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Sufficient conditions for a graph to be Hamilton-connected and traceable from every vertex

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Abstract: A path passing through all the vertices of a graph is called a Hamilton path. The graph G is said to be Hamilton-connected if any two vertices of G are connected by a Hamilton path. The graph G is traceable from any vertex if it contains a Hamilton path from every vertex of G. In terms of the edge number, the spectral radius and the signless Laplacian spectral radius of a graph, some sufficient conditions for the graph to be Hamilton-connected and to be traceable from every vertex were presented, respectively.

Key words: Hamilton-connected; traceable from every vertex; spectral radius; signless Laplacian spectral radius

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图的哈密尔顿连通性和从任意点出发都可迹的充分条件

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摘要:图的哈密尔顿路是指通过图的所有顶点的路.如果图 G 的任意两点都有一条哈密顿尔路,称此 G 为哈密尔顿连通的.如果图 G 从任意点出发都有一条哈密尔顿路,称 G 从任意点出发都是可迹的.根据图 G 的边数、谱半径和无符号拉普拉斯谱半径,分别给出哈密尔顿连通图以及从任意点出发都可迹图的一些充分条件.

关键词:哈密尔顿连通;可迹;谱半径;无符号拉普拉斯谱半径

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0 Introduction

Let $G=(V(G), E(G))$ be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$, denote by $e(G)=|E(G)|$ the number of edges of G. Let $v_i \in V(G)$, we denote the degree of v_i by $d_i=d_{v_i}=d_G(v_i)$, the minimum degree of G by $\delta(G)$, the degree sequence of G by $(d_1, d_2, \cdots, d_n)$, where $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$. The set of neighbours of a vertex v in G is denoted by $N_G(v)$. Let K_n be a complete graph of order n. Let G and H be two disjoint graphs, the union of G and H, denoted by $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Specially if $G_1=G_2=\cdots=G_k$, we denote $G_1 \cup G_2 \cup \cdots \cup G_k$ by kG_1. The join of G and H, denoted by $G \vee H$, is the graph obtained by joining every vertex of G to every vertex of H.

The adjacency matrix of G is defined to be a matrix $A(G)=(a_{ij})$ of order n, where $a_{ij}=1$ if v_i is adjacent to v_j, and $a_{ij}=0$ otherwise. The degree matrix of G is denoted by $D(G)=\mathrm{diag}(d_G(v_1), d_G(v_2), \cdots, d_G(v_n))$. The matrix $Q(G)=D(G)+A(G)$ is called the signless Laplacian matrix of G. The largest eigenvalue of $A(G)$, denoted by $\rho(G)$, is called the spectral radius of G. The largest eigenvalue of $Q(G)$, denoted by $q(G)$, is called the signless Laplacian spectral radius of G.

A cycle(path) passing through all the vertices of a graph is called a Hamilton cycle(path). The graph G is said to be Hamiltonian(traceable) if it contains a Hamilton cycle(path). The graph G is said to be Hamilton-connected if any two vertices of G are connected by a Hamilton path. The graph G is traceable from every vertex if it contains a Hamilton path from every vertex of G. We note that for an integer k, $kK_1 \vee K_k$ is not Hamilton-connected and $kK_1 \vee K_{k-1}$ is not traceable from every vertex.

The problem of deciding whether a given graph is Hamiltonian, traceable or Hamilton-connected is one of the most difficult classical problems in graph theory, which is in fact NP-complete. Recently there are many reasonable sufficient and necessary conditions for a graph to be Hamiltonian and traceable. Yu and Fan[1] established the spectral conditions for a graph to be Hamilton-connected in terms of the spectral radius of the adjacency matrix and signless Laplacian of the graph and its complement. Ho et al.[2] gave a sufficient condition for a graph to be Hamilton-connected in terms of the number of edges of the graph. Zhou and Wang[3] showed spectral conditions for a graph with minimum degree $\delta(G) \geqslant 3$ to be Hamilton-connected and traceable from every vertex in terms of the spectral radius and signless Laplacian spectral radius of the graph. Chen and Zhang[4] studied sufficient spectral conditions for a graph with large minimum degree to be Hamilton-connected in terms of spectral radius, signless Laplacian spectral radius of the graph and its complement.

In this paper, motivated by the ideas in Refs. [3-4], we continue to study the problem. In terms of the edge number, the spectral radius and the signless Laplacian spectral radius of a graph with minimum degree $\delta(G) \geqslant 3$, we establish some sufficient conditions for the graph to be Hamilton-connected and traceable from every vertex, respectively.

1 Preliminaries

We first give some lemmas that will be used later.

Lemma 1.1[5] Let G be a graph of order $n \geqslant 3$ with degree sequence $(d_1, d_2, \cdots, d_n)$, where $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$. If there is no integer $2 \leqslant k \leqslant \frac{n}{2}$ such that $d_{k-1} \leqslant k$ and $d_{n-k} \leqslant n-k$, then G is Hamliton-connected.

Lemma 1.2[6] Let G be a graph of order $n \geqslant 3$. Then G is traceable from every vertex if and only if $G \vee K_1$ is Hamliton-connected.

Lemma 1.3[7] Let G be a graph of order n with $\delta(G)=\delta$. Then

$$\rho(G)\leqslant\frac{\delta-1}{2}+\sqrt{2e(G)-n\delta+\frac{(\delta+1)^2}{4}}.$$

Let

$$f(x)=\frac{x-1}{2}+\sqrt{2e-nx+\frac{(x+1)^2}{4}},$$

it is easy to see that the function $f(x)$ is decreasing in x for $x\in[k, n-1]$ with $k\geqslant 2$ and $2e\leqslant n(n-1)$.

Lemma 1.4[8] Let G be a graph of order n. Then

$$q(G)\leqslant\frac{2e(G)}{n-1}+n-2.$$

2 Main results

Theorem 2.1 Let G be a connected graph of order $n\geqslant 6$ with $\delta(G)\geqslant 3$, and only one vertex of degree three when $\delta(G)=3$. If $e(G)\geqslant\frac{n^2-\frac{11}{2}n+24}{2}$, then G is Hamilton-connected unless $G=4K_1\vee K_4$, or $G=5K_1\vee K_5$, or $G=6K_1\vee K_6$.

Proof Suppose that $e(G)\geqslant\frac{n^2-\frac{11}{2}n+24}{2}$ and G is not Hamilton-connected. Let $d=(d_1, d_2, \cdots, d_n)$ be the degree sequence of G with $d_1\leqslant d_2\leqslant\cdots\leqslant d_n$ and v_i be the vertex of degree d_i for all $1\leqslant i\leqslant n$. By Lemma 1.1, there is an integer $2\leqslant k\leqslant\frac{n}{2}$, such that $d_{k-1}\leqslant k$ and $d_{n-k}\leqslant n-k$. Then

$$2e(G)=\sum_{i=1}^{n}d_i\leqslant (k-1)k+(n-2k+1)(n-k)+k(n-1)= n^2+(1-2k)n+3k^2-3k.$$

Thus

$$n^2+(1-2k)n+3k^2-3k\geqslant 2e(G)\geqslant n^2-\frac{11}{2}n+24,$$

$$3k^2-3k-24\geqslant(2k-\frac{13}{2})n.$$

When $k=2$, $d_1\leqslant 2$, when $k=3$, $d_1\leqslant d_2\leqslant 3$, which are inconsistent with the conditions of theory. When $k\geqslant 4$, $3k^2-3k-24\geqslant(2k-\frac{13}{2})n\geqslant 2k(2k-\frac{13}{2})$, i.e. $k^2-10k+24\leqslant 0$, we have $4\leqslant k\leqslant 6$. Then we discuss the following three cases.

Case 1 Assume $k=4$. In this case we have $n\geqslant 8$. Since $n^2-\frac{11}{2}n+24\leqslant 2e(G)\leqslant n^2-7n+36$, we also can get $n\leqslant 8$. Thus $n=8$ and $e(G)=22$, which imply the degree sequence $(4, 4, 4, 4, 7, 7, 7, 7)$. The four vertices of degree 7 must be adjacent to every vertex, so they induce a K_4. The remaining four vertices now have degree 4, so they induce a $4K_1$. Then the graph must be $4K_1\vee K_4$, which is not Hamilton-connected.

Case 2 Assume $k=5$. In this case we have $n\geqslant 10$. Since

$$n^2-\frac{11}{2}n+24\leqslant 2e(G)\leqslant n^2-9n+60,$$

we also can get $n\leqslant\frac{72}{7}$. Thus $n=10$ and $e(G)=35$, which imply the degree sequence $(5, 5, 5, 5, 5, 9, 9, 9, 9, 9)$. The five vertices of degree 9 must be adjacent to every vertex, so they induce a K_5. The remaining five vertices now have degree 5, so they induce a $5K_1$. Then the graph must be $5K_1\vee K_5$, which is not Hamilton-connected.

Case 3 Assume $k=6$. In this case we have $n\geqslant 12$. Since $n^2-\frac{11}{2}n+24\leqslant 2e(G)\leqslant n^2-11n+90$, we also can get $n\leqslant 12$. Thus $n=12$ and $e(G)=51$, which imply the degree sequence $(6, 6, 6, 6, 6, 6, 11, 11, 11, 11, 11, 11)$. The six vertices of degree 11 must be adjacent to every vertex, so they induce a K_6. The remaining six vertices now have degree 6, so they induce a $6K_1$. Then the graph must be $6K_1\vee K_6$, which is not Hamilton-connected.

The proof is completed.

Theorem 2.2 Let G be a connected graph of order $n\geqslant 6$ with $\delta(G)\geqslant 3$, and only one vertex of degree three when $\delta(G)=3$.

① If $\rho(G)\geqslant 1+\sqrt{n^2-\frac{17}{2}n+28}$, then G is

Hamilton-connected.

② If $q(G) \geqslant 2n - \frac{13}{2} + \frac{39}{2(n-1)}$, then G is Hamilton-connected.

Proof ① Suppose that

$$\rho(G) \geqslant 1 + \sqrt{n^2 - \frac{17}{2}n + 28}$$

and G is not Hamilton-connected. By Lemma 1.3, we have

$$1 + \sqrt{n^2 - \frac{17}{2}n + 28} \leqslant \rho(G) \leqslant 1 + \sqrt{2e(G) - 3n + 4}.$$

Thus $e(G) \geqslant \frac{n^2 - \frac{11}{2}n + 24}{2}$. By Theorem 2.1, $G = 4K_1 \vee K_4$, or $G = 5K_1 \vee K_5$, or $G = 6K_1 \vee K_6$. By direct calculation $\rho(4K_1 \vee K_4) = 5.772$, $\rho(5K_1 \vee K_5) = 7.3852$, $\rho(6K_1 \vee K_6) = 9$, which do not satisfy $\rho(G) \geqslant 1 + \sqrt{n^2 - \frac{17}{2}n + 28}$, a contradiction.

② Suppose that $q(G) \geqslant 2n - \frac{13}{2} + \frac{39}{2(n-1)}$ and G is not Hamilton-connected. By Lemma 1.4, we have

$$2n - \frac{13}{2} + \frac{39}{2(n-1)} \leqslant q(G) \leqslant \frac{2e(G)}{n-1} + n - 2.$$

Thus $e(G) \geqslant \frac{n^2 - \frac{11}{2}n + 24}{2}$. By Theorem 2.1, $G = 4K_1 \vee K_4$, or $G = 5K_1 \vee K_5$, or $G = 6K_1 \vee K_6$. By direct calculation $q(4K_1 \vee K_4) = 12$, $q(5K_1 \vee K_5) = 15.4031$, $q(6K_1 \vee K_6) = 11.3383$, which do not satisfy $q(G) \geqslant 2n - \frac{13}{2} + \frac{39}{2(n-1)}$, a contradiction.

The proof is completed.

Chen and Zhang[4] have given sufficient spectral conditions for a graph with large minimum degree to be Hamilton-connected in terms of spectral radius and signless Laplacian spectral radius of the graph. We use $cl_k(G)$ be the graph which obtained from G by successively joining pairs of nonadjacent vertices x, $y \in V(G)$ whose degree sum is at least k until no such pair remains.

Theorem 2.3[4] Let G be a connected graph of order $n \geqslant 6k^2 - 8k + 5$ with $\delta(G) \geqslant k \geqslant 2$.

① If

$$\rho(G) > \frac{k-1}{2} + \sqrt{n^2 - (3k-1)n + \frac{k^2 + 10k - 15}{4}},$$

then G is Hamilton-connected unless $cl_{n+1}(G) = K_2 \vee (K_{n-k-1} \cup K_{k-1})$ or $cl_{n+1}(G) = K_k \vee (K_{n-2k+1} \cup \overline{K}_{k-1})$.

② If $q(G) > 2n - 2k - \frac{2}{n-1}$, then G is Hamilton-connected unless $cl_{n+1}(G) = K_2 \vee (K_{n-k-1} \vee K_{k-1})$ or $cl_{n+1}(G) = K_k \vee (K_{n-2k+1} \cup \overline{K}_{k-1})$.

Remark 2.1 We now compare Theorems 2.2 and 2.3 when $k = 3$.

① If $n > 44$, we get

$$\frac{k-1}{2} + \sqrt{n^2 - (3k-1)n + \frac{k^2 + 10k - 15}{4}} = 1 + \sqrt{n^2 - 8n + 6} > 1 + \sqrt{n^2 - \frac{17}{2}n + 28}.$$

② If $n > 44$, we get

$$2n - 2k - \frac{2}{n-1} = 2n - 6 - \frac{2}{(n-1)} > 2n - \frac{13}{2} + \frac{39}{2(n-1)}.$$

That is to say, Theorem 2.2 improves Theorem 2.3 when $k = 3$, only one vertex of degree three and $n > 44$.

By Lemma 1.2 and Theorem 2.1, we obtain a sufficient condition for a graph to be traceable from every vertex.

Theorem 2.4 Let G be a connected graph of order $n \geqslant 5$ with $\delta(G) \geqslant 2$, and only one vertex of degree two when $\delta(G) = 2$. If $e(G) \geqslant \frac{n^2 - \frac{11}{2}n + \frac{39}{2}}{2}$, then G is traceable from every vertex unless $G = 4K_1 \vee K_3$, or $G = 5K_1 \vee K_4$, or $G = 6K_1 \vee K_5$.

Proof Since $|V(G \vee K_1)| = n + 1$,

$$e(G\vee K_1)=e(G)+n\geqslant\frac{n^2-\frac{11}{2}n+\frac{39}{2}}{2}+n=\frac{(n+1)^2-\frac{11}{2}(n+1)+24}{2},$$

and only one vertex of degree three in $G\vee K_1$ when $\delta(G)=2$. By Theorem 2.1, $G\vee K_1$ is Hamilton-connected unless $G\vee K_1=4K_1\vee K_4$, or $G\vee K_1=5K_1\vee K_5$, or $G\vee K_1=6K_1\vee K_6$. According to Lemma 1.2, G is traceable from every vertex unless $G=4K_1\vee K_3$, or $G=5K_1\vee K_4$, or $G=6K_1\vee K_5$.

The result follows.

Theorem 2.5 Let G be a connected graph of order $n\geqslant 5$ with $\delta(G)\geqslant 2$, and only one vertex of degree two when $\delta(G)=2$.

① If $\rho(G)\geqslant\frac{1}{2}+\sqrt{n^2-\frac{15}{2}n+\frac{87}{4}}$, then G is traceable from every vertex.

② If $q(G)\geqslant 2n-\frac{13}{2}+\frac{15}{(n-1)}$, then G is traceable from every vertex.

Proof ① Suppose that $\rho(G)\geqslant\frac{1}{2}+\sqrt{n^2-\frac{15}{2}n+\frac{87}{4}}$ and G is not traceable from every vertex. By Lemma 1.3, we have

$$\frac{1}{2}+\sqrt{n^2-\frac{15}{2}n+\frac{87}{4}}\leqslant\rho(G)\leqslant\frac{1}{2}+\sqrt{2e(G)-2n+\frac{9}{4}}.$$

Thus $e(G)\geqslant\frac{n^2-\frac{11}{2}n+\frac{39}{2}}{2}$. By Theorem 2.4, $G=4K_1\vee K_3$, or $G=5K_1\vee K_4$, or $G=6K_1\vee K_5$. By direct calculation $\rho(4K_1\vee K_3)=4.6056$, $\rho(5K_1\vee K_4)=6.2170$, $\rho(6K_1\vee K_5)=7.8310$, which do not satisfy

$$\rho(G)\geqslant\frac{1}{2}+\sqrt{n^2-\frac{15}{2}n+\frac{87}{4}},$$

a contradiction.

② Suppose that $q(G)\geqslant 2n-\frac{13}{2}+\frac{15}{(n-1)}$ and G is not traceable from every vertex. By Lemma 1.4, we have

$$2n-\frac{13}{2}+\frac{15}{(n-1)}\leqslant q(G)\leqslant\frac{2e(G)}{n-1}+n-2.$$

Thus $e(G)\geqslant\frac{n^2-\frac{11}{2}n+\frac{39}{2}}{2}$. By Theorem 2.4, $G=4K_1\vee K_3$, or $G=5K_1\vee K_4$, or $G=6K_1\vee K_5$. By direct calculation $q(G=4K_1\vee K_3)=9.7720$, $q(5K_1\vee K_4)=13.1789$, $q(6K_1\vee K_5)=16.5887$, which do not satisfy $q(G)\geqslant 2n-\frac{13}{2}+\frac{15}{n-1}$, a contradiction.

The proof is completed.

Zhou and Wang[3] have given sufficient spectral conditions for a graph with minimum degree $\delta(G)\geqslant 2$ to be traceable from every vertex in terms of the spectral radius and signless Laplacian spectral radius of the graph.

Theorem 2.6[3] Let G be a connected graph of order $n\geqslant 5$ with $\delta(G)\geqslant 2$.

① If $\rho(G)\geqslant\sqrt{n^2-6n+15}$, then G is traceable from every vertex.

② If $q(G)\geqslant 2n-6+\frac{10}{(n-1)}$, then G is traceable from every vertex.

Remark 2.2 We now compare Theorems 2.5 and 2.6.

① If $n\geqslant 9$, we get

$$\sqrt{n^2-6n+15}>\frac{1}{2}+\sqrt{n^2-\frac{15}{2}n+\frac{87}{4}}.$$

② If $n\geqslant 12$, we get

$$2n-6+\frac{10}{(n-1)}>2n-\frac{13}{2}+\frac{15}{(n-1)}.$$

That is to say, Theorem 2.5 greatly improves Theorem 2.6 when $n\geqslant 12$ and $\delta(G)\geqslant 3$.

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