第51卷第3期 中国科学技术大学学报 Vol. 51, No. 3
2021年3月 JOURNAL OF UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA Mar. 2021
Received: 2020-07-24; Revised: 2021-03-12 doi:10.52396/JUST-2020-1119

Hyperplane arrangement complement with top degree Betti number being small

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Abstract: The deletion-restriction method was used to classify hyperplane arrangements with the top degree Betti number of its complements being small.

Keywords: hyperplane arrangement; deletion-restriction method; Betti number
CLC number: O189.22 **Document code**: A
2010 Mathematics Subject Classification: 52C35; 55N25; 32S20

1 Introduction

Given a hyperplane arrangement $\mathscr{A} = \{H_0, \cdots, H_d\}$ in $\mathbb{CP}^n$, let $M(\mathscr{A}) = \mathbb{CP}^n \setminus \cup H_i$ be the complement of $\mathscr{A}$ and $b_n(\mathscr{A}) = b_n(M)$ be the n-th Betti number of M. Dimca and Papadima classified the hyperplane arrangements in the cases $b_n(\mathscr{A}) = 1$ or 2 [1]. The purpose of this paper is to extend their results to the cases $b_n(\mathscr{A}) = 3, 4$ or 5 using the deletion-restriction method.

We use $[x_0, \cdots, x_n]$ to denote the coordinates in $\mathbb{CP}^n$. For each hyperplane H_i, let l_i denote its reduced defining equation.

Theorem 1.1 Let $\mathscr{A}$ be a hyperplane arrangement in $\mathbb{CP}^n$ and $b_n(\mathscr{A})$ be its n-th Betti number of its complement.

① $b_n(\mathscr{A}) = 3$ if and only if it is one of the following cases, up to a change of coordinates and reordering of the hyperplanes:

(i) $l_i(x) = x_i$ for $0 \leqslant i \leqslant n$, $l_{n+1}(x) = x_0 + x_1 + x_2$.

(ii) $l_i(x) = x_i$ for $0 \leqslant i \leqslant n$, $l_{n+1}(x) = x_0 + x_1$, $l_{n+2}(x) = a_0 x_0 + x_1$ with $a_0 \neq 0, 1$.

② $b_n(\mathscr{A}) = 4$ if and only if it is one of the following cases, up to a change of coordinates and reordering of the hyperplanes:

(i) $l_i(x) = x_i$ for $0 \leqslant i \leqslant n$, $l_{n+1}(x) = x_0 + x_1 + x_2 + x_3$.

(ii) $l_i(x) = x_i$ for $0 \leqslant i \leqslant n$, $l_{n+1}(x) = x_0 + x_1$, $l_{n+2}(x) = x_0 + x_2$.

(iii) $l_i(x) = x_i$ for $0 \leqslant i \leqslant n$, $l_{n+1}(x) = x_0 + x_1$, $l_{n+2}(x) = x_2 + x_3$.

(iv) $l_i(x) = x_i$ for $0 \leqslant i \leqslant n$, $l_{n+1}(x) = x_0 + x_1$, $l_{n+2}(x) = a_0 x_0 + x_1$ and $l_{n+3}(x) = b_0 x_0 + x_1$ with $a_0, b_0 \neq 0, 1$ and $a_0 \neq b_0$.

③ $b_n(\mathscr{A}) = 5$ if and only if it is one of the following cases, up to a change of coordinates and reordering of the hyperplanes:

(i) $l_i(x) = x_i$ for $0 \leqslant i \leqslant n$, $l_{n+1}(x) = x_0 + x_1 + x_2 + x_3 + x_4$.

(ii) $l_i(x) = x_i$ for $0 \leqslant i \leqslant n$, $l_{n+1}(x) = x_0 + x_1 + x_2$, $l_{n+2}(x) = a_0 x_0 + x_1 + x_2$ with $a_0 \neq 0, 1$.

(iii) $l_i(x) = x_i$ for $0 \leqslant i \leqslant n$,
$l_{n+1}(x) = x_0 + x_1$, $l_{n+2}(x) = a_0 x_0 + x_1$,
$l_{n+3}(x) = b_0 x_0 + x_1$, $l_{n+4}(x) = c_0 x_0 + x_1$.
Here $a_0, b_0, c_0 \neq 0, 1$ and a_0, b_0 and c_0 are all different numbers.

2 Preliminaries

We introduce some definitions and notations as in Ref. [2]. For $\mathscr{A}$ a non-empty hyperplane arrangement in $\mathbb{CP}^n$, we fix $H_0 \in \mathscr{A}$ as the hyperplane at infinity. Then we can define an affine arrangement $(\mathscr{A}, H_0)^a$, where the total space of $(\mathscr{A}, H_0)^a$ is $\mathbb{C}^n = \mathbb{CP}^n \setminus H_0$ and the hyperplanes of $(\mathscr{A}, H_0)^a$ are $\{H_i \cap (\mathbb{CP}^n \setminus H_0) \mid H_i \neq H_0\}$. Note that the hyperplane arrangements $\mathscr{A}$ and $(\mathscr{A}, H_0)^a$ have the same complement space. It is more convenient to use the affine arrangement $(\mathscr{A}, H_0)^a$ to compute the Betti number $b_n(\mathscr{A})$. In the rest of the paper, we always assume the infinity hyperplane H_0 is defined by $x_0 = 0$ and abuse our language to identify the hyperplane arrangements $\mathscr{A}$ with $(\mathscr{A}, H_0)^a$. When we say two hyperplanes are parallel to each other, we mean

Citation: LI Fenglin. Hyperplane arrangement complement with top degree Betti number being small. J. Univ. Sci. Tech. China, 2021, 51(3): 193-195,258.

that they are parallel in the affine space $\mathbb{C}^n = \mathbb{CP}^n \backslash H_0$.

For any hyperplane in $\mathscr{A}$, say H_1, we can use it to define two hyperplane arrangements $\mathscr{A}' := \mathscr{A} \backslash \{H_1\}$ and $\mathscr{A}'' := \{H \cap H_1 \neq \varnothing \mid H \neq H_1\}$ in H_1, which are called deleted arrangement and restricted arrangement respectively. Then $(\mathscr{A}, \mathscr{A}', \mathscr{A}'')$ is called a triple of arrangements with respect to the distinguished hyperplane H_1. Note that such triples $(\mathscr{A}, \mathscr{A}', \mathscr{A}'')$ are very useful in proofs by induction, both arrangements $\mathscr{A}'$ and $\mathscr{A}''$ have less hyperplanes than $\mathscr{A}$. This method is so called the deletion-restriction method[3].

Dimca and Papadima classified hyperplane arrangements $\mathscr{A}$ when $b_n(\mathscr{A}) = 1$ or 2. We recall the related results here.

Lemma 2.1[1, Corollory 4] With the above notations and assumptions, we have the following:

① $b_n(\mathscr{A}) = b_n(\mathscr{A}') + b_{n-1}(\mathscr{A}'')$.

② $b_n(\mathscr{A}) > 0$ if and only if $\mathscr{A}$ is essential.

③ If $b_n(\mathscr{A}) > 0$, then $d \leqslant n + b_n(\mathscr{A}) - 1$.

Lemma 2.2[2, Proposition 1.1] If $\mathscr{A}$ is an arrangement in $\mathbb{C}^2$, then $b_2(\mathscr{A}) = \sum_{k \geqslant 2} n_k(k-1)$, where n_k denotes the number of k-fold intersection points.

Theorem 2.1[1, Corollory 4] Let $\mathscr{A}$ be a hyperplane arrangement in $\mathbb{CP}^n$.

① $b_n(\mathscr{A}) = 1$ if and only if $d = n$ and up to a linear coordinate change we have $l_i(x) = x_i$ for all $0 \leqslant i \leqslant n$.

② $b_n(\mathscr{A}) = 2$ if and only if $d = n+1$ and up to a linear coordinate change and reordering of the hyperplanes we have $l_i(x) = x_i$ for $0 \leqslant i \leqslant n$ and $l_{n+1}(x) = x_0 + x_1$.

3 Proof of Theorem 1.1

By Lemma 2.1③, we know that $d \leqslant n + b_n(\mathscr{A}) - 1$. On the other hand, $b_n(\mathscr{A}) > 0$ implies rank $(\mathscr{A}) = n$, so $d \geqslant n$. To classify the hyperplane arrangements when $b_n(\mathscr{A}) = 3, 4$ or 5, we analysis all the hyperplane arrangements with $n \leqslant d \leqslant n+4$.

3.1 Two formulas of top degree Betti number

In this subsection, we compute the top degree Betti number of hyperplane arrangement complement for $d = n, n+1, n+2$.

① If $d = n$ and rank $(\mathscr{A}) = n$, then $b_n(\mathscr{A}) = 1$. Since $b_n(\mathscr{A}) = 1$ implies $d \leqslant n$, the converse statement is also true. Hence up to a coordinate change, there is only one case when $b_n(\mathscr{A}) = 1$:

$$d = n \text{ and } l_i(x) = x_i \text{ for all } 0 \leqslant i \leqslant n.$$

② When $d = n+1$, after a change of coordinate we may assume $l_i(x) = x_i$ for $0 \leqslant i \leqslant n$ and $l_{n+1}(x) = a_0 x_0 + \cdots + a_n x_n$.

Theorem 3.1 Let $\mathscr{A}$ be a hyperplane arrangement as above, then $b_n(\mathscr{A}) = \#\{a_i \mid a_i \neq 0\}$.

Proof After a coordinate change, we may write $l_{n+1}(x) = a_0 x_0 + \cdots + a_m x_m$ with $a_i \neq 0$ for $0 \leqslant i \leqslant m$ and $a_i = 0$ for $i > m$. It is easy to see $M(\mathscr{A}) = M(\mathscr{A}_m) \times (\mathbb{C}^*)^{n-m}$, where

$$\mathscr{A}_m = \{ l_i(x) = x_i \text{ for } 0 \leqslant i \leqslant m, \; l_{m+1}(x) = a_0 x_0 + \cdots + a_m x_m \}.$$

Hence $b_n(\mathscr{A}) = b_m(\mathscr{A}_m)$ by Künneth formula, so we only need to compute $b_m(\mathscr{A}_m)$. Using hyperplane $\{x_m = 0\}$ to produce a triple of arrangement, we get $b_m(\mathscr{A}_m) = 1 + b_{m-1}(\mathscr{A}_{m-1})$, where

$$\mathscr{A}_{m-1} = \{ l_i(x) = x_i \text{ for } 0 \leqslant i \leqslant m-1, \; l_m(x) = a_0 x_0 + \cdots + a_{m-1} x_{m-1} \}.$$

Clearly $\mathscr{A}_{m-1}$ has the same type of $\mathscr{A}_m$, so we can continue the induction. Note that $b_2(\mathscr{A}_2) = 3$ by Lemma 2.2, hence $b_m(\mathscr{A}_m) = m+1$.

③ For a hyperplane arrangement with $d = n+2$ and $b_n(\mathscr{A}) > 0$, up to a change of coordinates we assume

$$\mathscr{A} = \{ l_i(x) = x_i \text{ for } 0 \leqslant i \leqslant n, \; l_{n+1}(x) = a_0 x_0 + \cdots + a_n x_n, \; l_{n+2}(x) = b_0 x_0 + \cdots + b_n x_n \}.$$

It is easy to see after another coordinate change, we may write $\begin{pmatrix} a_0 \cdots a_n \\ b_0 \cdots b_n \end{pmatrix}$ as

$$\begin{array}{ccccccc} r_1 & r_2 & \cdots & r_v & s & t & w \\ \end{array}$$
$$\begin{pmatrix} 1 \cdots 1 & 1 \cdots 1 & \cdots & 1 \cdots 1 & 1 \cdots 1 & 0 \cdots 0 & 0 \cdots 0 \\ k_1 \cdots k_1 & k_2 \cdots k_2 & \cdots & k_v \cdots k_v & 0 \cdots 0 & 1 \cdots 1 & 0 \cdots 0 \end{pmatrix},$$

where all k_i are different nonzero numbers and $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_v$. Set $u = r_1 + r_2 + \cdots + r_v$.

Theorem 3.2 For the above hyperplane arrangement, we have

$$b_n(\mathscr{A}) = \sum_{1 \leqslant i < j \leqslant v} r_i r_j + u + st + tu + su.$$

By convention, if $v = 0$ or 1, then $\sum_{1 \leqslant i < j \leqslant v} r_i r_j$ is 0.

Proof We first introduce some notations. For $m \leqslant n$, set

$$\mathscr{A}_m = \{ l_i(x) = x_i \text{ for } 1 \leqslant i \leqslant m, \; l_{m+1}(x) = a_0 + a_1 x_1 + \cdots + a_m x_m, \; l_{m+2}(x) = b_0 + b_1 x_1 + \cdots + b_m x_m \},$$

where $a_0, \cdots, a_n, b_0, \cdots, b_n$ are exactly the fixed elements in the above matrix.

For hyperplane arrangement $\mathscr{A}_m$, we always use hyperplane $\{x_m = 0\}$ to produce triple of arrangement. For $\mathscr{A}_m{}'$, if $b_m \neq 0$, then we can find a coordinate change B_m such that $x_0, \cdots, x_{m-1}$ being fixed and $l_{m+2}(x)$ becoming x_m. If $a_m \neq 0$, let A_m denote the similar coordinate change for $l_{n+1}(x)$.

Since $M(\mathscr{A}) = M(\mathscr{A}_{u+s+t-1}) \times (\mathbb{C}^*)^w$, by Künneth formula we get $b_n(\mathscr{A}) = b_{u+s+t-1}(\mathscr{A}_{u+s+t-1})$. Hence we can eliminate the last w coordinates.

For $\mathscr{A}'_{u+s+t-1}$, using coordinate change $B_{u+s+t-1}$ does not change the equation of $l_{u+s+t}(x)$. Note that $l_{u+s+t}(x)$ has $s+u$ coefficients being non-zero. Then by Theorem 3.1 we get $b_{u+s+t-1}(\mathscr{A}'_{u+s+t-1}) = s+u$. Since $\mathscr{A}''_{u+s+t-1} = \mathscr{A}_{u+s+t-2}$, by induction we have

$$b_n(\mathscr{A}) = (s+u)t + b_{s+u-1}(\mathscr{A}_{s+u-1}).$$

Using a similar argument for s we get

$$b_n(\mathscr{A}) = st + tu + su + b_{u-1}(\mathscr{A}_{u-1}).$$

For $\mathscr{A}'_{u-1}$, after using A_{u-1} the new equation of $l_{u+1}(x)$ is

$$(k_1 - k_v) + (k_1 - k_v)x_1 + \cdots + (k_{v-1} - k_v)x_{r_1+\cdots+r_{v-1}-1} + k_v x_{u-1}.$$

By Theorem 3.1 we know $b_{u-1}(\mathscr{A}'_{u-1}) = r_1 + \cdots + r_{v-1} + 1$. We can repeat this argument and use induction to compute $b_{u-2}(\mathscr{A}''_{u-1})$. Note that when we eliminate the last $r_2 + \cdots + r_v$ coordinates, $\{1 + x_1 + \cdots + x_{r_1-1} = 0\}$ and $\{k_1 + k_1x_1 + \cdots + k_1x_{r_1-1} = 0\}$ give the same hyperplane. By Theorem 3.1, the corresponding (r_1-1)-th Betti number is r_1. Hence we have

$$b_{u-1}(\mathscr{A}_{u-1}) = (r_1 + \cdots + r_{v-1} + 1)r_v + \cdots + (r_1+1)r_2 + r_1 = \sum_{1\leqslant i<j\leqslant v} r_ir_j + \sum_{i=1}^{v} r_i = \sum_{1\leqslant i<j\leqslant v} r_ir_j + u.$$

This completes our proof.

3.2 Classification of hyperplane arrangements when $b_n(\mathscr{A}) \leqslant 5$

In this subsection, we use Theorems 3.1 and 3.2 to prove Theorem 1.1. In the rest of paper, we always assume $l_i(x) = x_i$ for $0 \leqslant i \leqslant n$.

When $d = n$ or $n+1$, the conclusion is clear.

When $d = n+2$ we use Theorem 3.2 to find out all hyperplane arrangements with $b_n(\mathscr{A}) \leqslant 5$. We use the same notations as in Theorem 3.2. Note that $u \geqslant v$. If $v \geqslant 3$, then $b_n(\mathscr{A}) \geqslant 6$. So we only need to analysis hyperplane arrangements with $v \leqslant 2$.

① The case $v = 2$.

If $r_1 \geqslant 2$, then there is only one case satisfying $b_n(\mathscr{A}) \leqslant 5$:

(ⅰ) $b_n(\mathscr{A}) = 5$, $l_{n+1}(x) = x_0 + x_1 + x_2$ and $l_{n+2}(x) = a_0x_0 + x_1 + x_2$, $a_0 \neq 0,1$.

If $r_1 = r_2 = 1$ and $s + t \geqslant 2$, then $b_n(\mathscr{A}) \geqslant 7$. Hence there are only two cases under this condition:

(ⅰ′) $b_n(\mathscr{A}) = 5$, $l_{n+1}(x) = x_0 + x_1 + x_2$ and $l_{n+2}(x) = a_0x_0 + x_1, a_0 \neq 0,1$, which is equivalent to the case (ⅰ) after a coordinate change.

(ⅱ) $b_n(\mathscr{A}) = 3$, $l_{n+1}(x) = x_0 + x_1$ and $l_{n+2}(x) = a_0x_0 + x_1, a_0 \neq 0,1$.

② The case $v = 1$.

If $u \geqslant 3$, then $b_n(\mathscr{A}) \geqslant 6$. If $u = 2$ and $s+t \geqslant 2$, then we also have $b_n(\mathscr{A}) \geqslant 6$.

Since hyperplanes $l_{n+1}(x)$ and $l_{n+2}(x)$ are different, under the conditions $v = 1$ and $u = 2$ there is only one case:

(ⅲ) $b_n(\mathscr{A}) = 4$, $l_{n+1}(x) = x_0 + x_1 + x_2$ and $l_{n+2}(x) = x_0 + x_1$.

Since hyperplanes $l_{n+1}(x)$ and $l_{n+2}(x)$ are not coordinate hyperplanes, then we have $s \geqslant 1$ and $t \geqslant 1$ if $u = 1$. If $s \geqslant 2$ or $t \geqslant 2$, then $b_n(\mathscr{A}) \geqslant 6$. Hence we must have $s = 1 = t$:

(ⅲ′) $b_n(\mathscr{A}) = 4$, $l_{n+1}(x) = x_0 + x_1$ and $l_{n+2}(x) = x_0 + x_2$. It easy to see after a change of coordinate this is equivalent to (ⅲ).

③ The case $v = 0$.

When $v = 0$ we must have $s = 2 = t$:

(ⅳ) $b_n(\mathscr{A}) = 4$, $l_{n+1}(x) = x_0 + x_1$ and $l_{n+2}(x) = x_2 + x_3$.

Note that in Case (ⅲ) there exists a point which has exactly n hyperplanes passing through it, but in this case there is no such points. Hence Cases (ⅲ) and (ⅳ) are not projective equivalent.

We now analyze hyperplane arrangements with $d = n+3$. To have $b_n(\mathscr{A}) \leqslant 5$, the deleted arrangement $\mathscr{A}'$ with respect to $l_{n+3}(x)$ is a hyperplane arrangement with $n+2$ hyperplanes. Note that $b_n(\mathscr{A}') < 5$ by Lemma 2.1①. Hence there are only three possible cases for $\mathscr{A}'$ by the above discussion. For Case (iii) or (iv), we can use the hyperplane $l_{n+3}(x)$ to produce a triple of arrangement. The coordinate hyperplanes restricting to $l_{n+3}(x)$ contributes at least $n-1$ hyperplanes. If this number is n, then by Theorem 3.1 we have $b_n(\mathscr{A}) \geqslant 6$. If this number is $n-1$, then $l_{n+3}(x)$ must have the form $ax_i + x_j$. But in this case $l_{n+1}(x)$ and $l_{n+2}(x)$ contributes at least another hyperplane for the restricted arrangement, so we always have $b_n(\mathscr{A}) \geqslant 6$. For Case (ii), if $l_{n+3}(x)$ is not parallel to $\{x_1 = 0\}$, then up to a coordinate change we can reduce it to the above discussion. Hence the only possibility is Case (ii) with $l_{n+3}(x)$ being parallel to $\{x_1 = 0\}$:

(ⅴ) $b_n(\mathscr{A}) = 4$ and $d = n+3$, $l_{n+1}(x) = x_0 + x_1$, $l_{n+2}(x) = a_0x_0 + x_1$, $l_{n+3}(x) = b_0x_0 + x_1, a_0, b_0 \neq 0,1$, $a_0 \neq b_0$.

For the hyperplane arrangements with $d = n+4$ and $b_n(\mathscr{A}) \leqslant 5$, using a similar argument as above discussion we get that there is only one possible case:

(ⅵ) $b_n(\mathscr{A}) = 5$ and $d = n+4$,

$$l_{n+1}(x) = x_0 + x_1,\ l_{n+2}(x) = a_0x_0 + x_1,$$
$$l_{n+3}(x) = b_0x_0 + x_1,\ l_{n+4}(x) = c_0x_0 + x_1.$$

Here $a_0, b_0, c_0 \neq 0,1$ and a_0, b_0 and c_0 are all different numbers.

Putting all these results together, Theorem 1.1 follows.

Conflict of interest

The author declares no conflict of interest.

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一种预测高频价格的端到端双目标多任务方法

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摘要:高频价格变动预测是预测价格在短时间内(比如 1 min 内)的变化方向(上涨、不变或下跌). 用历史的高频交易数据去预测价格变化是一个比较困难的任务,这是因为二者之间的关系是高噪声、非线性和复杂的. 为提高高频价格预测准确率,提出了一个端到端的双目标多任务方法. 该方法引进了一个辅助目标(高频价格变化率),它和主目标(高频价格变化方向)是高度相关的并且能够提高主目标的预测准确率. 此外,每一个任务都有一个基于循环神经网络和卷积神经网络的特征提取模块,它可以学习出历史交易数据和两个目标之间的高噪声、非线性和复杂的时空相依关系. 为了缓解多任务方法的潜在的负迁移问题,每个任务的任务间共享部分和任务特有部分被显式地分开. 而且,通过一种梯度平衡方法利用两个目标之间的高相关性过滤掉从不一致性中学到的噪声的同时保留从一致性中学到的相依规律,从而提高高频价格变化方向预测准确率. 在真实数据集上的实验结果表明:所提方法能够利用高度相关的辅助目标帮助主任务的特征提取模块去学习出更有泛化能力的时空相依规律,最终提高高频价格变化方向预测准确率. 此外,辅助目标(高频价格变化率)不仅能够提高特征提取模块的总体效果,而且也提高特征提取模块的不同部分的效果.

关键词:多任务学习;细粒度辅助目标;特征提取;共享方法;负迁移;高频价格动态预测

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具有较小的最高阶 Betti 数的超平面配置补空间

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摘要:使用删除限制方法对补空间最高阶 Betti 数较小的超平面配置进行了分类.

关键词:超平面配置;删除限制方法;Betti 数