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# Riemann-Hilbert approach for a mixed coupled nonlinear Schrödinger equations and its soliton solutions

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**Abstract**: The integrable mixed coupled nonlinear Schrödinger (MCNLS) equations is studied, which describes the propagation of an optical pulse in a birefringent optical fiber. By the Riemann-Hilbert (RH) approach, the *N*-soliton solutions of the MCNLS equations can be expressed explicitly when the jump matrix of a constructed RH problem is a 3×3 unit matrix. As a special example, the expression of one soliton and two solitons are displayed explicitly. More generally, as a promotion, an integrable generalized multi-component NLS system with its linear spectral problem is discussed.

**Keywords**: Lax pair; Riemann-Hilbert approach; mixed coupled nonlinear Schrödinger (MCNLS) equations; soliton solution; boundary conditions
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## 1 Introduction

It is well known that solving nonlinear evolution equations becomes a challenging task due to the complexity of nonlinear systems. In particular, to seek exact solutions is crucial for research in various fields. Through many years of efforts of mathematicians and physicists, a variety of methods for exact solutions have been established, such as the inverse scattering transform (IST) method[1,2], the bilinear derivative approach[3], the Darboux transformation (DT)[4] and others[5]. In recent years, with the development of the soliton theory, more and more scholars have paid attention to Riemann-Hilbert (RH) approach[6], which is a new powerful approach to solve integrable partial differential equations (PDEs)[7-12]. The main idea of this method is to establish a corresponding matrix RH problem on the Lax pair of integrable equations. Furthermore, the RH approach is also an effective way to examine the initial-boundary value problems (IBVPs)[13-18] and the asymptotic properties[19-20] of solutions for the integrable equations.

The famous integrable nonlinear Schrödinger (NLS) equation

$$\mathrm{i}q_t \pm q_{zz} + |q|^2 q = 0 \tag{1}$$

arises in various physical backgrounds involving hydrodynamics, plasma physics, Bose-Einstein condensation, nonlinear optics, and other physical fields. However, a slice of phenomena in the real world and physical experiments can no longer be described by NLS Eq. (1). Accordingly, quite a few individuals began to examine the two-component case of NLS Eq. (1) (also known as the Manakov equations) to illustrate these phenomena,

$$\begin{cases} \mathrm{i}q_{1t} + \dfrac{1}{2}q_{1zz} + \epsilon(|q_1|^2 + |q_2|^2)q_1 = 0, \\ \mathrm{i}q_{2t} + \dfrac{1}{2}q_{2zz} + \epsilon(|q_1|^2 + |q_2|^2)q_2 = 0, \end{cases} \quad \epsilon = \pm 1 \tag{2}$$

where $\epsilon = -1$ and $\epsilon = 1$ represents focusing and defocusing cases, respectively. Indeed, Eq. (2) can be used to describe the propagation of optical pulses in birefringent fibers[21], which was first proposed by Manakov in 1974. Moreover, Eq. (2) also provides convenience for mathematically extending the local linearization analysis to the whole nonlinear unstable manifold under oscillatory waves.

In the present paper, we investigated the coupled focusing-defocusing NLS system, called the mixed coupled nonlinear Schrödinger (MCNLS) equations

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$$\begin{cases} \mathrm{i}q_{1t} + \dfrac{1}{2}q_{1zz} + (|q_2|^2 - |q_1|^2)q_1 = 0, \\ \mathrm{i}q_{2t} + \dfrac{1}{2}q_{2zz} + (|q_2|^2 - |q_1|^2)q_2 = 0 \end{cases} \quad (3)$$

based on RH method, where $t$ and $z$ represent time variables and propagation direction, respectively. In fact, system (3) is completely integrable, and quite a few of its properties have been widely discussed. As an example, Kanna et al. adopt intensity redistribution to analyze the shape change of soliton collisions[22]. Vijayajayanthi et al. obtained the bright and dark solitons of mixed N-coupled NLS equations and discussed their collision properties[23]. Ling et al. gave the bright dark rogue wave solutions, the type I and type II vector rogue wave solutions through DT method[24]. Recently, Tian discussed the IBVPs by the Fokas method[25]. However, according to the authors, the $N$-soliton solutions of the system (3) via the RH approach has not been solved before.

The organization of this work is as follows. In Section 2, we will construct a specific RH problem by the IST approach. In Section 3, we compute soliton solutions via this specific RH problem, which possesses the identity jump matrix on the real axis. Discussions and conclusions are given in the final section.

## 2 The Riemann-Hilbert problem

The MCNLS equations (3) admit the following Lax pair[25]:

$$\Phi_z = M(z,t,\theta)\Phi = (-\mathrm{i}\theta\Lambda + \mathrm{i}Q)\Phi \quad (4a)$$

$$\Phi_t = N(z,t,\theta)\Phi = \left(-\mathrm{i}\theta^2\Lambda + \mathrm{i}\theta Q + \frac{1}{2}(\mathrm{i}Q^2 - Q_z)\right)\Phi \quad (4b)$$

where $\theta$ is an iso-spectral parameter and

$$\Lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ Q = \begin{pmatrix} 0 & -q_1^* & q_2^* \\ q_1 & 0 & 0 \\ q_2 & 0 & 0 \end{pmatrix} \quad (5)$$

Eq. (4a)-(4b) can be written as

$$\Phi_z + \mathrm{i}\theta\Lambda\Phi = Q_1\Phi \quad (6a)$$

$$\Phi_t + \mathrm{i}\theta^2\Lambda\Phi = Q_2\Phi \quad (6b)$$

where

$$Q_1 = \mathrm{i}Q,\ Q_2 = \mathrm{i}\theta Q + \frac{1}{2}(\mathrm{i}Q^2 - Q_z).$$

Suppose $\widetilde{A}(z,t,\theta) = \mathrm{e}^{-(\mathrm{i}\theta z + \mathrm{i}\theta^2)\Lambda t}$, we find that $\widetilde{A}(z,t,\theta)$ is a solution for the (6a) and (6b). Let us introduce a new function

$$\Psi(z,t,\theta) = F(z,t,\theta)\widetilde{A}(z,t,\theta),$$

then we have

$$F_z + \mathrm{i}\theta[\Lambda, F] = Q_1 F \quad (7a)$$

$$F_t + \mathrm{i}\theta^2[\Lambda, F] = Q_2 F \quad (7b)$$

For $\theta \in \mathbb{R}$, one can construct two Jost solutions $F_\pm = F_\pm(z,\theta)$ of (7a):

$$F_+ = ([F_+]_1, [F_+]_2, [F_+]_3) \quad (8a)$$

$$F_- = ([F_-]_1, [F_-]_2, [F_-]_3) \quad (8b)$$

with the boundary conditions

$$F_+ \to I,\ z \to +\infty \quad (9a)$$

$$F_- \to I,\ z \to -\infty \quad (9b)$$

where $\{[F_\pm]_n\}_1^3$ represents the $n$-th column vector of $F_\pm$, $I = \mathrm{diag}\{1,1,1\}$ is a 3×3 unit matrix. In fact, the Jost solutions $F_\pm = F_\pm(z,\theta)$ of Eq. (7a) for $\theta \in \mathbb{R}$ are well-defined by

$$F_+(z,\theta) = I - \int_z^{+\infty} \mathrm{e}^{-\mathrm{i}\theta\widehat{\Lambda}(z-\xi)} Q_1(\xi) F_+(\xi,\theta)\mathrm{d}\xi \quad (10a)$$

$$F_-(z,\theta) = I + \int_{-\infty}^{z} \mathrm{e}^{-\mathrm{i}\theta\widehat{\Lambda}(z-\xi)} Q_1(\xi) F_-(\xi,\theta)\mathrm{d}\xi \quad (10b)$$

where $\widehat{\Lambda}$ is a matrix operator, such as $\widehat{\Lambda}Y = [\Lambda, Y]$ and $\mathrm{e}^{z\widehat{\Lambda}}Y = \mathrm{e}^{z\Lambda}Y\mathrm{e}^{-z\Lambda}$.

Thus, by further analysis, we know that $[F_+]_1$, $[F_-]_2$, $[F_-]_3$ enjoy analytic prolongations to the upper half $\theta$-plane $\mathbb{C}_+$. On the other hand, $[F_-]_1$, $[F_+]_2$, $[F_+]_3$ enjoy analytic prolongations to the lower half $\theta$-plane $\mathbb{C}_-$.

Next, we discuss the properties of $F_\pm$. Due to the Abel's identity and $\mathrm{tr}(Q) = 0$, we find that the determinants of $I_\pm$ are constants for all $z$. From the boundary conditions (9a) and (9b), we have

$$\det F_\pm = 1,\ \theta \in \mathbb{R} \quad (11)$$

Introducing another new function $A(z,\theta) = \mathrm{e}^{-\mathrm{i}\theta\Lambda z}$, then we know that the spectral problem (7a) has two fundamental matrix solutions $F_+A$ and $F_-A$, which are not independent and are linear associated by a 3×3 scattering matrix $S(\theta)$:

$$F_- = F_+ A \cdot S(\theta) A^{-1},\ \theta \in \mathbb{R} \quad (12)$$

It follows from (11) and (12) we know that

$$\det S(\theta) = 1 \quad (13)$$

Moreover, let $z \to +\infty$, the 3×3 scattering matrix $S(\theta)$ is defined by

$$S(\theta) = (s_{ij})_{3\times 3} = \lim_{z\to+\infty} A^{-1} F_- A = I + \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\theta\widehat{\Lambda}\xi} Q_1 F_- \mathrm{d}\xi,\ \theta \in \mathbb{R} \quad (14)$$

It follows from the analytic property of $F_-$ that $s_{22}$, $s_{23}$, $s_{32}$ and $s_{33}$ can be analytically prolonged to $\mathbb{C}_+$, $s_{11}$ allows analytic prolongations to $\mathbb{C}_-$. Generally speaking, $s_{12}$, $s_{13}$, $s_{21}$ and $s_{31}$ cannot be extended off the real $z$-axis.

In order to obtain behavior of Jost solutions for a very large $\theta$, we substitute the asymptotic expansion

$$F = F_0 + \frac{F_1}{\theta} + \frac{F_2}{\theta^2} + \frac{F_3}{\theta^3} + \frac{F_4}{\theta^4} + \cdots,\ \theta \to \infty \quad (15)$$

into the Eq. (7a) and compare coefficients

$$O(\theta^1): \mathrm{i}[\Lambda, F_0] = 0 \quad (16a)$$

$$O(\theta^0): F_{0,z} + \mathrm{i}[\Lambda, F_1] - Q_1 F_0 = 0 \quad (16b)$$

$$O(\theta^{-1}): F_{1,z} + \mathrm{i}[\Lambda, F_2] - Q_1 F_1 = 0 \quad (16c)$$

from $O(\theta^1)$ and $O(\theta^0)$ we find

$$\mathrm{i}[\Lambda, F_1] = Q_1 F_0,\ F_{0,z} = 0 \quad (17)$$

To establish the RH problem of the MCNLS equations, we define another new Jost solution for Eq. (7a) by

$$G_+ = ([F_+]_1, [F_-]_2, [F_-]_3) = F_+ A S_+ A^{-1} = F_+ A \begin{pmatrix} 1 & s_{12} & s_{13} \\ 0 & s_{22} & s_{23} \\ 0 & s_{32} & s_{33} \end{pmatrix} A^{-1} \quad (18)$$

which is analytic for $\theta \in \mathrm{C}_+$ and enjoys asymptotic behavior for very large $\theta$ as

$$G_+ \to I,\ \theta \to +\infty,\ \theta \in \mathrm{C}_+ \quad (19)$$

Furthermore, to obtain the analysis of $G_-$ in $\mathrm{C}_-$ which counterpart is $G_+$, we also need to consider the adjoint scattering equation of Eq. (7a):

$$J_z + \mathrm{i}\theta[\Lambda, J] = -JQ_1 \quad (20)$$

Similarly, one can define the inverse matrices $F_\pm^{-1}$ as

$$[F_+]^{-1} = \begin{pmatrix} [F_+^{-1}]^1 \\ [F_+^{-1}]^2 \\ [F_+^{-1}]^3 \end{pmatrix},\ [F_-]^{-1} = \begin{pmatrix} [F_-^{-1}]^1 \\ [F_-^{-1}]^2 \\ [F_-^{-1}]^3 \end{pmatrix} \quad (21)$$

where $[F_\pm^{-1}]^n$ represents the $n$-th row vector of $F_\pm^{-1}$, $F_\pm^{-1}$ satisfies this adjoint equation (20). Then we find that $[F_+^{-1}]^1$, $[F_-^{-1}]^2$ and $[F_-^{-1}]^3$ enjoy analytic continuations to $\mathrm{C}_-$ as well as $[F_-^{-1}]^1$, $[F_+^{-1}]^2$ and $[F_+^{-1}]^3$ enjoy analytic prolongations to the $\mathrm{C}_+$.

Moreover, it is not difficult to see that the inverse matrices $F_\pm^{-1}$ admits the following boundary conditions.

$$F_\pm^{-1} \to I,\ \theta \to \mp\infty \quad (22)$$

In addition, we define a matrix function $G_-$ expressed by

$$G_- = \begin{pmatrix} [F_+^{-1}]^1 \\ [F_-^{-1}]^2 \\ [F_-^{-1}]^3 \end{pmatrix} \quad (23)$$

With techniques similar to those used above, one can demonstrate that the adjoint Jost solutions $G_-$ are analytic in $\mathrm{C}_-$ and

$$G_- \to I,\ \theta \to -\infty,\ \theta \in \mathrm{C}_- \quad (24)$$

Suppose $R(\theta) = S^{-1}(\theta)$, we find

$$F_-^{-1} = AR(\theta)A^{-1}F_+^{-1} \quad (25)$$

and

$$G_- = \begin{pmatrix} [F_+^{-1}]^1 \\ [F_-^{-1}]^2 \\ [F_-^{-1}]^3 \end{pmatrix} = AR_+ A^{-1} F_+^{-1} = A \begin{pmatrix} 1 & 0 & 0 \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} A^{-1} F_+^{-1} \quad (26)$$

So far, we have obtained two analytic matrix-value functions $G_+(z,\theta)$ and $G_-(z,\theta)$ for $\theta$ in $\mathrm{C}_+$ and $\mathrm{C}_-$, respectively. In fact, two matrix functions $G_+(z,\theta)$ and $G_-(z,\theta)$ can be established by a $3\times 3$ special RH problem as follows

$$G_-(z,\theta) G_+(z,\theta) = T(z,\theta),\ \theta \in \mathrm{C}_- \quad (27)$$

where

$$T(z,\theta) = AR_+ S_+ A^{-1} = \begin{pmatrix} 1 & s_{12}\mathrm{e}^{2\mathrm{i}\theta z} & s_{13}\mathrm{e}^{2\mathrm{i}\theta z} \\ r_{21}\mathrm{e}^{-2\mathrm{i}\theta z} & 1 & 0 \\ r_{31}\mathrm{e}^{-2\mathrm{i}\theta z} & 0 & 1 \end{pmatrix},\ \theta \in \mathrm{C}_- \quad (28)$$

and the identity $r_{11}s_{11} + r_{12}s_{21} + r_{13}s_{31} = 1$ holds in (28).

On the other hand, owing to the fact that $F_-$ admits the following equation

$$F_{-,t} + \mathrm{i}\theta^2[\Lambda, F_-] = Q_2 F_- \quad (29)$$

we find that

$$F_- A = F_+ AS,\ (F_+ AS)_t + \mathrm{i}\theta^2[\Lambda, F_+ AS] = Q_2 F_+ AS \quad (30)$$

assuming that $q_1$ and $q_2$ have sufficient smoothness and decay as $z \to \infty$, we find that $Q_2 \to 0$ as $z \to \pm\infty$. Thus taking the limit $z \to +\infty$ of Eq. (30), we arrive at

$$S_t = -\mathrm{i}\theta^2[\Lambda, S] \quad (31)$$

which means that the scattering data $s_{11}, s_{22}, s_{33}, s_{23}, s_{32}$ are time independent, and

$$s_{12}(t,\theta) = s_{12}(0,\theta)\mathrm{e}^{2\mathrm{i}\theta^2 t},\ s_{13}(t,\theta) = s_{13}(0,\theta)\mathrm{e}^{2\mathrm{i}\theta^2 t},$$
$$s_{21}(t,\theta) = s_{21}(0,\theta)\mathrm{e}^{-2\mathrm{i}\theta^2 t},\ s_{31}(t,\theta) = s_{31}(0,\theta)\mathrm{e}^{-2\mathrm{i}\theta^2 t}.$$

## 3 The soliton solutions

The definitions of $G_\pm$, $F_\pm$ admits the scattering relationship (12) in Section 2, it is not difficulty to see that

$$\det G_+(z,\theta) = r_{11}(\theta),\ \det G_-(z,\theta) = s_{11}(\theta) \quad (32)$$

where $r_{11} = s_{22}s_{33} - s_{23}s_{32}$, so the det $G_+$ and $r_{11}(\theta)$ have the same zeros, as det $G_-$ and $s_{11}(\theta)$. In fact, since the scattering data $r_{11}$ and $s_{11}$ are time independent, we find that the roots of $r_{11} = 0$ and $s_{11} = 0$ are also time independent. Furthermore, as $Q^\dagger = \sigma Q \sigma$, ($\sigma = \mathrm{diag}\{1, -1, 1\}$). It is easy to know that

$$F_\pm^{-1}(z,t,\theta) = \sigma F_\pm^\dagger(z,t,\theta^*)\sigma \quad (33)$$

and

$$S^{-1}(\theta) = \sigma S^\dagger(\theta^*)\sigma,\ G_-(z,\theta) = \sigma G_+^\dagger(z,\theta^*)\sigma \quad (34)$$

Assuming that $r_{11}$ enjoys $N \geqslant 0$ feasible zeros in $\mathrm{C}_+$ expressed by $\{\theta_m, 1 \leqslant m \leqslant N\}$, and $s_{11}$ enjoys $N \geqslant 0$ feasible zeros in $\mathrm{C}_-$ expressed by $\{\widehat{\theta}_m, 1 \leqslant m \leqslant N\}$, one can set that all zeros $\{(\theta_m, \widehat{\theta}_m), m = 1, 2, \cdots, N\}$ are simple zeros of $r_{11}$ and $s_{11}$. In this event, each of Ker $G_+(\theta_m)$ only includes a single column vector $v_m$ and each of Ker $G_-(\widehat{\theta}_m)$ only includes a single row vector $\widehat{v}_m$. That is to say

$$G_+(\theta_m)v_m=0,\ \hat{v}_m G_-(\hat{\theta}_m)=0 \tag{35}$$

Since $G_+(\theta)$ is the solution of Eq. (7a), we suppose that the asymptotic expansion of $G_+(\theta)$ at large $\theta$ is

$$G_+=I+\frac{G_+^{(1)}}{\theta}+O(\theta^{-2}),\ \theta\to\infty \tag{36}$$

substituting this expansion into (7a) and (7b) and comparing $O(1)$ terms yields

$$Q_1=\mathrm{i}[\Lambda,G_+^{(1)}]=\begin{pmatrix}0 & -2\mathrm{i}(G_+^{(1)})_{12} & -2\mathrm{i}(G_+^{(1)})_{13}\\ 2\mathrm{i}(G_+^{(1)})_{21} & 0 & 0\\ 2\mathrm{i}(G_+^{(1)})_{31} & 0 & 0\end{pmatrix} \tag{37}$$

then the potential functions $q_1,q_2$ can be expressed by

$$q_1=2(G_+^{(1)})_{21},\ q_2=2(G_+^{(1)})_{31} \tag{38}$$

where $G_+^{(1)}=(G_+^{(1)})_{3\times3}$ and $(G_+^{(1)})_{ij}$ is the $(i;j)$-entry of $G_+^{(1)}$, $i,j=1,2,3$.

To obtain the spatial evolutions for $v_m(z,t)$, on the one hand, from $G_+v_m=0$ derivativing about $z$ and with the help of Eq. (7a) yields

$$G_+v_{m,z}+\mathrm{i}\theta_m G_+\Lambda v_m=0 \tag{39}$$

thus

$$v_{m,z}=-\mathrm{i}\theta_m\Lambda v_m \tag{40}$$

on the other hand, from $G_+v_m=0$ derivativing about $t$ and with the help of Eq. (7b) yields

$$G_+v_{m,t}+\mathrm{i}\theta_m^{\,2}G_+\Lambda v_m=0 \tag{41}$$

thus

$$v_{m,t}=-\mathrm{i}\theta_m^{\,2}\Lambda v_m \tag{42}$$

solving (40) and (42), obtains

$$\left.\begin{aligned}v_m(z,t)&=\mathrm{e}^{\mathrm{i}\theta_m\Lambda z-\mathrm{i}\theta_m^2\Lambda t}v_{m0},\\ \hat{v}_m(z,t)&=v_m^\dagger\sigma=\hat{v}_{m0}\mathrm{e}^{\mathrm{i}\theta_m^*\Lambda z+\mathrm{i}\theta_m^{*2}\Lambda t}\sigma\end{aligned}\right\} \tag{43}$$

In order to compute multi-soliton solutions for the MCNLS equations (3), one can choose the jump matrix $T=I$, which is a 3×3 unit matrix in (27). In this case, the unique solution to this special RH problem has been solved in Ref. [6], and the result is

$$G_+(\theta)=I-\sum_{m=1}^{N}\sum_{n=1}^{N}\frac{v_m(P^{-1})_{mn}\hat{v}_n}{\theta-\hat{\theta}_m} \tag{44}$$

where matrix $P=(p_{mn})_{N\times N}$ is given as

$$p_{mn}=\frac{\hat{v}_m v_n}{\theta_m^*-\theta_n},\ 1\leqslant m,n\leqslant N \tag{45}$$

Therefore, from (44), we obtain

$$G_+^{(1)}=\sum_{m=1}^{N}\sum_{n=1}^{N}v_m(P^{-1})_{mn}\hat{v}_n \tag{46}$$

we chose $v_{n0}=[1,c_n,d_n]^{\mathrm{T}}$, it follows from (46) that the general $N$-soliton solution for the MCNLS equations (3) is

$$q_1=2\sum_{m=1}^{N}\sum_{n=1}^{N}c_m\mathrm{e}^{\tau_m-\tau_n^*}(P^{-1})_{mn} \tag{47a}$$

$$q_2=2\sum_{m=1}^{N}\sum_{n=1}^{N}d_m\mathrm{e}^{\tau_m-\tau_n^*}(P^{-1})_{mn} \tag{47b}$$

and $P=(p_{mn})_{N\times N}$ is defined as

$$\left.\begin{aligned}p_{mn}&=\frac{\mathrm{e}^{-(\tau_m^*+\tau_n)}-(c_m^*c_n-d_m^*d_n)\mathrm{e}^{\tau_m^*+\tau_n}}{\theta_m^*-\theta_n},\\ &1\leqslant m,n\leqslant N\end{aligned}\right\} \tag{48}$$

with $\tau_n=-\mathrm{i}\theta_n z-\mathrm{i}\theta_n^{\,2}t$.

As a special example, let $N=1$ in (47a) and (47b) and with (45), one can arrives at the one-soliton solution expressed as

$$q_1(z,t)=\frac{2c_1\mathrm{e}^{\tau_1-\tau_1^*}(\theta_1^*-\theta_1)}{\mathrm{e}^{-(\tau_1+\tau_1^*)}-(|c_1|^2-|d_1|^2)\mathrm{e}^{\tau_1+\tau_1^*}} \tag{49a}$$

$$q_2(z,t)=\frac{2d_1\mathrm{e}^{\tau_1-\tau_1^*}(\theta_1^*-\theta_1)}{\mathrm{e}^{-(\tau_1+\tau_1^*)}-(|c_1|^2-|d_1|^2)\mathrm{e}^{\tau_1+\tau_1^*}} \tag{49b}$$

Let $\theta_1=\theta_{11}+\mathrm{i}\theta_{12}$, then the one-soliton solution (49a) and (49b) turn into

$$q_1(z,t)=2\mathrm{i}c_1\theta_{12}\mathrm{e}^{\tau_1-\tau_1^*-\xi_1}\mathrm{csch}(\tau_1+\tau_1^*+\xi_1) \tag{50a}$$

$$q_2(z,t)=2\mathrm{i}d_1\theta_{12}\mathrm{e}^{\tau_1-\tau_1^*-\xi_1}\mathrm{csch}(\tau_1+\tau_1^*+\xi_1) \tag{50b}$$

where

$$\tau_1-\tau_1^*=-2\mathrm{i}\theta_{11}z-2\mathrm{i}(\theta_{11}^{\,2}-\theta_{12}^2)t,$$
$$\tau_1+\tau_1^*=2\mathrm{i}\theta_{12}z+4\theta_{11}\theta_{12}t,$$

and $\xi_1$ admits $|c_1|^2-|d_1|^2=\mathrm{e}^{2\xi_1}$.

As another special example, let $N=2$ in (47a) and (47b) and with (45), one can obtain the two-soliton solution expressed as

$$\begin{aligned}q_1(z,t)=2[&c_1\mathrm{e}^{\tau_1-\tau_1^*}(P^{-1})_{11}+c_1\mathrm{e}^{\tau_1-\tau_2^*}(P^{-1})_{12}+\\ &c_2\mathrm{e}^{\tau_2-\tau_1^*}(P^{-1})_{21}+c_2\mathrm{e}^{\tau_2-\tau_2^*}(P^{-1})_{22}]\end{aligned} \tag{51a}$$

$$\begin{aligned}q_2(z,t)=2[&d_1\mathrm{e}^{\tau_1-\tau_1^*}(P^{-1})_{11}+d_1\mathrm{e}^{\tau_1-\tau_2^*}(P^{-1})_{12}+\\ &d_2\mathrm{e}^{\tau_2-\tau_1^*}(P^{-1})_{21}+d_2\mathrm{e}^{\tau_2-\tau_2^*}(P^{-1})_{22}]\end{aligned} \tag{51b}$$

where $P=(p_{mn})_{2\times2}$ is defined as

$$p_{11}=\frac{-2\mathrm{e}^{\xi_1}}{\theta_1^*-\theta_1}\sinh(\tau_1^*+\tau_1+\xi_1),$$
$$p_{12}=\frac{-2\mathrm{e}^{\xi_2}}{\theta_1^*-\theta_2}\sinh(\tau_1^*+\tau_2+\xi_2),$$
$$p_{21}=\frac{-2\mathrm{e}^{\xi_2^*}}{\theta_2^*-\theta_1}\sinh(\tau_1+\tau_2^*+\xi_2^*),$$
$$p_{22}=\frac{-2\mathrm{e}^{\xi_3}}{\theta_2^*-\theta_2}\sinh(\tau_2^*+\tau_2+\xi_3)$$

and $\xi_j(j=2,3)$ admit

$$\mathrm{e}^{2\xi_2}=c_1^*c_2+d_1^*d_2,\ \mathrm{e}^{2\xi_3}=|c_2|^2+|d_2|^2.$$

## 4 Discussions and conclusions

As an extension, the integrable Manokov system (2) or the MCNLS equation (3) can be extended to the integrable generalized multi-component NLS system as follows:

$$\mathrm{i}q_t + \frac{1}{2}q_{zz} + \epsilon q q^{\dagger}\Omega q = 0, \epsilon = \pm 1 \tag{52}$$

where $q=(q_1, q_2, \cdots, q_N)^{\mathrm{T}}$, $\Omega=\operatorname{diag}(\omega_1, \omega_2, \cdots, \omega_N)$, which enjoy the following Lax pair for $\epsilon=-1$

$$\Phi_z = (-\mathrm{i}\theta\Lambda + \mathrm{i}Q)\Phi \tag{53a}$$

$$\Phi_t = \left(-\mathrm{i}\theta^2\Lambda + \mathrm{i}\theta Q - \frac{1}{2}(\mathrm{i}\Lambda Q^2 - \Lambda Q_z)\right)\Phi \tag{53b}$$

where $\theta \in \mathrm{C}$ is an iso-spectral parameter and

$$\Lambda = \begin{pmatrix} -1 & 0_{1\times N} \\ 0_{N\times 1} & I_{N\times N} \end{pmatrix}, \ Q = \begin{pmatrix} 0 & -q^{\dagger} \\ q & 0_{N\times N} \end{pmatrix} \tag{54}$$

Indeed, if all $\omega_i=1$, which meets with the focusing case, if all $\omega_i=-1$, which meets with the defocusing case, or otherwise the mixed case. Accordingly, one can also examine the $N$-soliton solutions to the integrable generalized multi-component NLS system in the same way in Section 3. But we don't study them here since the procedure is mechanical. However, for other integrable systems, can we seek their multi-soliton solutions in the complex $\theta$-plane according to the RH approach?

Moreover, based on the $3\times 3$ matrix RH problem of the MCNLS equations[25], one can examine the asymptotic properties of the solutions for MCNLS equations through the Deift-Zhou approach[19]. This two questions will be solved in our future paper.

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## Conflict of interest

The authors declare no conflict of interest.

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# 混合耦合非线性 Schrödinger 方程的 Riemann-Hilbert 方法及其孤子解

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**摘要**: 研究了可积混合耦合的非线性 Schrödinger(MCNLS)方程, 该方程可以用来描述双折射光纤中光脉冲的传播. 基于 Riemann-Hilbert(RH)方法, 在构造的矩阵 RH 问题的跳跃矩阵为 3×3 单位矩阵时, 给出了 MCNLS 方程 *N*-孤子解的显式表达式, 作为例子说明, 给出了 1-孤子和 2-孤子的显式表达式. 更一般地, 作为推广,还讨论了可积广义多分量 NLS 系统的线性谱问题.

**关键词**: Lax 对; Riemann-Hilbert 方法; 混合耦合非线性 Schrödinger 方程; 孤子解; 边界条件