1\left(\frac{I^{k-t}}{I^{k-t}\mathfrak{m}^t}\right)\to H_{\mathfrak{m}}^1\left(\frac{S}{I^{k-t}\mathfrak{m}^t}\right)\to H_{\mathfrak{m}}^1\left(\frac{S}{I^{k-t}}\right)\to H_{\mathfrak{m}}^2\left(\frac{I^{k-t}}{I^{k-t}\mathfrak{m}^t}\right)\to\cdots.$$

Since $\frac{I^{k-i}}{I^{k-i}\mathfrak{m}^i}$ is $\mathfrak{m}$-torsion for $1\leqslant i\leqslant k$, we have $H_{\mathfrak{m}}^1\left(\frac{I^{k-i}}{I^{k-i}\mathfrak{m}^i}\right)=0=H_{\mathfrak{m}}^2\left(\frac{I^{k-i}}{I^{k-i}\mathfrak{m}^i}\right)$ by 2.3 (e). Consequently

$$H_{\mathfrak{m}}^1\left(\frac{S}{I^{k-t}\mathfrak{m}^t}\right)\cong H_{\mathfrak{m}}^1\left(\frac{S}{I^{k-t}}\right),$$

and hence $a_1(S/\mathfrak{m}^tI^{k-t})=a_1(S/I^{k-t})$.

(b) We will prove this after Lemma 3.2.

(c) The short exact sequence

$$0\to I\to S\to S/I\to 0$$

yields a long exact sequence

$$\cdots\to H_{\mathfrak{m}}^1(S)\to H_{\mathfrak{m}}^1(S/I)\to H_{\mathfrak{m}}^2(I)\to H_{\mathfrak{m}}^2(S)\to\cdots.$$

Applying a graded version of Lemma 2.7, we get $H_{\mathfrak{m}}^1(S)=H_{\mathfrak{m}}^2(S)=0$. As a result, $a_1(S/I)=a_2(I)$.

(d) When $j\geqslant 1$, we have

$$\frac{T}{F^k}\otimes\check{C}^j(x_1,\cdots,x_s,y_1,\cdots,y_r)=\bigoplus_{|\sigma\cup\delta|=j}\frac{T}{(I+J+\mathfrak{mn})^k}[m_{\sigma\delta}^{-1}].$$

If both σ and δ are nonempty, then

$$\frac{T}{(I+J+\mathfrak{mn})^k}[m_{\sigma\delta}^{-1}]=0.$$

Therefore, the module $\bigoplus_{|\sigma\cup\delta|=j}\frac{T}{(I+J+\mathfrak{mn})^k}[m_{\sigma\delta}^{-1}]$ is simply

$$\left(\bigoplus_{|\sigma|=j}\frac{T}{F^k}[m_{\sigma\varnothing}^{-1}]\right)\oplus\left(\bigoplus_{|\delta|=j}\frac{T}{F^k}[m_{\varnothing\delta}^{-1}]\right).$$

Since $\mathfrak{m}\,T[m_{\sigma\varnothing}^{-1}]=T[m_{\sigma\varnothing}^{-1}]$ and $J+\mathfrak{n}=\mathfrak{n}$, we have

$$F^kT[m_{\sigma\varnothing}^{-1}]=(I+J+\mathfrak{mn})^kT[m_{\sigma\varnothing}^{-1}]=(I+\mathfrak{n})^kT[m_{\sigma\varnothing}^{-1}].$$

This means

$$\frac{T}{F^k}[m_{\sigma\varnothing}^{-1}]=\frac{T}{(I+\mathfrak{n})^k}[m_{\sigma\varnothing}^{-1}].$$

Likewise,

$$\frac{T}{F^k}[m_{\varnothing\delta}^{-1}]=\frac{T}{(J+\mathfrak{m})^k}[m_{\varnothing\delta}^{-1}].$$

So the Čech complex at the positions from j to $j+1$ can be written as

$$\cdots\to\left(\bigoplus_{|\sigma|=j}\frac{T}{(I+\mathfrak{n})^k}[m_{\sigma\varnothing}^{-1}]\right)\oplus\left(\bigoplus_{|\delta|=j}\frac{T}{(J+\mathfrak{m})^k}[m_{\varnothing\delta}^{-1}]\right)\xrightarrow{\partial_j}\left(\bigoplus_{|\sigma|=j+1}\frac{T}{(I+\mathfrak{n})^k}[m_{\sigma\varnothing}^{-1}]\right)\oplus\left(\bigoplus_{|\delta|=j+1}\frac{T}{(J+\mathfrak{m})^k}[m_{\varnothing\delta}^{-1}]\right)\to\cdots.$$

Furthermore, when $j > 1$, $\partial_j\left(\frac{T}{(I+\mathfrak{n})^k}[m_{\sigma\emptyset}^{-1}]\right)$ is a subset of

$$\Big(\bigoplus_{\substack{|\sigma|=j,\\ i\in[s]\setminus\sigma}}\frac{T}{F^k}[m_{\sigma\emptyset}^{-1}x_i^{-1}]\Big)\oplus\Big(\bigoplus_{\substack{|\delta|=j,\\ i\in[r]\setminus\delta}}\frac{T}{F^k}[m_{\emptyset\delta}^{-1}y_i^{-1}]\Big)=$$

$$\bigoplus_{|\sigma|=j+1}\frac{T}{(I+\mathfrak{n})^k}[m_{\sigma\emptyset}^{-1}].$$

by (3) and (4). Then

$$\mathrm{im}\partial_j^1\subseteq\bigoplus_{|\sigma|=j+1}\frac{T}{(I+\mathfrak{n})^k}[m_{\sigma\emptyset}^{-1}].$$

Likewise,

$$\mathrm{im}\partial_j^2\subseteq\bigoplus_{|\delta|=j+1}\frac{T}{(J+\mathfrak{m})^k}[m_{\emptyset\delta}^{-1}].$$

Thus $\partial_j=\partial_j^1\oplus\partial_j^2$ for each integer $j\geqslant 1$.

Now, we are ready to present the first main result of this paper.

Theorem 2.9 Take the assumptions as in setting 2.1.

(a) If $j\geqslant 2$, then

$$a_j(T/F^k)=\max\{a_j(S/I^{k-t})+t,\ a_j(R/J^{k-t})+t:0\leqslant t\leqslant k-1\}.$$

(b) Assume in addition that $\dim(S)>2$, $\dim(R)>2$, $I\subseteq\mathfrak{m}^2$, $\sqrt{I}\neq\mathfrak{m}$, $J\subseteq\mathfrak{n}^2$ and $\sqrt{J}\neq\mathfrak{n}$. Then,

$$a_1(T/F^k)=\max\{2k-2, a_1(S/I^{k-t})+t,\ a_1(R/J^{k-t})+t\ :0\leqslant t\leqslant k-1\}.$$

Proof Assume that $j\geqslant 1$. Then Lemma 2.5 says

$$H_{\mathfrak{m}+\mathfrak{n}}^j(T/F^k)=H^j((T/F^k)\otimes\check{C}^{\cdot}(x_1,\cdots,x_s,y_1,\cdots,y_r)).$$

Set $m_{\sigma\delta}=\prod_{i\in\sigma}x_i\cdot\prod_{i\in\delta}y_i\in T$ for $\sigma\subseteq[s]$ and $\delta\subseteq[r]$. We have

$$\frac{T}{F^k}\otimes\check{C}^j(x_1,\cdots,x_s,y_1,\cdots,y_r)=$$

$$\bigoplus_{|\sigma\cup\delta|=j}\frac{T}{(I+J+\mathfrak{m}\mathfrak{n})^k}[m_{\sigma\delta}^{-1}].$$

Notice that both S and R have multigraded structures respectively. Hence T will have inherited multigrading, bigrading and standard grading. We will use this fact freely in the following proof.

(a) When $j\geqslant 2$, we have the following bigraded decomposition via Lemma 2.3(b), Lemma 2.5 and Proposition 2.8(d):

$$H_{\mathfrak{m}+\mathfrak{n}}^j\Big(\frac{T}{(I+J+\mathfrak{m}\mathfrak{n})^k}\Big)=$$

$$\frac{\ker(\partial_j)}{\mathrm{im}(\partial_{j-1})}=\frac{\ker(\partial_j^1\oplus\partial_j^2)}{\mathrm{im}(\partial_{j-1}^1\oplus\partial_{j-1}^2)}\cong$$

$$\frac{\ker(\partial_j^1)\}}{im(\partial_{j-1}^1)}\oplus\frac{ker(\partial_j^2)\}}{\mathrm{im}(\partial_{j-1}^2)}\cong$$

$$H_{\mathfrak{m}T}^j\Big(\frac{T}{(I+\mathfrak{n})^k}\Big)\oplus H_{\mathfrak{n}T}{}^j\Big(\frac{T}{(J+\mathfrak{m})^k}\Big)=$$

$$H_{\mathfrak{m}}^j\Big(\frac{T}{(I+\mathfrak{n})^k}\Big)\oplus H_{\mathfrak{n}}^j\Big(\frac{T}{(J+\mathfrak{m})^k}\Big).$$

This implies

$$a_j(T/(I+J+\mathfrak{m}\mathfrak{n})^k)=\max\{a_j(T/(I+\mathfrak{n})^k),\ a_j(T/(J+\mathfrak{m})^k)\}.$$

As $S-$modules, we have the following bigraded isomorphism:

$$\frac{T}{(I+\mathfrak{n})^k}\cong\bigoplus_{\beta\in\mathbb{N}^r,|\beta|<k}\frac{S}{I^{k-|\beta|}}(0,-|\beta|).$$

Then the canonical epimorphism $T\to S$ induces an isomorphism

$$H_{\mathfrak{m}}^j\Big(\frac{T}{(I+\mathfrak{n})^k}\Big)\cong H_{\mathfrak{m}}^j\Big(\bigoplus_{\beta\in\mathbb{N}^r,|\beta|<k}\frac{S}{I^{k-|\beta|}}(0,-|\beta|)\Big)\cong$$

$$\bigoplus_{\beta\in\mathbb{N}^r,|\beta|<k}H_{\mathfrak{m}}^j\Big(\frac{S}{I^{k-|\beta|}}\Big)(0,-|\beta|),$$

via Lemma 2.3(a). Hence

$$a_j(T/(I+\mathfrak{n})^k)=\max\{a_j(S/I^{k-t})+t:\ 0\leqslant t\leqslant k-1\}.$$

Likewise,

$$a_j(T/(J+\mathfrak{m})^k)=\max\{a_j(R/\{J^{k-t})+t:\ 0\leqslant t\leqslant k-1\}.$$

Therefore, when $j\geqslant 2$, we arrive at the conclusion

$$a_j(T/F^k)=\max\{a_j(S/I^{k-t})+t,\ a_j(R/J^{k-t})+t:0\leqslant t\leqslant k-1.$$

(b) Now we consider the case with $j=1$. The proof will be divided into three steps.

Claim 1 $a_1(T/F^k)\geqslant\max\{a_1(S/I^{k-t})+t,\ a_1(R/J^{k-t})+t\ :0\leqslant t\leqslant k-1\}$.

Since $\partial_1=\partial_1^1\oplus\partial_1^2$ by Proposition 2.8(d), one has $\ker\partial_1=\ker\partial_1^1\oplus\ker\partial_1^2$. Let ∂_0^1 be the composition of ∂_0 with the projection map from $\bigoplus_{|\sigma\cup\delta|=1}\frac{T}{(I+J+\mathfrak{m}\mathfrak{n})^k}[m_{\sigma\delta}^{-1}]$ to its direct summand $\bigoplus_{|\sigma|=1}\frac{T}{(I+J+\mathfrak{m}\mathfrak{n})^k}[m_{\sigma\emptyset}^{-1}]$), and similarly define

∂_0^2. It is clear that $\mathrm{im}(\partial_0)\subseteq\mathrm{im}(\partial_0^1)\oplus\mathrm{im}(\partial_0^2)$ holds. Therefore,

$$\frac{\ker(\partial_1^1)\oplus\ker(\partial_1^2)}{\mathrm{im}(\partial_0^1)\oplus\mathrm{im}(\partial_0^2)}\cong\frac{\ker(\partial_1^1)}{im(\partial_0^1)}\oplus\frac{ker(\partial_1^2)}{\mathrm{im}(\partial_0^2)}$$

is an epimorphic image of $H^1_{\mathfrak{m}+\mathfrak{n}}(T/F^k)\cong\frac{\ker(\partial_1)}{im(\partial_0)}$, which in turn implies

$$a_1(T/F^k)\geqslant\max\{l\in\mathbb{Z}:(\frac{\ker(\partial_1^1)}{\mathrm{im}(\partial_0^1)})_l\neq 0 \text{ or } (\frac{\ker(\partial_1^2)}{im(\partial_0^2)})_l\neq 0\}.$$

Then

$$\max\{l\in\mathbb{Z}:(\frac{\ker(\partial_1^1)}{\mathrm{im}(\partial_0^1)})_l\neq 0\}=$$
$$\max\{l\in\mathbb{Z}:H^j_{\mathfrak{m}}(\frac{T}{(I+\mathfrak{n})^k})_l\neq 0\}=$$
$$\max\{l\in\mathbb{Z}:\bigoplus_{\beta\in\mathbb{N}^{\prime},|\beta|<k}H^1_{\mathfrak{m}}(\frac{S}{I^{k-|\beta|}}(0,-|\beta|))_l\neq 0\}=$$
$$\max\{a_1(S/I^{k-t})+t:0\leqslant t\leqslant k-1\}.$$

Similarly, one has

$$\max\{l\in\mathbb{Z}:(\frac{\ker\partial_1^2}{\mathrm{im}\partial_0^2})_l\neq 0\}=$$
$$\max\{a_1(R/J^{k-t})+t:0\leqslant t\leqslant k-1\}.$$

Thus, $a_1(T/F^k)\geqslant\max\{a_1(S/I^{k-t})+t,\ a_1(R/J^{k-t})+t\ :0\leqslant t\leqslant k-1\}$, establishing the first claim.

Claim 2 $a_1(T/F^k)\geqslant 2k-2$.

It is sufficient to find a bigraded element $u\in\ker(\partial_1)$ such that its total degree $\deg(u)=2k-2$ and $u\notin\mathrm{im}(\partial_0)$. For any $v\in T$, let $[v]$, $[v]_{x_i}$ and $[v]_{y_i}$ be the equivalence classes of v in $\frac{T}{F^k}$, $\frac{T}{F^k}[x_i^{-1}]$ and $\frac{T}{F^k}[y_i^{-1}]$ respectively.

Suppose $f\in S$ and $g\in R$ with $\deg(f)=\deg(g)=k-1$, then

$$[fg]_{x_i}\neq[0]_{x_i}\Leftrightarrow x_i^l fg\notin(I+\mathfrak{n})^k \text{ for } l\geqslant 0$$

by the equality (4). Since $\deg(g)=k-1$ and $g\in R$, we have $g\notin\mathfrak{n}^k$ and $g\in\mathfrak{n}^p$ for any $1\leqslant p\leqslant k-1$. Meanwhile, it is clear that $I^k\subseteq I^{k-1}\subseteq\cdots\subseteq I^2\subseteq I$ holds. Therefore, the above equivalent statements can be further simplified into saying $x_i^l f\notin I$ for $l\geqslant 0$, i. e., $f\notin I:(x_i)^\infty$.

As $\sqrt{I}\neq\mathfrak{m}$, we can find some homogeneous element $f\in S$ of degree $k-1$ satisfying $f\notin I:\mathfrak{m}^\infty$. Similarly, we can find some homogeneous element $g\in R$ of degree $k-1$ satisfying $g\notin J:\mathfrak{n}^\infty$. We will verify that $u=(\bigoplus_{i=1}^{s}[fg]_{x_i})\oplus(\bigoplus_{i=1}^{r}[0]_{y_i})$ is the expected element.

To see this, notice first that $u\in\ker(\partial_1)$ and u is bi-homogeneous of degree $(k-1,k-1)$. Consequently, the total degree $\deg(u)=2(k-1)$. Thus, it remains to show that $u\notin\mathrm{im}(\partial_0)$. Assume to the contrary that there exists an element $h\in T$ such that $\partial_0([h])=(\bigoplus_{i=1}^{s}[fg]_{x_i})\oplus(\bigoplus_{i=1}^{r}[0]_{y_i})$. Without loss of generality, we may assume that h is homogeneous of bidegree $(k-1,k-1)$. Whence, $[h\text{-}fg]_{x_i}=[0]_{x_i}$ and $[h]_{y_j}=[0]_{y_j}$ for each $1\leqslant i\leqslant s$ and $1\leqslant j\leqslant r$. Notice that

$$[h\text{-}fg]_{x_i}=[0]_{x_i}\Leftrightarrow h\text{-}fg\in(I+\mathfrak{n})^k:(x_i)^\infty$$

holds for $1\leqslant i\leqslant s$. Consequently, we have

$$h-fg\in(I+\mathfrak{n})^k:\mathfrak{m}^\infty\subseteq(I+\mathfrak{n}^k):\mathfrak{m}^\infty.$$

Since the partial degree $\deg_y(h-fg)=k-1$, it is clear that $h-fg\notin\mathfrak{n}^k:\mathfrak{m}^\infty$ unless $h-fg=0$. So by bigrading, $h-fg\in I:\mathfrak{m}^\infty$. Likewise, we will have $h\in J:\mathfrak{n}^\infty$. As a result, $fg=h-(h-fg)\in(I:\mathfrak{m}^\infty)+(J:\mathfrak{n}^\infty)$. Then, again, by bigrading, we will have $f\in I:\mathfrak{m}^\infty$ or $g\in J:\mathfrak{n}^\infty$, a contradiction. And this completes our proof for the second claim.

So far, we have proved

$$a_1(T/F^k)\geqslant\max\{2k-2,a_1(S/I^{k-t})+t,\ a_1(R/J^{k-t})+t\ :0\leqslant t\leqslant k-1\}. \tag{5}$$

Claim 3 The converse direction of the inequality (5) also holds.

Let $H=I+\mathfrak{m}\mathfrak{n}$ and $G_t=H^k+\sum_{i=1}^{t}(\mathfrak{m}\mathfrak{n})^{k-i}J^i$ for $0\leqslant t\leqslant k$. Since

$$G_{t-1}\cap(\mathfrak{m}\,\mathfrak{n})^{k-t}J^t=\mathfrak{m}^{k-t+1}\mathfrak{n}^{k-t}J^t$$

for $1\leqslant t\leqslant k$ by Lemma 2.6, the following short exact sequence arises

$$0\to\frac{(\mathfrak{m}\mathfrak{n})^{k-t}J^t}{\mathfrak{m}^{k-t+1}\mathfrak{n}^{k-t}J^t}\to\frac{T}{G_{t-1}}\to\frac{T}{G_t}\to 0,$$

which induces a long exact sequence

$$\cdots\to H^1_{\mathfrak{m}+\mathfrak{n}}(\frac{T}{G_{t-1}})\to H^1_{\mathfrak{m}+\mathfrak{n}}(\frac{T}{G_t})\to$$

$$H^2_{\mathfrak{m}+\mathfrak{n}}(\frac{(\mathfrak{m}\mathfrak{n})^{k-t}J^t}{\mathfrak{m}^{k-t+1}\mathfrak{n}^{k-t}J^t}) \to \cdots$$

by Lemma 2.3(c). Since $\frac{(\mathfrak{m}\mathfrak{n})^{k-t}J^t}{\mathfrak{m}^{k-t+1}\mathfrak{n}^{k-t}J^t}$ is an $\mathfrak{m}$ T-torsion T-module, according to Lemma 2.3(b) and (d) we have

$$H^2_{(\mathfrak{m}+\mathfrak{n})T}(\frac{(\mathfrak{m}\mathfrak{n})^{k-t}J^t}{\mathfrak{m}^{k-t+1}\mathfrak{n}^{k-t}J^t}) \cong H^2_{\mathfrak{n}T}(\frac{(\mathfrak{m}\mathfrak{n})^{k-t}J^t}{\mathfrak{m}^{k-t+1}\mathfrak{n}^{k-t}J^t}) = H^2_{\mathfrak{n}}(\frac{(\mathfrak{m}\mathfrak{n})^{k-t}J^t}{\mathfrak{m}^{k-t+1}\mathfrak{n}^{k-t}J^t}) \cong H^2_{\mathfrak{n}}(\frac{\mathfrak{m}^{k-t}}{\mathfrak{m}^{k-t+1}} \otimes \mathfrak{n}^{k-t}J^t) \cong \frac{\mathfrak{m}^{k-t}}{\mathfrak{m}^{k-t+1}} \otimes H^2_{\mathfrak{n}}(\mathfrak{n}^{k-t}J^t).$$

Hence,

$$a_1(T/G_t) \leqslant \max\{a_1(T/G_{t-1}), a_2(\{(\mathfrak{m}\mathfrak{n})^{k-t}J^t\}/\{\mathfrak{m}^{k-t+1}\mathfrak{n}^{k-t}J^t\})\} = \max\{a_1(T/G_{t-1}), a_2(\mathfrak{n}^{k-t}J^t)+k-t\} = \max\{a_1(T/G_{t-1}), a_1(R/J^t)+k-t\}.$$

The last equality holds via Proposition 2.8 and (c). Thus we can conclude

$$a_1(T/(I+J+\mathfrak{m}\mathfrak{n})^k) = a_1(T/G_k) \leqslant \max\{a_1(T/G_0), a_1(R/J^t)+k-t: 1 \leqslant t \leqslant k\} = \max\{a_1(T/H^k), a_1(R/J^t)+k-t: 1 \leqslant t \leqslant k\}.$$

Notice that $H = I + \mathfrak{m}\mathfrak{n}$ can also be viewed as the fiber product of $I \subseteq S$ and $(0) \subseteq R$. With a similar argument, we can get

$$a_1(T/H^k) \leqslant \max\{a_1(T/\mathfrak{m}^k\mathfrak{n}^k), a_1(S/I^t)+k-t: 1 \leqslant t \leqslant k\} = \max\{2k-2, a_1(S/I^t)+k-t: 1 \leqslant t \leqslant k\}$$

by Proposition 2.8(b). These arguments altogether yield

$$a_1(T/F^k) \leqslant \max\{2k-2, a_1(S/I^{k-t})+t, a_1(R/J^{k-t})+t: 0 \leqslant t \leqslant k-1\}, \tag{6}$$

which establishes the third claim. Now, combing the inequalities (5) with (6), we complete the proof.

Remark 2.10 In Theorem 2.9(b), if $T = S[y]$ and $J = 0$, then the conditions $\dim(S) > 2$ and $I \subseteq \mathfrak{m}^2$ can be removed. The proof is only slightly different and will be omitted here.

3 Top dimensional a_i-invariants of $S/I_\Delta^{(n)}$

Let Δ be a k-dimensional simplicial complex over $[s] := \{1,2,\cdots,s\}$ for some positive integers k and s. Assume that $S = \mathbb{K}[x_1,\cdots,x_s]$ is a polynomial ring over a field $\mathbb{K}$ and $I_\Delta \subseteq S$ is the Stanley-Reisner ideal associated to Δ. The main task of this section is to investigate the a_{k+1}-invariants associated to its power I_Δ^n and its symbolic power $I_\Delta^{(n)}$. The case when $k=1$ has already been considered in Ref.[4]. There, it was shown that $a_2(S/I_\Delta^{(n)}) = a_2(S/I_\Delta^n)$ holds when Δ has no isolated vertex. We will generalize this result here by showing $a_{k+1}(S/I_\Delta^{(n)}) = a_{k+1}(S/I_\Delta^n)$ for any k-dimensional simplicial complex Δ. After that, we will characterize when $a_{k+1}(S/I_\Delta^{(n)})$ is maximal.

Let us start with reviewing some basic notions. Recall that Δ is a simplicial complex on $[s]$ if Δ is a collection of subsets of $[s]$ such that if $F \in \Delta$ and $F' \subseteq F$ then $F' \in \Delta$. Each element $F \in \Delta$ is called a face of Δ. The dimension of F is defined to be $\dim(F) := |F|-1$ and the dimension of Δ is defined to be $\dim(\Delta) := \max\{\dim(F): F \in \Delta\}$. A facet is a maximal face of Δ with respect to inclusion. We will use $\mathscr{F}(\Delta)$ to denote the set of facets of Δ. Meanwhile, a non-face of Δ is a subset F of $[s]$ with $F \notin \Delta$. We will use $\mathfrak{N}(\Delta)$ to denote the set of minimal non-faces of Δ.

In order to describe $a_i(S/I_\Delta^{(n)})$, we need to know when $H^i_{\mathfrak{m}}(S/I_\Delta^{(n)})_\alpha$ is vanishing. Due to a formula in Ref.[17] of Takayama, this problem boils down to computing the dimension of the simplicial homology of the degree complex. Set $G_\alpha := \{i \in [s]: \alpha_i < 0\}$. Recall that the degree complex $\Delta_\alpha(I)$ of a monomial ideal I is given by

$$\Delta_\alpha(I) := \{F \subseteq [s] \backslash G_\alpha : \boldsymbol{x}^\alpha \notin IS_{F \cup G_\alpha}\}.$$

Here, $S_{F\cup G_\alpha} = S[x_i^{-1}: i \in F \cup G_\alpha]$ and $\boldsymbol{x}^\alpha = x_1^{\alpha_1} \cdots x_s^{\alpha_s}$.

For each monomial ideal I, consider a simplicial complex

$$\Delta(I) := \{F \subseteq [s]: \boldsymbol{x}_F \notin \sqrt{I}\}.$$

It is clear that $\Delta(I) = \Delta(\sqrt{I})$ holds. And when I is square free, it is exactly the Stanley-Reisner complex of I. We also have $\Delta(S) = \emptyset$ since for any $F \subset [s]$, $x_F \in \sqrt{S} = S$. %Here is the

Takayama's formula.

Lemma 3.1 (Takayama) Let I be a monomial ideal in S and α a vector in $\mathbb{Z}^s$. Then

$$\dim_{\mathbb{K}} H_{\mathfrak{m}}^{i}(S/I)_{\alpha}=\begin{cases}\dim_{\mathbb{K}}\widetilde{H}_{i-|G_{\alpha}|-1}(\Delta_{\alpha}(I)), \\ \qquad \text{if } G_{\alpha}\in\Delta(I), \\ 0, \text{ otherwise.}\end{cases}$$

Here, $\widetilde{H}_i(\Delta_{\alpha}(I))$ is the i-th reduced simplicial homology group of the complex $\Delta_{\alpha}(I)$ over $\mathbb{K}$.

Lemma 3.2[18] Take the assumptions as in setting 2.1. Assume in addition that $I\subseteq S$ and $J\subseteq R$ are two monomial ideals. Then, for any $\alpha\in\mathbb{Z}^s$, $\beta\in\mathbb{Z}^r$ and $\gamma=(\alpha,\beta)\in\mathbb{Z}^{s+r}$, we have the following two cases:

(a) if $p=1$ while both $\Delta_{\alpha}(I^{s-|\beta|})$ and $\Delta_{\beta}(I^{s-|\alpha|})$ are nonempty, then

$$\dim_{\mathbb{K}} H_{\mathfrak{m}+\mathfrak{n}}{}^{p}(T/(I+J+\mathfrak{m}\mathfrak{n})^k)_{\gamma}= \dim_{\mathbb{K}} H_{\mathfrak{m}}{}^{p}(S/I^{k-|\beta|})_{\alpha}+ \dim_{\mathbb{K}} H_{\mathfrak{n}}{}^{p}(R/J^{k-|\alpha|})_{\beta}+1;$$

(b) otherwise,

$$\dim_{\mathbb{K}} H_{\mathfrak{m}+\mathfrak{n}}{}^{p}(T/(I+J+\mathfrak{m}\mathfrak{n})^k)_{\gamma}= \dim_{\mathbb{K}} H_{\mathfrak{m}}{}^{p}(S/I^{k-|\beta|})_{\alpha}+\dim_{\mathbb{K}} H_{\mathfrak{n}}{}^{p}(R/J^{k-|\alpha|})_{\beta}.$$

Here, $|\alpha|=\sum_{i=1}^{s}\alpha_i$ for $\alpha=(\alpha_1,\cdots,\alpha_s)$ and one can similarly define $|\beta|$.

As a quick application of the above two lemmas, we finish the proof of Proposition 2.8.

Proof(Proof of Proposition 2.8(b)) Notice that the ideal $\mathfrak{m}^k\mathfrak{n}^k\subseteq T$ is the fiber product of $I=(0)\subseteq S$ and $J=(0)\subseteq R$. Now, take arbitrary $\alpha\in\mathbb{Z}^s$ and $\beta\in\mathbb{Z}^r$.

First, we consider $H_{\mathfrak{m}}^{1}(S/I^{k-|\beta|})_{\alpha}$. If $|\beta|<k$, then $I^{k-|\beta|}=0$. Hence $H_{\mathfrak{m}}^{1}(S/I^{k-|\beta|})_{\alpha}=H_{\mathfrak{m}}^{1}(S)_{\alpha}=0$ by Lemma 2.7. When $|\beta|\geqslant k$, we have $I^{k-|\beta|}=S$. Then $H_{\mathfrak{m}}^{1}(S/I^{k-|\beta|})_{\alpha}=H_{\mathfrak{m}}^{1}(0)_{\alpha}=0$. Thus for any α and β, $H_{\mathfrak{m}}^{1}(S/I^{k-|\beta|})_{\alpha}=0$. Likewise, $H_{\mathfrak{n}}{}^{1}(R/J^{k-|\alpha|})_{\beta}=0$.

So, according to Lemma 2.6, if $H_{\mathfrak{m}+\mathfrak{n}}^{1}(T/(I+J+\mathfrak{m}\mathfrak{n})^k)_{(\alpha,\beta)}\neq 0$, then both $\Delta_{\alpha}(I^{k-|\beta|})$ and $\Delta_{\beta}(I^{k-|\alpha|})$ are nonempty. In the following, we will suppose that this is the case.

Notice that if $|\beta|\geqslant k$, then $I^{k-|\beta|}=S$. Whence, for any $F\subseteq[s]\backslash G_{\alpha}$, we have $\boldsymbol{x}^{\alpha}\in S_{F\cup G_{\alpha}}$, which implies $F\notin\Delta_{\alpha}(S)$ by definition. So $\Delta_{\alpha}(I^{k-|\beta|})=\varnothing$, contradicting the assumption. Thus $|\beta|\leqslant k-1$ and similarly $|\alpha|\leqslant k-1$. Consequently, $a_1(\frac{T}{\mathfrak{m}^k\mathfrak{n}^k})\leqslant 2k-2$.

On the other hand, let $\alpha=(k-1,0,\cdots,0)\in\mathbb{Z}^s$ and $\beta=(k-1,0,\cdots,0)\in\mathbb{Z}^r$. We have $[s]\backslash\{1\}\in\Delta_{\alpha}(I^{k-|\beta|})$ since $x_1{}^{k-1}\notin(0)S_{[s]\backslash\{1\}}$. So $\Delta_{\alpha}(I^{k-|\beta|})$ is nonempty. Likewise, $\Delta_{\beta}(J^{k-|\alpha|})$ is nonempty. Thus, $H_{\mathfrak{m}+\mathfrak{n}}^{1}(T/(\mathfrak{m}\mathfrak{n})^k)_{(\alpha,\beta)}\neq 0$ via Lemma 2.6, meaning $a_1(\frac{T}{\mathfrak{m}^k\mathfrak{n}^k})\geqslant 2k-2$. So $a_1(\frac{T}{\mathfrak{m}^k\mathfrak{n}^k})=2k-2$, completing the proof.

The following lemma gives a precise description of $\Delta_{\alpha}(I_{\Delta}^{(n)})$.

Lemma 3.3[1] Assume that $G_{\alpha}\in\Delta$ for some $\alpha\in\mathbb{Z}^s$. Then

$$\mathscr{F}(\Delta_{\alpha}(I_{\Delta}^{(n)}))= \{F\in\mathscr{F}(\mathrm{link}_{\Delta}(G_{\alpha})):\sum_{i\notin F\cup G_{\alpha}}a_i\leqslant n-1\}.$$

The concept of monomial localization was introduced in Ref. [19] as a simplification of the localization. Fix a subset $F\subseteq[s]$. Let $\pi_F: S\to\mathbb{K}[x_i: i\in[s]\backslash F]$ be the $\mathbb{K}$-algebra homomorphism sending x_i to x_i for $i\in[s]\backslash F$ and x_i to 1 for $i\in F$. The image of a monomial ideal I of S under the map π_F is called the monomial localization of I with respect to F and will be denoted by $I[F]$. It is clear that if I and J are both monomial ideals of S, then $(IJ)[F]=I[F]J[F]$ and $(I\cap J)[F]=I[F]\cap J[F]$.

The degree complex can be expressed using the monomial localization as follows.

Lemma 3.4[4,6] Let I be a monomial ideal in $S=\mathbb{K}[x_1,\cdots,x_s]$ and $\alpha=(\alpha_1,\cdots,\alpha_s)$ be a vector in $\mathbb{Z}^s$. Define α_+ to be the non-negative part of α, i.e., $\alpha_+:=(\alpha_1',\cdots,\alpha_s')$ where $\alpha_i'=\max(0,\alpha_i)$ for each i.

(a) $\Delta_{\alpha}(I)$ is a subcomplex of $\Delta(I)$. Moreover, if I has no embedded associated prime and $\alpha\in\mathbb{N}^s$, then $\mathscr{F}(\Delta_{\alpha}(I))\subseteq\mathscr{F}(\Delta(I))$.

(b) $\Delta_{\alpha}(I)=\{F\subseteq[s]\backslash G_{\alpha}: \boldsymbol{x}^{\alpha_+}\notin I[F\cup G_{\alpha}]$

S}.

(c) If $G_\alpha\neq\emptyset$, then $\Delta_\alpha(I)=\mathrm{link}_{\Delta_{\alpha_+}(I)}(G_\alpha)$.

Lemma 3.5 Let Δ be a k-dimensional complex on $[s]$. If $F\in\Delta$ with $\dim(F)=k$, then

$$I_\Delta[F]=(x_i:i\in[s]\backslash F).$$

Proof As $1\notin I_\Delta[F]$, it follows that $I_\Delta[F]\subseteq(x_i:i\in[s]\backslash F)$ holds. Conversely, for each $i\in[s]\backslash F$, $F\cup\{i\}\notin\Delta$ since $\dim(\Delta)=k$. So $\boldsymbol{x}_{F'\cup\{i\}}\in I_\Delta$ for some $F'\subseteq F$, implying $x_i\in I_\Delta[F]$. Since this holds for any $i\in[s]\backslash F$, we have the converse containment $I_\Delta[F]\supseteq(x_i:i\in[s]\backslash F)$.

Proposition 3.6 Let Δ be a k-dimensional complex on $[s]$. For each $\alpha=(\alpha_1,\cdots,\alpha_s)\in\mathbb{N}^s$ and each k-dimensional simplex $F\subseteq[s]$, the following statements are equivalent:

(a) $F\in\Delta_\alpha(I_\Delta^n)$;

(b) $F\in\Delta$ and $\sum\limits_{i\in[s]\backslash F}\alpha_i\leqslant n-1$.

Proof First assume that $F\in\Delta_\alpha(I_\Delta^n)$ holds. Since $G_\alpha=\emptyset$, according to Lemma 3.4(a) we have $F\in\Delta_\alpha(I_\Delta^n)\subseteq\Delta(I_\Delta^n)=\Delta(I_\Delta)=\Delta$. It follows from Lemma 3.4(b) and Lemma 3.5 that

$$\boldsymbol{x}^\alpha\notin I_\Delta^n[F]S=(I_\Delta[F])^nS=(x_i:i\in[s]\backslash F)^nS$$

holds. Hence $\sum\limits_{i\in[s]\backslash F}\alpha_i\leqslant n-1$ and this proves (a)⇒(b).

Conversely, assume that $F\in\Delta$ and $\sum\limits_{i\in[s]\backslash F}\alpha_i\leqslant n-1$ hold. Then by Lemma 3.5,

$$\boldsymbol{x}^\alpha\notin(x_i:i\in[s]\backslash F)^n=(I_\Delta[F])^nS=I_\Delta^n[F]S.$$

Thus $F\in\Delta_\alpha(I_\Delta^n)$ via Lemma 3.4(b), which proves (b)⇒(a).

Lemma 3.7 If Δ is a simplicial complex on $[s]$ and I_Δ is its Stanley-Reisner ideal in $S=\mathbb{K}[x_1,\cdots,x_s]$ over a field $\mathbb{K}$, then $\Delta=\Delta(I_\Delta^n)=\Delta(I_\Delta^{(n)})$ for all positive integer n.

Proof It follows from (1) and (2) that

$$\sqrt{I_\Delta^{(n)}}=\sqrt{\bigcap_{F\in\mathcal{F}(\Delta)}P_F^n}=\bigcap_{F\in\mathcal{F}(\Delta)}\sqrt{P_F^n}=\bigcap_{F\in\mathcal{F}(\Delta)}\sqrt{P_F}=\sqrt{\bigcap_{F\in\mathcal{F}(\Delta)}P_F}=\sqrt{I_\Delta}=\sqrt{I_\Delta^n}$$

holds. Therefore,

$$\Delta(I_\Delta^{(n)})=\Delta(\sqrt{I_\Delta^{(n)}})=\Delta(\sqrt{I_\Delta^n})=\Delta(I_\Delta^n),$$

and they agree with $\Delta(\sqrt{I_\Delta})=\Delta(I_\Delta)=\Delta$.

Recall that the pure ith skeleton of Δ is the pure simplicial complex $\Delta^{(i)}$ whose facets are the faces F of Δ with $\dim(F)=i$.

Proposition 3.8 Let Δ be a k-dimensional simplicial complex over $[s]$. For any $\alpha\in\mathbb{Z}^s$ with $G_\alpha\in\Delta$, we have $\Delta_\alpha(I_\Delta^n)^{(k)}=\Delta_\alpha(I_\Delta^{(n)})^{(k)}$.

Proof If $G_\alpha\neq\emptyset$, then $\dim(\Delta_\alpha(I_\Delta^{(n)}))\leqslant\dim(\mathrm{link}_\Delta(G_\alpha))<\dim(\Delta)=k$ by Lemma 3.3. Similarly, if $G_\alpha\neq\emptyset$, then $\dim(\Delta_\alpha(I_\Delta^n))<\dim(\Delta_{\alpha_+}(I_\Delta^n))\leqslant\dim(\Delta(I_\Delta^n))=\dim(\Delta)=k$ by Lemma 3.4(a) and (c). Therefore, $\Delta_\alpha(I_\Delta^n)^{(k)}=\{\emptyset\}=\Delta_\alpha(I_\Delta^{(n)})^{(k)}$.

When $G_\alpha=\emptyset$, then $\mathrm{link}_\Delta(G_\alpha)=\Delta$. Now, for each $F\in\Delta$ with $\dim(F)=k$, we have $F\in\Delta_\alpha(I_\Delta^{(n)})\Leftrightarrow F$ is a facet of Δ and $\sum\limits_{i\in[s]\backslash F}\alpha_i\leqslant n-1\Leftrightarrow \mathrm{F}\in\Delta_\alpha(\mathrm{I}_\Delta^n)$.

The first equivalence comes from Lemma 3.3 and the second comes from Proposition 3.6. So we can conclude safely with $\Delta_\alpha(I_\Delta^n)^{(k)}=\Delta_\alpha(I_\Delta^{(n)})^{(k)}$.

Now, we can state the second main result of this paper.

Theorem 3.9 Let Δ be a k-dimensional complex on $[s]$. If I_Δ is the Stanley-Reisner ideal in the polynomial ring $S=\mathbb{K}[x_1,\cdots,x_s]$ over a field $\mathbb{K}$, then $a_{k+1}(S/I_\Delta^{(n)})=a_{k+1}(S/I_\Delta^n)$ for all $n\geqslant1$.

Proof We have already seen that $\Delta=\Delta(I_\Delta^n)=\Delta(I_\Delta^{(n)})$ holds by Lemma 3.7. Now, take an arbitrary $\alpha\in\mathbb{Z}$. If $G_\alpha\notin\Delta$, then

$$H_{\mathfrak{m}}^{k+1}(S/I_\Delta^{(n)})_\alpha=H_{\mathfrak{m}}^{k+1}(S/I_\Delta^n)_\alpha=0$$

by Lemma 3.1. Thus, we may assume instead $G_\alpha\in\Delta$. In this case, we claim

$$\Delta_\alpha(I_\Delta^n)^{(k-|G_\alpha|)}=\Delta_\alpha(I_\Delta^{(n)})^{(k-|G_\alpha|)}\tag{7}$$

Notice that

$$\Delta_{\alpha_+}(I_\Delta^n)^{(k)}=\Delta_{\alpha_+}(I_\Delta^{(n)})^{(k)}$$

holds by Lemma 3.8. Therefore, (7) holds when $|G_\alpha|=0$. In the following, we will assume additionally $|G_\alpha|\geqslant1$. Now, $\Delta_\alpha(I_\Delta^{(n)})^{(k-|G_\alpha|)}$ is actually a simplicial complex over $[s]\backslash G_\alpha$ by Lemma 3.3. Meanwhile, $\Delta_\alpha(I_\Delta^{(n)})^{k-|G_\alpha|}$ is also a simplicial complex over $[s]\backslash G_\alpha$ by Lemma 3.4(c). Hence, to establish (7) in this situation, we will take an arbitrary $(k-|G_\alpha|)$-dimensional face $A\in$

Δ with $A\cap G_\alpha=\emptyset$. Now,

$$A\in\Delta_\alpha(I_\Delta^{(n)})\Leftrightarrow A\in\mathscr{F}(\Delta_\alpha(I_\Delta^{(n)}))$$
$$\Leftrightarrow A\in\mathscr{F}(\mathrm{link}_\Delta(G_\alpha))\text{ and }\sum_{i\notin A\cup G_\alpha}\alpha_i\leqslant n-1\quad(8)$$
$$\Leftrightarrow A\cup G_\alpha\in\Delta\text{ and }\sum_{i\notin A\cup G_\alpha}\alpha_i\leqslant n-1$$
$$\Leftrightarrow A\cup G_\alpha\in\Delta_{\alpha_+}(I_\Delta^n)\quad(9)$$
$$\Leftrightarrow A\in\Delta_\alpha(I_\Delta^n).\quad(10)$$

The equivalences in (8), (9) and (10) come from Lemma 3.3, Proposition 3.6 and Lemma 3.4 (c) respectively. And this establishes the equality in Ref.[11].

Notice that $\dim(\Delta_\alpha(I_\Delta^n)^{(k-|G_\alpha|)})=\dim(\Delta_\alpha(I_\Delta^{(n)})^{(k-|G_\alpha|)})=k-|G_\alpha|$. Consequently, the boundaries

$$B_{k-|G_\alpha|}(\Delta_\alpha(I_\Delta^n)^{(k-|G_\alpha|)})=B_{k-|G_\alpha|}(\Delta_\alpha(I_\Delta^{(n)})^{(k-|G_\alpha|)})=0.$$

Thus, by (7), the simplicial homologies

$$H_{k-|G_\alpha|}(\Delta_\alpha(I_\Delta^n);\mathbb{K})=\frac{Z_{k-|G_\alpha|}(\Delta_\alpha(I_\Delta^n)^{(k-|G_\alpha|)})}{B_{k-|G_\alpha|}(\Delta_\alpha(I_\Delta^n)^{(k-|G_\alpha|)})}=Z_{k-|G_\alpha|}(\Delta_\alpha(I_\Delta^n)^{(k-|G_\alpha|)})=Z_{k-|G_\alpha|}(\Delta_\alpha(I_\Delta^{(n)})^{(k-|G_\alpha|)})=\frac{Z_{k-|G_\alpha|}(\Delta_\alpha(I_\Delta^{(n)})^{(k-|G_\alpha|)})}{B_{k-|G_\alpha|}(\Delta_\alpha(I_\Delta^{(n)})^{(k-|G_\alpha|)})}=H_{k-|G_\alpha|}(\Delta_\alpha(I_\Delta^{(n)});\mathbb{K}),$$

and consequently,

$$\widetilde{H}_{k-|G_\alpha|}(\Delta_\alpha(I_\Delta^n);\mathbb{K})=\widetilde{H}_{k-|G_\alpha|}(\Delta_\alpha(I_\Delta^{(n)});\mathbb{K}).$$

This equality together with Lemma 3.1 will yield

$$H_{\mathfrak{m}}^{k+1}(S/I_\Delta^{(n)})_\alpha\neq0\Leftrightarrow H_{\mathfrak{m}}^{k+1}(S/I_\Delta^n)_\alpha\neq0,$$

which finishes the proof.

In the rest of this paper, we will examine when $a_{k+1}(S/I_\Delta^{(n)})$ is maximal. The following lemma allows us to clarify some details.

Lemma 3.10 Let Δ be a k-dimensional complex on $[s]$ and I_Δ the Stanley-Reisner ideal in the polynomial ring $S=\mathbb{K}[x_1,\cdots,x_s]$ over a field $\mathbb{K}$. Suppose $\alpha=(\alpha_1,\cdots,\alpha_s)\in\mathbb{Z}$ such that $\widetilde{H}_{k-|G_\alpha|}(\Delta_\alpha(I_\Delta^{(n)}))\neq0$ and $G_\alpha\in\Delta$ hold. Then, $\alpha_i\leqslant n-1$ for each $i\in[s]$.

Proof Assume that this is not true. Without loss of generality, we may assume $\alpha_1\geqslant n$. Let $A_1,\cdots,A_l$ be the complete list of $(k-|G_\alpha|)$-dimensional faces in $\Delta_\alpha(I_\Delta^{(n)})$. Then, according to Lemma 3.3, we have $1\in A_i$ for each $1\leqslant i\leqslant l$. Hence, $\Delta_\alpha(I_\Delta^{(n)})^{(k-|G_\alpha|)}$ is a cone. Thanks to Ref.[7], $\widetilde{H}_{k-|G_\alpha|}(\Delta_\alpha(I_\Delta^{(n)})^{(k-|G_\alpha|)})=0$. Since $\Delta_\alpha(I_\Delta^{(n)})$ is $(k-|G_\alpha|)$-dimensional, this implies $\widetilde{H}_{k-|G_\alpha|}(\Delta_\alpha(I_\Delta^{(n)}))=0$, contradicting the assumption.

Finally, we are ready to present the last main result of this paper.

Theorem 3.11 Let Δ be a k-dimensional complex on $[s]$ and I_Δ the Stanley-Reisner ideal in the polynomial ring $S=\mathbb{K}[x_1,\cdots,x_s]$ over a field $\mathbb{K}$. Then

$$a_{k+1}(S/I_\Delta^{(n)})\leqslant(k+2)(n-1)$$

for each positive integer n. Furthermore, the following statements are equivalent for $n\geqslant2$:

(a) $a_{k+1}(S/I_\Delta^{(n)})=(k+2)(n-1)$;

(b) there exists a subset $B=\{p_1,\cdots,p_{k+2}\}\subseteq[s]$ such that $\mathscr{F}(\Delta|_B)$ is a k-dimensional sphere, namely

$$\mathscr{F}(\Delta|_B)=\{B\backslash\{p_i\}:1\leqslant i\leqslant k+2\}.$$

Proof Take arbitrary $\alpha\in\mathbb{Z}^s$ with $H_{\mathfrak{m}}^{k+1}(S/I_\Delta^{(n)})_\alpha\neq0$. According to Lemma 3.1, this simply means $\widetilde{H}_{k-|G_\alpha|}(\Delta_\alpha(I_\Delta^{(n)}))\neq0$ and $G_\alpha\in\Delta$. Since

$$\dim(\Delta_\alpha(I_\Delta^{(n)}))\leqslant\dim(\Delta)-|G_\alpha|=k-|G_\alpha|$$

by Lemma 3.3, the nonvanishing of $\widetilde{H}_{k-|G_\alpha|}(\Delta_\alpha(I_\Delta^{(n)}))$ implies particularly that $\dim(\Delta_\alpha(I_\Delta^{(n)}))=k-|G_\alpha|$ holds. Therefore, we can take some $F\in\Delta_\alpha(I_\Delta^{(n)})$ with $\dim(F)=k-|G_\alpha|$. Notice that $F\cap G_\alpha=\emptyset$, $\sum_{i\notin F\cup G_\alpha}\alpha_i\leqslant n-1$ and $\sum_{i\in F}\alpha_i\leqslant|F|(n-1)$ hold by Lemma 3.2 and Lemma 2.9 respectively. Henceforth,

$$|\alpha|=\sum_{i\in F}\alpha_i+\sum_{i\notin F\cup G_\alpha}\alpha_i+\sum_{i\in G_\alpha}\alpha_i\leqslant|F|(n-1)+(n-1)-|G_\alpha|=(k-|G_\alpha|+2)(n-1)-|G_\alpha|\leqslant(k+2)(n-1),$$

establishing the expected upper bound. It remains to prove the equivalence of (a) and (b) when $n\geqslant2$.

(a)⇒(b): Assume that $a_{k+1}(S/I_\Delta^{(n)})=(k+2)(n-1)$. Then, we can find some $\alpha\in\mathbb{Z}^s$ satisfying $H_{\mathfrak{m}}^{k+1}(S/I_\Delta^{(n)})_\alpha\neq0$ and $|\alpha|=(k+2)(n-1)$. From the above argument, we can see $|G_\alpha|=0$, i.e., $G_\alpha=\emptyset$. Furthermore, there exists some $F\in\Delta_\alpha(I_\Delta^{(n)})$ with $\dim(F)=k$. And since $\widetilde{H}_k(\Delta_\alpha(I_\Delta^{(n)}))\neq0$ holds, $\Delta_\alpha(I_\Delta^{(n)})\neq\langle F\rangle$. Assume

$F=[k+1]$ for convenience. Now, $\sum_{i\in[k+1]}\alpha_i=(k+1)(n-1)$ and $\sum_{i\notin[k+1]}\alpha_i=n-1$. Hence $\alpha_1=\alpha_2=\cdots=\alpha_{k+1}=n-1$ by Lemma 3.10. Our task is then reduced to describing α_j when $k+1<j\leqslant s$.

We claim that there exists precisely one j with $\alpha_j>0$ and $k+1<j\leqslant s$. If this is not true, we may assume that $\alpha_{k+2},\alpha_{k+3}>0$ holds. As $\Delta_\alpha(I_\Delta^{(n)})\neq\langle F\rangle$, we may find some facet G of $\Delta_\alpha(I_\Delta^{(n)})$ with $G\neq F$. Since $\dim(G)\leqslant\dim(F)=k$, either $|F\backslash G|\geqslant 2$ or $|\{k+2,k+3\}\backslash G|\geqslant|F\backslash G|=1$. In both cases, we have $\sum_{i\notin G}\alpha_i>n-1$, which implies $G\notin\Delta_\alpha(I_\Delta^{(n)})$ by Lemma 3.2, a contradiction to the choice of G. Therefore, we may assume $\alpha_{k+2}=n-1$ and $\alpha_h=0$ for $k+2<h\leqslant s$. An argument as in the previous paragraph also shows $G\subseteq B:=[k+2]$ and $\dim(G)=k$ for any $G\in\mathscr{F}(\Delta_\alpha(I_\Delta^{(n)}))$. If the pure simplicial complex $\Delta_\alpha(I_\Delta^{(n)})|_B$ is not a sphere, then it is collapsible and consequently $\widetilde{H}_k(\Delta_\alpha(I_\Delta^{(n)}))=0$, contradicting the assumption. Hence $\Delta|_B=\Delta_\alpha(I_\Delta^{(n)})|_B$ is indeed a sphere.

(b)⇒(a): Without loss of generality, we may assume that $B=[k+2]$ holds and $\Delta|_B$ is a k-dimensional sphere. Take $\alpha=(\alpha_1,\cdots,\alpha_s)\in\mathbb{N}^s$ where $\alpha_i=n-1$ for $0\leqslant i\leqslant k+2$ and $\alpha_i=0$ for $k+2<i\leqslant s$. Then by Lemma 3.3, we have $\Delta|_B=\Delta_\alpha(I_\Delta^{(n)})$ and $\widetilde{H}_k(\Delta_\alpha(I_\Delta^{(n)}))\neq 0$. So $H_{\mathfrak{m}}^{k+1}(S/I_\Delta^{(n)})_\alpha\neq 0$ via Lemma 3.1, which implies $\alpha_{k+1}(S/I_\Delta^{(n)})\geqslant(k+2)(n-1)$. Since $a_{k+1}(S/I_\Delta^{(n)})\leqslant(k+2)(n-1)$ holds in general, the proof is completed.

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