第50卷第2期 中国科学技术大学学报 Vol. 50, No. 2
2020年2月 JOURNAL OF UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA Feb. 2020

文章编号:0253-2778(2020)02-0087-07

Characterizing isolated singularities of conformal hyperbolic metrics

FENG Yu, SHI Yiqian, XU Bin

(*CAS Wu Wen-Tsun Key Laboratory of Mathematics, School of Mathematical Sciences, University of Science and Technology of China, Hefei* 230026, *China*)

Abstract: The explicit local models were classified for isolated singularities of complex one-dimensional hyperbolic metrics by complex analysis, which is interesting in its own way and could potentially be extended to higher dimensional cases.

Key words: conformal hyperbolic metrics; conical singularity; cusp singularity; developing map

CLC number: O186.1 **Document code**: A doi:10.3969/j.issn.0253-2778.2020.02.001

2010 Mathematics Subject Classification: Primary 51M10; Secondary 34M35

Citation: FENG Yu, SHI Yiqian, XU Bin. Characterizing isolated singularities of conformal hyperbolic metrics [J]. Journal of University of Science and Technology of China, 2020,50(2):87-93.
冯宇,史毅茜,许斌. 刻画共形双曲度量的孤立奇点[J]. 中国科学技术大学学报,2020,50(2):87-93.

刻画共形双曲度量的孤立奇点

冯 宇,史毅茜,许 斌

(中国科学技术大学数学科学学院中科院吴文俊数学重点实验室,安徽合肥 230026)

摘要:利用复分析方法给出了共形双曲度量的孤立奇点的局部模型.这种方法本身有趣,并且有可能推广到高维的情况.

关键词:共形双曲度量;锥奇点;尖奇点;展开映射

0 Introduction

Let Σ be a Riemann surface and $D=\sum_{i=1}^{\infty}(\theta_i-1)p_i$ a $\mathbb{R}$-divisor on Σ such that $0\leqslant\theta_i\neq 1$, where $\{p_i\}_{i=1}^{\infty}\subset\Sigma$ is a closed discrete subset. We denote by $M(\Sigma)$ the set of C^∞ conformal metrics of constant curvature -1 on a Riemann surface Σ. We call $\mathrm{d}\sigma^2$ a (singular) conformal hyperbolic metric representing D if and only if

① $\mathrm{d}\sigma^2\in M(\Sigma\backslash\mathrm{supp}\ D)$, where $\mathrm{supp}\ D=\{p_1,\cdots,p_n,\cdots\}$.

② If $\theta_i>0$, then $\mathrm{d}\sigma^2$ has a conical singularity at p_i with cone angle $2\pi\theta_i>0$. That is, in a neighborhood U of p_i, $\mathrm{d}\sigma^2=\mathrm{e}^{2u}|\mathrm{d}z|^2$, where z is a complex coordinate of U with $z(p_i)=0$ and $u-(\theta_i-1)\ln|z|$ extends to a continuous function in U.

③ If $\theta_i=0$, then $\mathrm{d}\sigma^2$ has a cusp singularity at p_i. That is, in a neighborhood V of p_i, $\mathrm{d}\sigma^2=\mathrm{e}^{2u}|\mathrm{d}z|^2$, where z is a complex coordinate of V with $z(p_i)=0$ and $u+\ln|z|+\ln(-\ln|z|)$

Received: 2019-12-02; **Revised**: 2020-02-10

Foundation item: Supported by NNSF of China (11931009, 11971450), Anhui Initiative in Quantum Information Technologies (AHY150200), the Fundamental Research Funds for the Central Universities.

Biography: FENG Yu, male, born in 1990, PhD candidate. Research field: Differential geometry. E-mail: yuf@mail.ustc.edu.cn

Corresponding author: XU Bin, PhD/ associate Prof. E-mail: bxu@ustc.edu.cn

extends to a continuous function in V.

There have been some studies on the local behavior of a conformal hyperbolic metric near an isolated singularity. Nitsche[1], Heins[2], Chou and Wan[3-4] proved that an isolated singularity of a conformal hyperbolic metric must be either a conical singularity or a cusp one. They all obtained the result by studying the behaviour of the solutions of the Liouville equation

$$\Delta u = e^{2u}$$

near isolated singularities. Yamada[5] considered the same problem from the perspective of complex analysis. However, all of them only gave an asymptotic model for a hyperbolic metric near an isolated singularity. We want to seek for an explicit local model.

For this purpose, some explorations were done. Firstly, in Ref. [6], by using PDEs the authors proved the following lemma:

Lemma 0.1 Let $d\sigma^2$ be a conformal hyperbolic metric on a Riemann surface Σ, and suppose $d\sigma^2$ represents a divisor $D=\sum_{i=1}^{\infty}(\theta_i-1)p_i$, $0\leqslant\theta_i\neq 1$. Suppose that $F:\Sigma\backslash\operatorname{supp} D\to\mathbb{D}$ is a developing map of $d\sigma^2$. Then the Schwarzian derivative $\{F,z\}=\frac{F'''(z)}{F'(z)}-\frac{3}{2}\left(\frac{F''(z)}{F'(z)}\right)^2$ of F equals

$$\{F,z\}=\frac{1-\theta_i{}^2}{2z^2}+\frac{d_i}{z}+\phi_i(z)$$

in a neighborhood U_i of p_i with complex coordinate z and $z(p_i)=0$, where d_i are constants and ϕ_i are holomorphic functions in U_i, depending on the complex coordinate z.

Based on Lemma 0.1, we obtained the local expressions of developing maps near isolated singularities of hyperbolic metrics (see Ref. [7, Lemma 2.4]). In this process, we solved a Fuchsian equation of second order. Using these expressions, we finally got the local model of a singular conformal hyperbolic metric. Therefore, this proof process has some twists and turns. After reading Ref. [5], we speculated that there should be a direct proof using complex analysis only, which motivated this manuscript. In this note, we complete this modest project.

Below we present the main result of this note.

Theorem 0.1[7, Theorem 1.2] Let $d\sigma^2$ be a conformal hyperbolic metric on the punctured disk $\mathbb{D}^*=\{w\in\mathbb{C}\mid 0<|w|<1\}$. Then 0 is either a conical singularity or a cusp singularity of $d\sigma^2$. If $d\sigma^2$ has a conical singularity at $w=0$ with the angle $2\pi\alpha>0$, then there exists a complex coordinate z on $\Delta_\varepsilon=\{w\in\mathbb{C}\mid |w|<\varepsilon\}$ for some $\varepsilon>0$ with $z(0)=0$ such that

$$d\sigma^2\,|_{\Delta_\varepsilon}=\frac{4\alpha^2\,|z|^{2\alpha-2}}{(1-|z|^{2\alpha})^2}\,|dz|^2.$$

Moreover, z is unique up to replacement by λz where $|\lambda|=1$. If $d\sigma^2$ has a cusp singularity at $w=0$, then there exists a complex coordinate z on $\Delta_\varepsilon=\{w\in\mathbb{C}\mid |w|<\varepsilon\}$ for some $\varepsilon>0$ with $z(0)=0$ such that

$$d\sigma^2\,|_{\Delta_\varepsilon}=|z|^{-2}(\ln|z|)^{-2}\,|dz|^2.$$

Moreover, z is unique up to replacement by λz where $|\lambda|=1$.

From the proof of Theorem 0.1, we can directly obtain the local expressions of developing maps near isolated singularities of the metrics.

Theorem 0.2 Let $F:\Sigma\backslash\operatorname{supp} D\to\mathbb{D}$ be a developing map of the singular conformal hyperbolic metric $d\sigma^2$ representing D. If p is a conical singularity of $d\sigma^2$, then there exists a neighborhood U of p with complex coordinate z and $\mathscr{L}\in\mathrm{PSU}(1,1)$ such that $z(p)=0$ and $G=\mathscr{L}\circ F$ has the form $G(z)=z^\alpha$. If q is a cusp singularity of $d\sigma^2$, we assume $F:\Sigma\backslash\operatorname{supp} D\to\mathbb{H}$ for convenience, where $\mathbb{H}$ is the upper half-plane model of the hyperbolic plane, then there exists a neighborhood V of q with complex coordinate z and $\mathscr{L}\in\mathrm{PSL}(2,\mathbb{R})$, such that $z(q)=0$ and $G=\mathscr{L}\circ F$ has the form $G(z)=-\sqrt{-1}\log z$.

Moreover, the proof process of Theorem 0.1 provides a possible approach to studying the codimension-one singularities of complex hyperbolic metrics in higher dimension. We shall investigate the following problem in the near future.

Problem 0. 1 What is the asymptotic behavior of a complex hyperbolic metric on $\underbrace{\mathbb{D}\times\cdots\times\mathbb{D}}_{n}\setminus\{z_1z_2\cdots z_n=0\}$ for $n\geqslant 2$?

We organize the left part of the manuscript as follows. In Section 1, we at first give some knowledge of hyperbolic geometry that we need to use. Then we state the definition and properties of developing maps. Section 2 is the proofs for Theorem 0. 1 and Theorem 0. 2.

1 Preliminaries

1. 1 Conformal isometries of the hyperbolic plane

We will work with both the Poincaré disk model

$$\left(\mathbb{D}=\{z\in\mathbb{C}:|z|<1\},\ \mathrm{d}\sigma_{\mathbb{D}}^2=\frac{4\,|\mathrm{d}z|^2}{(1-|z|^2)^2}\right)$$

and the upper half-plane model

$$\left(\mathbb{H}=\{z\in\mathbb{C}:\mathrm{Im}\,z>0\},\ \mathrm{d}\sigma_{\mathbb{H}}^2=\frac{|\mathrm{d}z|^2}{(\mathrm{Im}\,z)^2}\right)$$

of the hyperbolic plane at convenience. We denote

$$\mathrm{PSU}(1,1)=\{z\mapsto\frac{az+b}{\bar{b}z+\bar{a}}:a,\ b\in\mathbb{C},\ |a|^2-|b|^2=1\}$$

and

$$\mathrm{PSL}(2,\mathbb{R})=\{z\mapsto\frac{az+b}{cz+d}:a,\ b,\ c,\ d\in\mathbb{R},\ ad-bc=1\}$$

the group of all orientation-preserving isometries of $\mathbb{D}$ and $\mathbb{H}$, respectively.

Definition 1. 1[8] If $\mathscr{L}$, $\mathscr{K}\in I(\mathbb{D})$ and $\mathscr{L}$ is not the identity map of $\mathbb{D}$, then $\mathscr{L}$ has a fixed point in $\overline{\mathbb{D}}$, where $I(\mathbb{D})$ is the isometry group of $\mathbb{D}$. The transformation $\mathscr{L}$ is said to be

① elliptic if $\mathscr{L}$ fixes a point of $\mathbb{D}$;

② parabolic if $\mathscr{L}$ fixes no point of $\mathbb{D}$ and fixes a unique point of $\partial\mathbb{D}$;

③ hyperbolic if $\mathscr{L}$ fixes no point of $\mathbb{D}$ and fixes two points of $\partial\mathbb{D}$.

Lemma 1. 1[9-10] In the Poincaré disk model $\mathbb{D}$, if $\mathscr{L}\in\mathrm{PSU}(1,1)$ is elliptic, then there exists $\mathscr{K}\in\mathrm{PSU}(1,1)$ such that $\mathscr{K}\circ\mathscr{L}\circ\mathscr{K}^{-1}(z)=\mathrm{e}^{\mathrm{i}\theta}z$ for some real number θ. In the upper half-plane model $\mathbb{H}$, if $\mathscr{L}\in\mathrm{PSL}(2,\mathbb{R})$ is parabolic, then there exists $\mathscr{K}\in\mathrm{PSL}(2,\mathbb{R})$ such that $\mathscr{K}\circ\mathscr{L}\circ\mathscr{K}^{-1}(z)=z+t$ for some real number t. In the upper half-plane model $\mathbb{H}$, if $\mathscr{L}\in\mathrm{PSL}(2,\mathbb{R})$ is hyperbolic, then there exists $\mathscr{K}\in\mathrm{PSL}(2,\mathbb{R})$ such that $\mathscr{K}\circ\mathscr{L}\circ\mathscr{K}^{-1}(z)=\lambda z$ for some positive real number λ.

Lemma 1. 2[9, Theorem 4. 16] Let $f:\mathbb{H}\to\mathbb{H}$ be holomorphic. If f is a homeomorphism, then $f\in\mathrm{PSL}(2,\mathbb{R})$. If f is not a homeomorphism, then $d_{\mathbb{H}}(f(z_1),f(z_2))<d_{\mathbb{H}}(z_1,z_2)$ for all distinct $z_1,z_2\in\mathbb{H}$, where $d_{\mathbb{H}}$ denotes the hyperbolic distance of $\mathbb{H}$.

Lemma 1. 3[10] For any isometry $\mathscr{L}$ let m be the infimum of $d_{\mathbb{H}}(z,\mathscr{L}z)$ taken with respect to $z\in\mathbb{H}$. Then $\mathscr{L}$ is hyperbolic if and only if $m>0$; if $m=0$, then $\mathscr{L}$ is elliptic when m is attained and parabolic when m is not attained.

1. 2 Developing map

A multi-valued locally univalent meromorphic function F on a Riemann surface Σ is said to be projective if any two function elements $\mathscr{F}_1$, $\mathscr{F}_2$ of F near a point $p\in\Sigma$ are related by a fractional linear transformation $\mathscr{L}\in\mathrm{PGL}(2,\mathbb{C})$, i. e., $\mathscr{F}_1=\mathscr{L}\circ\mathscr{F}_2$.

Definition 1. 2 Let $\mathrm{d}\sigma^2$ be a conformal hyperbolic metric on a Riemann surface Σ, not necessarily compact, representing the divisor D. We call a projective function $F:\Sigma\setminus\mathrm{supp}\ D\to\mathbb{D}$ a developing map of the metric $\mathrm{d}\sigma^2$ if $\mathrm{d}\sigma^2=F^*\mathrm{d}\sigma_{\mathbb{D}}^2$, where $\mathrm{d}\sigma_{\mathbb{D}}^2=\frac{|\mathrm{d}z|^2}{(1-|z|^2)^2}$ is the hyperbolic metric on the unit disc $\mathbb{D}$. And F can also be viewed as a locally schlicht holomorphic function from $\widetilde{\Sigma}$ to $\mathbb{D}$, where $\widetilde{\Sigma}$ is the universal cover of $\Sigma\setminus\mathrm{supp}\ D$.

Lemma 1. 4[7, Lemma 2. 1 and Lemma 2. 2] Let $\mathrm{d}\sigma^2$ be a conformal hyperbolic metric on a Riemann surface Σ, representing the divisor D. Then there exists a developing map F from $\Sigma\setminus\mathrm{supp}\ D$ to the unit disc $\mathbb{D}$ such that the monodromy of F belongs to $\mathrm{PSU}(1,1)$ and

$$\mathrm{d}\sigma^2=F^*\mathrm{d}\sigma_{\mathbb{D}}^2,$$

where $\mathrm{d}\sigma_{\mathbb{D}}^2=\frac{|\mathrm{d}z|^2}{(1-|z|^2)^2}$ is the hyperbolic metric on $\mathbb{D}$. Moreover, any two developing maps F_1, F_2

of the metric $d\sigma^2$ are related by a fractional linear transformation $\mathscr{L} \in \mathrm{PSU}(1,1)$, i. e., $F_2 = \mathscr{L} \circ F_1$.

Remark 1.1 There exists an analogue of the above lemma on the upper half-plane model $\mathbb{H}$.

2 Proofs for Theorem 0.1 and Theorem 0.2

Let $d\sigma^2 \in M(\mathbb{D}^*)$ and $\mathbb{H}$ be the upper half-plane. Consider the universal covering from $\mathbb{H}$ to $\mathbb{D}^*$, $z \mapsto e^{iz}$, whose covering group Γ is generated by $\tau(z) = z + 2\pi$. Since $(e^{iz})^* d\sigma^2 \in M(\mathbb{H})$, there exists a locally schlicht holomorphic function f from $\mathbb{H}$ to $\mathbb{D}$ such that $(e^{iz})^* d\sigma^2 = f^* d\sigma^2_{\mathbb{D}}$ by Lemma 1.4. Moreover, we obtain the monodromy homomorphism $\mathscr{M}: \Gamma \to \mathrm{PSU}(1,1)$. So we have $f \circ \tau = \mathscr{M}(\tau) \circ f$, set $\mathscr{L} = \mathscr{M}(\tau)$.

Lemma 2.1[5, Lemma 7] $\mathscr{L}$ is not a hyperbolic transformation.

Proof By Lemma 1.2,

$$d_{\mathbb{D}}(f(z), \mathscr{L} \circ f(z)) = d_{\mathbb{D}}(f(z), f(z+2\pi)) \leqslant d_{\mathbb{H}}(z, z+2\pi).$$

Let $z = iy$, $\gamma(t) = t + iy$, $0 \leqslant t \leqslant 2\pi$. Then the length of $\gamma(t)$ equals

$$l(\gamma(t)) = \int_0^{2\pi} \frac{1}{y} dt = \frac{2\pi}{y}.$$

So $d_{\mathbb{H}}(z, z+2\pi) \to 0$ as $y \to +\infty$. Hence $m = \inf d_{\mathbb{D}}(z, \mathscr{L} \circ z) = 0$, $\mathscr{L}$ is not hyperbolic by Lemma 1.3.

Lemma 2.2 The following expressions hold near the origin.

① If $\mathscr{L}$ is parabolic, then

$$d\sigma^2 |_{\Delta_\varepsilon} = |\xi|^{-2} (\ln|\xi|)^{-2} |d\xi|^2,$$

where $\Delta_\varepsilon = \{w \in \mathbb{C} \mid |w| < \varepsilon\}$ for some $\varepsilon > 0$. Moreover, ξ is unique up to replacement by $\lambda\xi$ where $|\lambda| = 1$.

② If $\mathscr{L}$ is elliptic, then

$$d\sigma^2 |_{\Delta_\varepsilon} = \frac{4(k+\alpha)^2 |\xi|^{2k+2\alpha-2}}{(1-|\xi|^{2k+2\alpha})^2} |d\xi|^2,$$

where $0 < \alpha < 1$, k is a nonnegative integer and $\Delta_\varepsilon = \{w \in \mathbb{C} \mid |w| < \varepsilon\}$ for some $\varepsilon > 0$. Moreover, ξ is unique up to replacement by $\lambda\xi$ where $|\lambda| = 1$.

③ If $\mathscr{L}$ is the identity, then

$$d\sigma^2 |_{\Delta_\varepsilon} = \frac{4k^2 |\xi|^{2k-2}}{(1-|\xi|^{2k})^2} |d\xi|^2,$$

where k is a positive integer and $\Delta_\varepsilon = \{w \in \mathbb{C} \mid |w| < \varepsilon\}$ for some $\varepsilon > 0$. Moreover, ξ is unique up to replacement by $\lambda\xi$ where $|\lambda| = 1$.

Proof ① Lemma 1.1 and Lemma 1.4 imply that there exists a locally schlicht function $f: \mathbb{H} \to \mathbb{H}$ such that $(e^{iz})^* d\sigma^2 = f^* d\sigma^2_{\mathbb{H}}$ and $f(z+2\pi) = f(z) + t$ for all $z \in \mathbb{H}$, where $t \neq 0$ is a real number.

(i) If $t < 0$, then

$$\widetilde{f} = \begin{pmatrix} \sqrt{\frac{2\pi}{-t}} & 0 \\ 0 & \sqrt{\frac{-t}{2\pi}} \end{pmatrix} \circ f = -\frac{2\pi}{t} f$$

is also a developing map by Lemma 1.4. We have $\widetilde{f}(z+2\pi) = \widetilde{f}(z) - 2\pi$. Let $g(z) = \widetilde{f}(z) + z$, then $g(z+2\pi) = g(z)$. And $g(z)$ is a simply periodic function with period 2π. Let $w = e^{iz}$, then there exists a unique holomorphic function G in $\mathbb{D}^* = \{w \mid 0 < |w| < 1\}$ such that $g(z) = G(w)$. Thus by Ref. [11] we have the complex Fourier development

$$g(z) = \sum_{n=-\infty}^{\infty} a_n e^{niz}.$$

So $\widetilde{f}(z) = \sum_{n=-\infty}^{\infty} a_n e^{niz} - z$, and $\mathrm{Im}\, \widetilde{f} > 0$, i. e.

$$\mathrm{Im} \sum_{n=-\infty}^{\infty} a_n w^n - \ln \frac{1}{|w|} > 0 \text{ for } 0 < |w| < 1 \tag{1}$$

We know that $G(w) = \sum_{n=-\infty}^{\infty} a_n w^n$ is a holomorphic function in $\mathbb{D}^*$.

If 0 is a removable singularity of $G(w)$, it contradicts the above inequality as w tends to 0.

Suppose that 0 is a pole of order m for $G(w)$. Then by (1) we have

$$\mathrm{Im} \sum_{n=-\infty}^{\infty} a_n w^n = \mathrm{Im}\, w^{-m} h(w) > 0$$

for $0 < |w| < 1$, where $h(w)$ is a holomorphic function in $\mathbb{D}$. Contradiction!

Suppose that 0 is an essential singularity of $G(w)$. Then it contradicts (1) by the Great

Picard Theorem.

Therefore, we have excluded case $t<0$.

(ii) If $t>0$, then

$$\widetilde{f}=\begin{pmatrix}\sqrt{\frac{2\pi}{t}} & 0\\ 0 & \sqrt{\frac{t}{2\pi}}\end{pmatrix}\circ f=\frac{2\pi}{t}f$$

is also a developing map by Lemma 1.4. We have $\widetilde{f}(z+2\pi)=\widetilde{f}(z)+2\pi$. Let $g(z)=\widetilde{f}(z)-z$, then $g(z+2\pi)=g(z)$. And $g(z)$ is a simply periodic function with period 2π. Let $w=e^{iz}$, then there exists a unique holomorphic function G in $\mathbb{D}^*=\{w\mid 0<|w|<1\}$ such that $g(z)=G(w)$. Thus we have the complex Fourier development

$$g(z)=\sum_{n=-\infty}^{\infty}a_n e^{niz}.$$

So $\widetilde{f}(z)=\sum_{n=-\infty}^{\infty}a_n e^{niz}+z$, and $\operatorname{Im}\widetilde{f}>0$, i.e.

$$\operatorname{Im}\sum_{n=-\infty}^{\infty}a_n w^n+\ln\frac{1}{|w|}>0 \text{ for } 0<|w|<1 \tag{2}$$

We know that $G(w)=\sum_{n=-\infty}^{\infty}a_n w^n$ is a holomorphic function on $\mathbb{D}^*$. By (2) we have

$$e^{\operatorname{Im}G(w)}=|e^{-\sqrt{-1}G(w)}|>|w| \text{ for } 0<|w|<1.$$

So $\left|\frac{e^{-\sqrt{-1}G(w)}}{w}\right|>1$ and $\frac{e^{-\sqrt{-1}G(w)}}{w}$ is a holomorphic function in $\mathbb{D}^*$. By the Great Picard Theorem, 0 is not an essential singularity of $\frac{e^{-\sqrt{-1}G(w)}}{w}$.

If $G(w)$ has an essential singularity at 0, consider any non-zero $c\in\mathbb{C}$. By Casorati-Weierstrass theorem[12, Theorem 3.3], there is a sequence $z_n\to 0$ such that $G(z_n)\to\sqrt{-1}\log c$. So $\exp(-\sqrt{-1}G(z_n))\to c$. Since this is true for all non-zero c, $\exp(-\sqrt{-1}G(w))$ must have an essential singularity at 0, then $\frac{e^{-\sqrt{-1}G(w)}}{w}$ does too. Contradiction!

Suppose that 0 is a pole of order m for $G(w)$. Then

$$-\sqrt{-1}G(w)=\frac{h(w)}{w^m},$$

where $h(w)$ is holomorhic on $\mathbb{D}$ and does not vanish near the origin. Let $h(0)=re^{\sqrt{-1}\theta}$, $r>0$. Consider the sequence

$$z_k=\frac{\exp(\sqrt{-1}\theta/m)}{k}$$

then $-\sqrt{-1}G(z_k)=h(z_k)\exp(-\sqrt{-1}\theta)k^m$. Since $h(z_k)\to re^{\sqrt{-1}\theta}$, $\exp(-\sqrt{-1}G(z_k))$ converges to $+\infty$. If we consider

$$w_k=\frac{\exp(\sqrt{-1}(\pi+\theta)/m)}{k},$$

then we have $\exp(-\sqrt{-1}G(w_k))$ converge to 0. So $\exp(-\sqrt{-1}G(w))$ have an essential singularity at 0, contradiction!

Hence $G(w)$ extends to $w=0$ holomorphically.

So $\widetilde{f}(z)=\sum_{n=k}^{\infty}a_n e^{niz}+z$, where $k\geqslant 0$. And $\widetilde{f}(w)=-\sqrt{-1}\log w+\sum_{n=k}^{\infty}a_n w^n$ can be viewed as a developing map from $\mathbb{D}^*$ to $\mathbb{H}$. So we can choose another complex coordinate ξ near 0 with $\xi(0)=0$ such that

$$\xi=w\cdot\exp\Big(\sqrt{-1}\sum_{n=k}^{\infty}a_n w^n\Big),$$

then

$$d\sigma^2|_{\Delta_\varepsilon}=\widetilde{f}^*d\sigma_{\mathbb{H}}^2=|\xi|^{-2}(\ln|\xi|)^{-2}|d\xi|^2,$$

where $\Delta_\varepsilon=\{w\in\mathbb{C}\mid |w|<\varepsilon\}$ for some $\varepsilon>0$.

Here we show the uniqueness of the complex coordinate ξ. Let ξ and $\tilde{\xi}$ be coordinates such that conditions of the lemma are satisfied, then

$$F(\xi)=-\sqrt{-1}\log\xi,\ \widetilde{f}(\tilde{\xi})=-\sqrt{-1}\log\tilde{\xi}$$

are all developing maps of $d\sigma^2$. By Remark 1.1, there exists $\mathscr{L}\in PSL(2,\mathbb{R})$ such that $\widetilde{F}=\mathscr{L}\circ F$, then $\widetilde{F}=\frac{aF+b}{cF+d}$, $ad-bc=1$. Since $\xi(0)=\tilde{\xi}(0)=0$, $F(0)=\widetilde{F}(0)=\infty$, we have $c=0$ by a calculation. Thus $\widetilde{F}=\frac{aF+b}{d}=a^2F+ab$, then

$$-\sqrt{-1}\log\tilde{\xi}=-a^2\sqrt{-1}\log\xi+ab,$$

$$\tilde{\xi}=\xi^{a^2}\cdot e^{\sqrt{-1}ab}.$$

So there exists an open disk V which is near 0 and does not contain 0 such that $a^2=1$, $\log\tilde{\xi}=\log\xi+ab\sqrt{-1}$. Therefore we have $\tilde{\xi}=\lambda\xi$ on V with $|\lambda|=1$. Since ξ, $\tilde{\xi}$ and w are coordinates near 0, z and $\tilde{\xi}$ are holomorphic functions of w, then $\tilde{\xi}=\lambda\xi$, $|\lambda|=1$ holds in a neighborhood of 0.

② As in case ① there exists a holomorphic function $f:\mathbb{H}\to\mathbb{D}$ such that $(e^{iz})^*d\sigma^2=f^*d\sigma^2_{\mathbb{D}}$ and that $f(z+2\pi)=e^{2\pi\alpha i}f(z)$, $0<\alpha<1$. Let $g(z)=f\cdot\exp(-i\alpha z)$, then $g(z+2\pi)=g(z)$. And $g(z)$ is a simply periodic function with period 2π. Let $w=e^{iz}$, then there exists a unique holomorphic function G in $\mathbb{D}^*=\{w\mid 0<|w|<1\}$ such that $g(z)=G(w)$. Thus we have the complex Fourier development

$$g(z)=\sum_{n=-\infty}^{\infty}a_n e^{niz}.$$

We know that $G(w)=\sum_{n=-\infty}^{\infty}a_n w^n$ is a holomorphic function in $\mathbb{D}^*$. We have $|G(w)|\cdot|w|^{\alpha}<1$ by the range of f, then we have that $w=0$ is a removable singularity of $G(w)$ by $|G(w)|<|w|^{-\alpha}$ and $0<\alpha<1$. So

$$f(w)=w^{\alpha}\sum_{n=k}^{\infty}a_n w^n,$$

where $k(\geqslant 0)$ is an integer and $a_k\neq 0$. And $f(w)$ can be viewed as a developing map from $\mathbb{D}^*$ to $\mathbb{D}$. So we can choose another complex coordinate ξ near 0 with $\xi(0)=0$ such that $\xi^{\alpha+k}=w^{\alpha}\sum_{n=k}^{\infty}a_n w^n$, then

$$d\sigma^2|_{\Delta_\varepsilon}=\frac{4(k+\alpha)^2|\xi|^{2k+2\alpha-2}}{(1-|\xi|^{2k+2\alpha})^2}|d\xi|^2,$$

where $\Delta_\varepsilon=\{w\in\mathbb{C}\mid |w|<\varepsilon\}$ for some $\varepsilon>0$.

We show the uniqueness of the complex coordinate ξ. Let ξ and $\tilde{\xi}$ be coordinates such that conditions of the lemma are satisfied, then $F(\xi)=\xi^{\alpha}$, $\tilde{F}(\tilde{\xi})=\tilde{\xi}^{\alpha}$ are all developing maps of $d\sigma^2$. By Lemma 1.4, there exists $\mathscr{L}\in PSU(1,1)$ such that $\tilde{F}=\mathscr{L}\circ F$, then $\tilde{F}=\frac{aF+b}{\bar{b}F+\bar{a}}$, $|a|^2-|b|^2=1$. Since $\xi(0)=\tilde{\xi}(0)=0$, $F(0)=\tilde{F}(0)=0$, we have $b=0$ by a calculation. Thus $\tilde{F}=\frac{a}{\bar{a}}F=\mu F$, $|\mu|=1$, then there exists an open disk V which is near 0 and does not contain 0 such that $\tilde{\xi}^{\alpha}=\mu\xi^{\alpha}$. Therefore we have $\tilde{\xi}=\lambda\xi$ on V with $|\lambda|=1$. Since ξ, $\tilde{\xi}$ and w are coordinates near 0, ξ and $\tilde{\xi}$ are holomorphic functions of w, then $\tilde{\xi}=\lambda\xi$, $|\lambda|=1$ holds in a neighborhood of 0.

③ Since $\mathscr{L}$ is the identity, $f(z+2\pi)=f(z)$, then $f(z)$ is a simply periodic function with period 2π. Let $w=e^{iz}$, then there exists a unique holomorphic function F in $\mathbb{D}^*=\{w\mid 0<|w|<1\}$ such that $f(z)=F(w)$. Thus we have the complex Fourier development

$$f(z)=\sum_{n=-\infty}^{\infty}a_n e^{niz}.$$

We know that $F(w)=\sum_{n=-\infty}^{\infty}a_n w^n$ is a holomorphic function in $\mathbb{D}^*$ and $|F|=|f|<1$, so $w=0$ is a removable singularity and $F(w)$ extends to $w=0$ holomorphically. Let

$$F(w)=\sum_{n=k}^{\infty}a_n w^n,$$

where $k(\geqslant 0)$ is an integer and $a_k\neq 0$. And $F(w)$ can be viewed as a developing map from $\mathbb{D}^*$ to $\mathbb{D}$. Since $\frac{aF+b}{\bar{b}F+\bar{a}}$ is also a developing, where a, $b\in\mathbb{C}$ and $|a|^2-|b|^2=1$, so we can set $F(0)=0$ without loss of generality. Then we can choose another complex coordinate ξ near 0 with $\xi(0)=0$ such that $\xi^k=\sum_{n=k}^{\infty}a_n w^n$, then

$$d\sigma^2|_{\Delta_\varepsilon}=\frac{4k^2|\xi|^{2k-2}}{(1-|\xi|^{2k})^2}|d\xi|^2,$$

where k is a positive integer, $\Delta_\varepsilon=\{w\in\mathbb{C}\mid |w|<\varepsilon\}$ for some $\varepsilon>0$.

The uniqueness of the coordinate ξ is similar to the above.

The proof is completed.

We have thus obtained Theorem 0.1. Note that in the proof of Lemma 2.2, we actually obtain the local expressions of $d\sigma^2$ near the origin by choosing a special developing map under a suitable complex coordinate. From the proof of the above lemma and Lemma 1.4, we can get the Theorem 0.2.

References

[1] NITSCHE J. Über die isolierten singularitäten der Lösungen von $\Delta u = e^u$[J]. Math Z, 1957, 68: 316-324.

[2] HEINS M. On a class of conformal metrics[J]. Nagoya Math J, 1962, 21: 1-60.

[3] CHOU K S, WAN T. Asymptotic radial symmetry for solutions of $\Delta u + e^u = 0$ in a punctured disc[J]. Pacific Journal of Mathematics, 1994, 163: 269-276.

[4] CHOU K S, WAN T. Correction to "Asymptotic radial symmetry for solutions of $\Delta u + e^u = 0$ in a punctured disc"[J]. Pacific Journal of Mathematics, 1995, 171: 589-590.

[5] YAMADA A. Bounded analytic functions and metrics of constant curvature of Riemann surfaces[J]. Kodai Math J, 1988, 11: 317-324.

[6] LI B, FENG Y, LI L, et al. Bounded projective functions and hyperbolic metrics with isolated singularities[J]. Annales Academiae Scientiarum Fennicae: Mathematica, 2020, 45: 687-698.

[7] FENG Y, SHI Y Q, XU B. Isolated singularities of conformal hyperbolic metrics[J]. Chinese Annals of Mathematics, 2019, 40: 15-26(in Chinese); Chinese J Contemp Math, 2019, 40: 15-26(English translation).

[8] RATCLIFFE J G. Foundations of Hyperbolic Manifolds[M]. 2nd ed. New York: Springer Science, 2006.

[9] ANDERSON J W. Hyperbolic Geometry[M]. 2nd ed. London: Springer-Verlag, 2005.

[10] BEARDON A F. The Geometry of Discrete Groups[M]. New York: Springer-Verlag, 1983.

[11] AHLFORS L V. Complex Analysis[M]. 3rd ed. New York: McGraw-Hill, 1979.

[12] STEIN E M, SHAKARCHI R. Complex Analysis[M]. Princeton, NJ: Princeton University Press, 2003.