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A fourth order linear parabolic equation on conical surfaces

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Abstract: A parabolic equation of fourth order on surfaces with conical singularities is considered. By the analysis of energy and approximations, the existence and uniqueness of the solution of this equation in a special space that has some approximation property are proved. Finally, it's proved that the property is equivalent to the finiteness of energy for some functions when $\beta \in (-1, 0)$.
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1 Introduction

Let M be a smooth Riemann surface, and $p_1, \cdots, p_l$ be finitely many points on M. For each point p_i, we assign a weight $\beta_i > -1$. We are interested in the class of metrics g which are smooth and compatible with the conformal structure of M away from p_i. Assume that in some neighborhood of p_i, g is given by

$$g = |z|^{2\beta_i} |dz|^2 \quad (1)$$

where z is a complex coordinate around p_i and $z(p_i) = 0$. A metric g that satisfies the above conditions is called a metric with conical singularities. Obviously, around p_i, (M, g) is isometric to a flat cone metric with total cone angle $2\pi(\beta_i+1)$, i.e.

$$r^{2\beta_i}(\mathrm{d}r^2 + r^2 \mathrm{d}\theta^2).$$

In this paper, we investigate a linear parabolic equation of fourth order:

$$u_t = \frac{1}{2} \mathrm{e}^{-2a} \triangle (\mathrm{e}^{-2a}(-\triangle u + K_0)) \quad (2)$$

where $\triangle$ is the Laplacian of g, a and K_0 are known coefficients. We will prove that under appropriate conditions the initial value problem of (2) has a solution, and the solution satisfies some estimates (see Theorem 1.1).

The purpose of discussing Equation (2) is to make preparations for investigating the conical Calabi flow. The Calabi flow was first proposed by Calabi[1] in 1982. Precisely, on a smooth surface, we define Calabi flow to be

$$\frac{\partial g}{\partial t} = \triangle_g K \cdot g \quad (3)$$

where K is the Gaussian curvature. For a smooth initial metric g_0, if $g(t) = \mathrm{e}^{2u(t)} g_0$ is a solution of (3), then

$$u_t = \frac{1}{2} \mathrm{e}^{-2u} \triangle_{g_0} (\mathrm{e}^{-2u}(-\triangle_{g_0} u + K_0)) \quad (4)$$

where K_0 is the Gaussian curvature of g_0. Chrusciel[2], Chen[3] and Struwe[4] independently proved the long time existence and convergence of Calabi flow on smooth surfaces. And Li et al.[5] obtained convergence theorems of the Calabi flow on extremal Kähler surfaces, under the assumption of global existence of the Calabi flow solutions. In the mean time, the topic of Ricci flow with conical singularities attracts the attention of many researchers[6-9]. In particular, Yin[10] proved the long time existence of the conical Ricci flow for general cone angle. And Zheng[11,12] also did some research on the conical Calabi flow.

We may also investigate the conical Calabi flow. To dicuss Equation (4) on conical surfaces, we will first need to study the corresponding linear equation (2). Although Equation (2) is linear, (M, g) is incomplete. Hence, we need to define some special function spaces.

We assume without loss of generality that there is only one singular point p of order β. In a neighborhood of p, let x, y be the real and imaginary part of z. We define the coordinate (ρ, θ) by the following equations:

$$x = r\cos\theta, \ y = r\sin\theta \quad (5)$$

and

$$\rho = \frac{1}{\beta + 1} r^{\beta+1} \quad (6)$$

It is not hard to see that ρ is the Riemannian distance to p with respect to g. We assume that (1) holds in $\{\rho <$

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$1\}$, and define $U=M\backslash\{\rho\leqslant\frac{1}{2}\}$.

Definition 1.1[10] Let (ρ,θ) and U be used as above, and $l\in\mathbb{N}$, $\alpha\in(0,1)$. For any function $u\in C^{l,\alpha}(M\backslash\{p\})$, we define

$$\|u\|_{\mathscr{E}^{l,\alpha}}:=\sup_{k\in\mathbb{N}}\|u(2^{-k}\rho,\theta)\|_{C^{l,\alpha}(B_1\backslash B_{1/2})}+\|u\|_{C^{l,\alpha}(U)} \tag{7}$$

where B_r is $\{(\rho,\theta)\mid\rho<r\}$. We define $\mathscr{E}^{l,\alpha}$ to be the set of functions u satisfying $\|u\|_{\mathscr{E}^{l,\alpha}}<+\infty$.

Similarly, we can define the parabolic version of the above weighted Hölder function space.

Definition 1.2[10] If u is a function defined on $M\backslash\{p\}\times[0,T]$, we define

$$\|u\|_{\mathscr{P}^{l,\alpha,[0,T]}}:=\sup_{k\in\mathbb{N}}\|u(2^{-k}\rho,\theta,16^{-k}t)\|_{C^{l,\alpha}(B_1\backslash B_{1/2}\times[0,16^kT])}+\|u\|_{C^{l,\alpha}(U\times[0,T])} \tag{8}$$

and $\mathscr{P}^{l,\alpha,[0,T]}$ to be the set of functions u satisfying $\|u\|_{\mathscr{P}^{l,\alpha,[0,T]}}<+\infty$.

It is not hard to see $\mathscr{E}^{l,\alpha}$ and $\mathscr{P}^{l,\alpha,[0,T]}$ are Banach spaces. Since the main tool used for proving the apriori estimate in this paper is the energy method, we need to add another constraint to the spaces above.

Definition 1.3 For a function u defined on $M\backslash\{p\}$, we define

$$[u]_X:=\left(\int(|u|^2+|\nabla u|^2+|\triangle u|^2)\mathrm{d}V\right)^{1/2} \tag{9}$$

For a function v defined on $(M\backslash\{p\})\times[0,T]$, we define

$$[v]_{X,T}=\sup_{t\in[0,T]}[v(t)]_X.$$

Definition 1.4 For $u\in C^{l,\alpha}(M\backslash\{p\})$, we say u has the property of approximations, if there is a sequence of functions u_i defined on $M\backslash\{p\}$ satisfying:

(i) For each i, there is a neighborhood of p, such that u_i in it are constant;

(ii) u_i converges to u in $C^{l,\alpha}_{\text{loc}}(M\backslash\{p\})$;

(iii) $\lim\limits_{i\to\infty}[u_i]_X=[u]_X$.

Similarly, for $v\in C^{l,\alpha}((M\backslash\{p\})\times[0,T])$, we say v has the property of approximations, if there is a sequence of v_i defined in $(M\backslash\{p\})\times[0,T]$ satisfying:

(i′) For each i, there is a neighborhood of p, such that for $t\in[0,T]$, $v_i(t)$ in it are constant;

(ii′) $v_i(t)$ converges to v in $C^{l,\alpha}_{\text{loc}}((M\backslash\{p\})\times[0,T])$;

(iii′) $\lim\limits_{i\to\infty}[a_i]_{X,T}=[a]_{X,T}$;

(iv′) there is a constant c (independent of v), such that

$$\|v_i\|_{C^0((M\backslash\{p\})\times[0,T])}\leqslant c\|v\|_{C^0((M\backslash\{p\})\times[0,T])},$$
$$\|\partial_t v_i\|_{C^0((M\backslash\{p\})\times[0,T])}\leqslant c\|\partial_t v\|_{C^0((M\backslash\{p\})\times[0,T])}.$$

With these definitions, we can state the main theorem of this paper as follows.

Theorem 1.1 Let (M,g) be a closed surface with conical metric and assume that p is the only cone point. Assume that $a\in\mathscr{P}^{2,\alpha,[0,T]}$, $\partial_t a\in C^0(M\times[0,T])$, $K_0\in\mathscr{E}^{2,\alpha}$, $u_0\in\mathscr{E}^{4,\alpha}$, and

(i) K_0 is identically 0 in a neighborhood U_K of p;

(ii) $[a]_{X,T}$, $[u_0]_X<\infty$;

(iii) a and u_0 have the property of approximations,

then there exists a solution $u\in\mathscr{P}^{4,\alpha,[0,T]}$ satisfying

$$u_t=\frac{1}{2}\mathrm{e}^{-2a}\triangle\left(\mathrm{e}^{-2a}(-\triangle u+K_0)\right) \tag{10}$$

and $u(0)=u_0$ such that

$$\|u\|_{\mathscr{P}^{4,\alpha,[0,T]}}\leqslant C(\|u_0\|_{\mathscr{E}^{4,\alpha}},\|a\|_{\mathscr{P}^{4,\alpha,[0,T]}},\|K_0\|_{\mathscr{E}^{2,\alpha}}) \tag{11}$$

and

$$[u]_{X,T}\leqslant C(\|a\|_{C^0},\|\partial_t a\|_{C^0},\|K_0\|_{\mathscr{E}^{2,\alpha}},[a]_{X,T},[u_0]_X,T) \tag{12}$$

To prove this theorem, we use a sequence of surfaces with boundary to approximate the surface with conical point. Specifically, we consider

$$M_k:=M\backslash\left\{(\rho,\theta)\mid\rho<\frac{1}{k}\right\}.$$

In this surface with boundary, we consider the same initial value problem, with some special boundary conditions:

$$\frac{\partial u}{\partial\nu}=\frac{\partial(\triangle u)}{\partial\nu}=0\ \text{on}\ \partial M_k.$$

By the boundary conditions, we can use the energy method to get some uniform apriori estimates (see Section 2). Based on this result, in Section 3, we finish the proof of Theorem 1.1 by taking $k\to\infty$. Finally, in Section 4, we discuss the property of approximations stated in Definition 1.4 and prove that the condition (iii) in Theorem 1.1 can be removed when $\beta\in(-1,0)$ and $\triangle u$ is bound. Specifically,

Theorem 1.2 Let $\beta\in(-1,0)$. If $u\in C^{4,\alpha}(M\backslash\{p\})$ satisfies $[u]_X<\infty$ and $\triangle u$ is bounded, then u has the property of approximations defined as Definition 1.4. Similarly, if $a\in C^{2,\alpha}((M\backslash\{p\})\times[0,T])$ satisfies $[a]_{X,T}<\infty$ and $\triangle a(t)$ is bounded, $\forall t\in[0,T]$, then a has the property of approximations.

2 Estimates of boundary value problem

In this section, M is a compact surface with nonempty boundary and a smooth Riemannian metric g. Consider the linear boundary value problem

$$\left\{\begin{aligned}&\partial_t u=\frac{1}{2}\mathrm{e}^{-2a}\triangle(\mathrm{e}^{-2a}(-\triangle u+K_0))\ \text{on}\ M\times[0,T],\\&u(0)=u_0\ \text{on}\ M,\\&\frac{\partial u}{\partial\nu}=\frac{\partial(\triangle u)}{\partial\nu}=0\ \text{on}\ \partial M\end{aligned}\right. \tag{13}$$

where ν is the outward normal vector to the boundary.

Theorem 2.1 Let $a\in C^{2,\alpha}(M\times[0,T])$, $K_0\in$

$C^{2,\alpha}(M)$, $u_0 \in C^{4,\alpha}(M)$. Assume that $a(t)$ and u_0 are constants in a neighborhood of ∂M. Then there is a unique solution $u(x,t) \in C^{4,\alpha}(M\times[0,T])$ to Equation (13) satisfying the initial condition $u(0)=u_0$. If M' containing the support of K_0 is a smooth domain with boundary satisfying $M' \cap \partial M = \emptyset$, then we have the following uniform estimate:

$$\max_{t\in[0,T]}\int_M (|u|^2 + |\nabla u|^2 + |\triangle u|^2)\mathrm{d}V \leqslant$$
$$C(\|a\|_{C^0}, \|K_0\|_{C^2}, \|\partial_t a\|_{C^0}, [a]_{X,T}, [u_0]_X, T, M') \tag{14}$$

where C depends on the geometric property of (M', g), such as Sobolev inequality, the coefficient of L^p estimates, but is independent of M.

Proof Since u_0 is constant around ∂M, the compatibility condition

$$\frac{\partial u_0}{\partial \nu} = \frac{\partial(\triangle u_0)}{\partial \nu} = 0 \text{ on } \partial M \tag{15}$$

holds. The existence and uniqueness of the solution u is well known from the classical theroy[13].

Next, we prove some uniform estimates of the L^2 norm of $\triangle u$, ∇u, u.

Differentiating directly and using integration by parts by boundary conditions, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\triangle u|^2 \mathrm{e}^{-2a}\mathrm{d}V =$$
$$2\int_M \triangle u \triangle \frac{\partial u}{\partial t}\mathrm{e}^{-2a}\mathrm{d}V - 2\int_M |\triangle u|^2 \mathrm{e}^{-2a}\partial_t a \mathrm{d}V \leqslant$$
$$-2\int_M \nabla(\triangle u \mathrm{e}^{-2a})\cdot\nabla\frac{\partial u}{\partial t}\mathrm{d}V + C_a\int_M |\triangle u|^2\mathrm{d}V,$$

where C_a is a constant depending on $\|a\|_{C^0}$ and $\|\partial_t a\|_{C^0}$. In the calculation below, each time C_a appears it may represent a different constant. If necessary, to specify a constant, we use double subscripts, such as C_{a1}.

Use integration by parts once again, substitute Equation (2) into the inequality, and apply Young's inequality, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\triangle u|^2\mathrm{e}^{-2a}\mathrm{d}V \leqslant 2\int_M \triangle(\triangle u \mathrm{e}^{-2a})\frac{\partial u}{\partial t}\mathrm{d}V +$$
$$C_a\int_M |\triangle u|^2\mathrm{d}V = -4\int_M |\partial_t u|^2\mathrm{e}^{2a}\mathrm{d}V +$$
$$2\int_M \triangle(\mathrm{e}^{-2a}K_0)\partial_t u\mathrm{d}V + C_a\int_M |\triangle u|^2\mathrm{d}V \leqslant$$
$$-C_{a1}\int_M |\partial_t u|^2\mathrm{d}V + \varepsilon\int_M |\partial_t u|^2\mathrm{d}V +$$
$$\frac{1}{\varepsilon}\int_M |\triangle(\mathrm{e}^{-2a}K_0)|^2\mathrm{d}V + C_a\int_M |\triangle u|^2\mathrm{d}V \leqslant$$
$$C_a\int_M |\triangle u|^2\mathrm{d}V + C_a\int_M |\triangle(\mathrm{e}^{-2a}K_0)|^2\mathrm{d}V.$$

Here in the last line above, we choose $\varepsilon < \frac{C_{a1}}{2}$. For the second term in the above inequality,

$$\int_M |\triangle(\mathrm{e}^{-2a}K_0)|^2\mathrm{d}V \leqslant$$
$$C_{K_0}\int_M (|\triangle a|^2 + |K_0|^2|\nabla a|^4 + |\nabla a|^2)\mathrm{d}V\square + C_{a,K_0} \tag{16}$$

where the meaning of subscript in C_{K_0} and C_{a,K_0} is understood in a similar way as in C_a. The items in parentheses above, except $|\nabla a|^4$, are controlled by $[a]_{X,T}$. Using Sobolev's embedding theorem and L^2 estimates in the support of K_0, we have

$$\int_M |K_0|^2|\nabla a|^4\mathrm{d}V \leqslant C_{K_0}\int_{M'} |\nabla a|^4\mathrm{d}V \leqslant$$
$$C_{K_0}C_{M'}\|a\|^4_{W^{2,2}(M')} \leqslant$$
$$C_{K_0}C_{M'}(\|\triangle a\|_{L^2(M')} + \|\nabla a\|_{L^2(M')} + \|a\|_{L^2(M')})^4 \leqslant$$
$$C_{K_0}C_{M'}[a]^4_{X,T}.$$

Base on the above, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\triangle u|^2\mathrm{e}^{-2a}\mathrm{d}V \leqslant$$
$$C_a\int_M |\triangle u|^2\mathrm{e}^{-2a}\mathrm{d}V + C_{a,K_0} + C_{a,K_0}C_{M'}[a]^4_{X,T}.$$

We can then get the uniform estimate of $\|\triangle u\|_{L^2(M)}$ by Gronwall's inequality.

Similarly, using Equation (2) and integration by parts twice (use the condition that a is constant around ∂M and boundary condition), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\nabla u|^2\mathrm{d}V = -2\int_M \triangle u\frac{\partial u}{\partial t}\mathrm{d}V =$$
$$\int_M \triangle u\mathrm{e}^{-2a}\triangle(\mathrm{e}^{-2a}\triangle u)\mathrm{d}V - \int_M \triangle u\mathrm{e}^{-2a}\triangle(\mathrm{e}^{-2a}K_0)\mathrm{d}V =$$
$$-\int_M |\nabla(\triangle u\mathrm{e}^{-2a})|^2\mathrm{d}V - \int_M \triangle u\mathrm{e}^{-2a}\triangle(\mathrm{e}^{-2a}K_0)\mathrm{d}V \leqslant$$
$$\frac{1}{2}\int_M |\triangle u|^2\mathrm{d}V + \frac{C_a}{2}\int_M |\triangle(\mathrm{e}^{-2a}K_0)|^2\mathrm{d}V.$$

In the last line above, we drop a negative term, and use the Schwarz's inequality. The first term on the far right of the above inequality is already estimated, and the second term is already discussed in (16).

Using Gronwall's inequality once again, we get the estimate of $\int_M |\nabla u|^2\mathrm{d}V$.

To get the estimate of L^2 norm of u, noticing that $\partial_\nu a = \partial_\nu K_0 = \partial_\nu(\triangle u) = \partial_\nu u = 0$, we use integration by parts twice,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M |u|^2\mathrm{e}^{2a}\mathrm{d}V = 2\int_M |u|^2\mathrm{e}^{2a}\partial_t a\mathrm{d}V +$$
$$\int_M u\triangle(\mathrm{e}^{-2a}(-\triangle u + K_0))\mathrm{d}V \leqslant$$
$$C_a\int_M |u|^2\mathrm{d}V - \int_M \nabla u\cdot\nabla(\mathrm{e}^{-2a}(-\triangle u + K_0))\mathrm{d}V =$$
$$C_a\int_M |u|^2\mathrm{d}V + \int_M \triangle u\mathrm{e}^{-2a}(-\triangle u + K_0)\mathrm{d}V.$$

By the Young's inequality,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M |u|^2\mathrm{e}^{2a}\mathrm{d}V \leqslant$$

$$C_a\int_M |u|^2 \mathrm{d}V - C_a\int_M |\triangle u|^2 + C_{a,K_0} \leqslant C_a\int_M |u|^2\mathrm{d}V + C_{a,K_0}.$$

By the Gronwall's inequality, and the proof is done.

3 Solution of the linear equation via approximations

The purpose of this section is to prove Theorem 1.1 by Theorem 2.1.

Recall the definition

$$M_k := M\setminus\{(\rho,\theta) \mid \rho < \frac{1}{k}\} \tag{17}$$

Due to the assumption (iii) in Theorem 1.1, by taking a subsequence, we can assume without loss of generality that there are $u_{0;k} \in C^{4,\alpha}(X\setminus\{p\})$ such that $u_{0;k}$ is constant around ∂M_k, similarly, $a_k \in C^{2,\alpha}((M\setminus\{p\})\times[0,T])$ satisfying that a_k is constant around ∂M_k for $t\in[0,T]$. Furthermore, due to the assumption (i) in Theorem 1.1, we can assume that $M' := M\setminus U_K$ is compactly contained in M_k when k goes to infinity.

By the above discussion, we can apply Theorem 2.1 to the boundary value problem

$$\left\{\begin{aligned} &\partial_t u = \frac{1}{2}\mathrm{e}^{-2a_k}\triangle(\mathrm{e}^{-2a_k}(-\triangle u + K_0)) \text{ on } M_k\times[0,T],\\ &u(0) = u_{0;k} \text{ on } M_k,\\ &\frac{\partial u}{\partial\nu} = \frac{\partial(\triangle u)}{\partial\nu} = 0 \text{ on } \partial M_k \end{aligned}\right. \tag{18}$$

and denote the solution by u_k. It is defined in $M_k\times[0,T]$, and satisfies the uniform estimate (14):

$$\max_{t\in[0,T]}\int_{M_k}(|u_k|^2 + |\nabla u_k|^2 + |\triangle u_k|^2)\mathrm{d}V \leqslant C(\|a_k\|_{C^0}, \|K_0\|_{C^2}, \|\partial_t a_k\|_{C^0}, [a_k]_{X,T}, [u_{0;k}]_X, T, M') \tag{19}$$

Meanwhile, for any fixed compact set $W\subset M\setminus\{p\}$, a_k converges to a in $C^{2,\alpha}(W\times[0,T])$, $u_{0;k}$ converges to u_0 in $C^{2,\alpha}(W)$. After taking subsequence if necessary, we might as well call it u_k, it converges to a function $u(x,t)$ defined in $(M\setminus\{p\})\times[0,T]$, and u is a classical solution to the initial value problem of Equation (2).

Since $a\in\mathscr{P}^{2,\alpha,[0,T]}$, $u_0\in\mathscr{E}^{4,\alpha}$ and $K_0\in\mathscr{E}^{2,\alpha}$, we may apply the Schauder interior estimates to (2) to obtain that $u\in\mathscr{P}^{4,\alpha,[0,T]}$ and (11).

Meanwhile, since u_k satisfies (14), by the definition of the property of approximations, we have

$$\|a_k\|_{C^0} \leqslant c\|a\|_{C^0},$$
$$\|\partial_t a_k\|_{C^0} \leqslant c\|\partial_t a\|_{C^0},$$
$$\lim_{k\to\infty}[a_k]_{X,T} = [a]_{X,T},$$
$$\lim_{k\to\infty}[u_{0;k}]_X = [u_0]_X.$$

Let k go to ∞, and we get (12). Hence we finish the proof of Theorem 1.1.

4 About the property of approximations

The aim of this section is to prove Theorem 1.2. We will give the approximation sequence by explicit construction. The proof is divided into two parts. First, we prove the theorem for $u\in C^{4,\alpha}(M\setminus\{p\})$.

In order to define the approximation sequence, we first give some properties of the function u near the cone point p. Although the function class $C^{4,\alpha}(M\setminus\{p\})$ puts few restrictions on the properties of the function near p, the condition

$$\int_M(|u|^2 + |\nabla u|^2 + |\triangle u|^2)\mathrm{d}V < \infty$$

implies a lot, which is summarized in the following lemma.

First, we need a lemma about the integration by parts.

Lemma 4.1 A function u is defined on M. If u is bounded and $\int_M |\nabla u|^2\mathrm{d}V$ is bounded, then

$$\int_M \triangle u\mathrm{d}V = 0 \text{ and } \int_M u\cdot\triangle u\mathrm{d}V = -\int_M |\nabla u|^2\mathrm{d}V.$$

Lemma 4.2 Let u satisfy the requirements of Theorem 1.2, then

(i) there is $\gamma\in(0,1)$ which depends only on β, such that u is in $C^{0,\gamma}$ in the coordinate z;

(ii) $|\nabla u|^2$ is bounded.

Proof We denote the flat metric $\mathrm{d}x^2+\mathrm{d}y^2$ by g_s and write $W^{2,p}(g_s)$ for the Sobolev space with respect to g_s.

Letting $f=\triangle u$, in the neighborhood $B:=\{\rho<1/2\}$ of p, by (1), we have

$$\triangle_{g_s} u = |z|^{2\beta} f \tag{20}$$

Meanwhile, by the assumption $\int_M |\triangle u|^2\mathrm{d}V < \infty$, we have

$$\int_B |f|^2 |z|^{2\beta}\mathrm{d}x\mathrm{d}y < \infty \tag{21}$$

Since $\beta\in(-1,0)$, there exists $q>2$, such that $|z|^\beta$ is in $L^q(g_s)$. With (21), we deduce that the right hand side of (20) is in $L^{\frac{2q}{2+q}}$. By $\int_M |\nabla u|^2\mathrm{d}V < \infty$, we get that u is a weak solution to (20), and then by L^p estimates and Sobolev's imbedding theorem, there exists $\gamma\in(0,1)$, such that u is in $C^{0,\gamma}$ in $\{\rho<1/4\}$ as a function of z.

Next we prove (ii). Assume first that $-\frac{1}{2}<\beta<0$. In this case, $\triangle_{g_s}u=|z|^{2\beta}f\in L^q(g_s)$, for some $q>2$.

We claim that $u\in W^{2,q}(g_s)$. To see this, let $\bar{u}$ be the usual solution of

$$\triangle_{g_s}\bar{u} = |z|^{2\beta}f$$

with boundary value $\bar{u}=u$ on $\{r=1\}$. We know $\bar{u}\in W^{2,q}(g_s)$. Meanwhile, the difference $\bar{u}-u$ is a harmonic function defined on $\{0<r<1\}$ and vanish on $\{r=1\}$. Moreover, it is bounded and $\int_M |\nabla(u-\bar{u})|^2 dV$ is bounded. Hence it is zero by Lemma 4.1 and $u=\bar{u}$.

By Sobolev's embedding theorem, $\partial_x u$ and $\partial_y u$ are bounded. Hence, $|\nabla u|^2$ is bounded because

$$|\nabla u|^2 = |z|^{-2\beta}(|\partial_x u|^2 + |\partial_y u|^2).$$

If $-1<\beta\leqslant -\frac{1}{2}$, we can find positive integer m and $\beta_0 \in (-\frac{1}{2}, 0]$ such that $1+\beta_0 = m(1+\beta)$. Hence, we may consider a cone of order β_0, which is m-fold cover of the original one. Then the lemma with cone of order β follows from the cone of β_0. Precisely, by setting $\rho = \frac{1}{\beta+1}$, we have

$$g = d\rho^2 + (1+\beta)^2 \rho^2 d\theta^2.$$

Consider another cone of order β_0, whose metric is given by

$$\hat{g} = d\rho^2 + (1+\beta_0)^2 \rho^2 d\eta^2.$$

The map Ψ from (ρ, η) to $(\rho, m\eta \bmod 2\pi)$ is an m-fold isometric covering. By setting $\hat{u} = u\circ\Psi$ and $\hat{f} = f\circ\Psi$, we have

$$\triangle_{\hat{g}} \hat{u} = \hat{f}.$$

Since $\hat{f}, \hat{u}$ are bounded and $\int |\nabla \hat{u}|^2 dV_{\hat{g}}$ is bounded, we know $|\hat{\nabla}\hat{u}|^2$ is bounded. So is u and the lemma is proved.

To define the approximation sequence, we need a sequence of cut-off functions φ_i satisfying:

(C1) for any $\rho \in [0, \frac{1}{2}]$, $\varphi_i(\rho) \in [0,1]$;

(C2) there is $\delta_i > 0$, $\varphi_i \equiv 1$ in $[0, \delta_i]$;

(C3) for any $\rho > \frac{1}{i}$, $\varphi_i(\rho) = 0$;

(C4) φ_i is smooth, and

$$\lim_{i\to\infty} \int_0^{\frac{1}{2}} |\varphi_i'(\rho)|^2 \rho d\rho = 0.$$

(C5) there exists $c>0$, such that

$$\sup_{[0,\frac{1}{2}]} |\varphi_i''(\rho)| \leqslant \frac{c}{\rho^2}.$$

We claim that φ_i satisfying the above conditions exists. To see this, for any $m>1$, we choose a smooth function $\psi: \mathbb{R} \to \mathbb{R}$ such that

$$\psi(s) \equiv 1 \ \forall x \geqslant m+1,$$
$$\psi(s) \equiv 0 \ \forall x \leqslant m.$$

Naturally we have

$$\sup_{\mathbb{R}} (|\psi'| + |\psi''|) \leqslant 4.$$

For any i, we choose m which is large enough (dependent of i), and define

$$\varphi_i(\rho) = \psi(\log(-\log\rho)).$$

Due to the equations

$$\varphi_i'(\rho) = \begin{cases} \psi'(\log(-\log\rho)) \dfrac{1}{\rho\log\rho}, \\ \quad m < \log(-\log\rho) < m+1; \\ 0, \text{ otherwise}; \end{cases}$$

and

$$\varphi_i''(\rho) = \psi''(\log(-\log\rho)) \frac{1}{\rho^2(\log\rho)^2} + \psi'(\log(-\log\rho)) \frac{-1-\log\rho}{\rho^2(\log\rho)^2},$$

we obtain that if m is large enough, (C1)-(C5) hold.

By Lemma 4.2, we can write

$$u(\rho,\theta) = u(p) + \tilde{u}(\rho,\theta),$$

where $u(p)$ is the value of u at p, and

$$|\tilde{u}|(\rho,\theta) \leqslant C\rho^\alpha \tag{22}$$

for some $\alpha > 1$. This is very important for later estimates.

We define

$$u_i = u(p) + (1-\varphi_i)\tilde{u} \tag{23}$$

We just need to verify that u_i meets the requirements of Definition 1.4, where (i) and (ii) therein are direct consequences of (C2) and (C3). Hence, it suffices to show

$$\lim_{i\to\infty} \int_{B_{1/2}} (|\varphi_i \tilde{u}|^2 + |\nabla(\varphi_i \tilde{u})|^2 + |\triangle(\varphi_i \tilde{u})|^2) dV_g = 0 \tag{24}$$

By the dominated convergence theorem,

$$\lim_{i\to\infty} \int_{B_{1/2}} |\varphi_i \tilde{u}|^2 dV_g = 0$$

is obvious. Meanwhile,

$$\int_{B_{1/2}} |\nabla(\varphi_i \tilde{u})|^2 dV_g \leqslant 2\int_{B_{1/2}} (|\nabla\varphi_i|^2 |\tilde{u}|^2 + \varphi_i^2 |\nabla\tilde{u}|^2) dV_g.$$

By (C4), we get that the right hand side of the above equation goes to 0 when $i\to\infty$.

Finally,

$$|\triangle(\varphi_i \tilde{u})|^2 \leqslant c(|\triangle\varphi_i|^2 \tilde{u}^2 + |\nabla\varphi_i|^2 |\nabla\tilde{u}|^2 + \varphi_i^2 |\triangle\tilde{u}|^2).$$

It is obvious that the integral of the last term in the right hand side of the above inequality goes to 0, and the integral of the second term also goes to 0 because of (ii) in Lemma 4.2 and (C4). To estimate the first one, we use (C5) and (22),

$$\int_{B_{1/2}} |\triangle\varphi_i|^2 \tilde{u}^2 dV_g \leqslant C\int_0^{1/2} (\varphi_i'' + \frac{1}{\rho}\varphi_i')^2 \rho^{2\alpha} \rho d\rho.$$

Notice that $2\alpha > 2$, and the domain of the above integral is really just $[0, \frac{1}{i}]$, we get this term also goes to 0 when $i\to\infty$. Therefore we finish the proof of the property of approximations for u.

Let a satisfy the assumptions of Theorem 1.2. Naturally, $a(t)$ as a function defined on $M\setminus\{p\}$, satisfies that $[a(t)]_X$ is finite, then Lemma 4.2 holds for $a(t)$. Hence we can write

$$a = a(p,t) + \tilde{a}.$$

Next set

$$a_i = a(p,t) + (1-\varphi_i)\tilde{a} \tag{25}$$

For a fixed t by repeating the proof above, we obtain that (i′)-(iii′) in Definition 1.4 hold for a_i. To show (iv′), take the C^0 norm of (25),

$$\| a_i(t) \|_{C^0(M\setminus\{p\})} \leqslant | a(p,t) | + \| \tilde{a}(t) \|_{C^0(M\setminus\{p\})} \leqslant 3 \| a(t) \|_{C^0(M\setminus\{p\})}.$$

Take the derivative of (25) with respect to t, and take C^0 norm again,

$$\| \partial_t a_i(t) \|_{C^0(M\setminus\{p\})} \leqslant | \partial_t a(p,t) | + \| \partial_t \tilde{a}(t) \|_{C^0(M\setminus\{p\})} \leqslant 3 \| \partial_t a(t) \|_{C^0(M\setminus\{p\})}.$$

Conflict of interest

The author declares no conflict of interest.

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一个带锥曲面上的四阶线性抛物方程

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摘要：考虑一个带锥曲面上的四阶线性抛物方程，利用能量分析和逼近的方法，证明了方程在一个具有逼近性质的空间上的解的存在唯一性. 最后，证明了当 $\beta\in(-1,0)$ 时，对于一类函数这个性质等价于能量有限.

关键词：抛物方程;Calabi 流;锥奇点