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Global dynamics of an SEIQR model with saturation incidence rate and hybrid strategies

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Abstract: An SEIQR epidemic model with the saturation incidence rate and hybrid strategies was proposed, and the stability of the model was analyzed theoretically and numerically. Firstly, the basic reproduction number R_0 was derived, which determines whether the disease was extinct or not. Secondly, through LaSalle's invariance principle, it was proved that the disease-free equilibrium is globally asymptotically stable and the disease generally dies out when $R_0<1$. By Routh-Hurwitz criterion theory, it was proved that the disease-free equilibrium is unstable and the unique endemic equilibrium is locally asymptotically stable when $R_0>1$. Thirdly, according to the periodic orbit stability theory and the second additive compound matrix, it was proved that the unique endemic equilibrium is globally asymptotically stable and the disease persists at this endemic equilibrium if it initially exists when $R_0>1$. Finally, some numerical simulations were carried out to illustrate the results.

Keywords: basic reproductive number; equilibrium; stability; saturation incidence rate
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1 Introduction

Epidemiology is the study of hot spots of the spread of infectious disease, with the objective to trace factors that contribute to their occurrence. For a long time, mathematical models describing the population dynamics of infectious diseases have been playing an important role in a better understanding of the disease control and epidemic patterns. In order to predict the spread of infectious diseases among the areas, the transmission dynamics of infectious diseases is studied by many epidemic models in host populations. However, many infectious diseases, such as pertussis, diphtheria, SARS, viral hepatitis and so on, incubate inside the population for a period of time before becoming infectious. Therefore, the systems that are more general than SIR or SIRS types need to study the role of incubation in the spread of infectious diseases. It may be assumed that a susceptible individual first goes through a latent period before becoming infectious. The present model is of SEIR or SEIRS class, depending on whether the adaptive immunity is permanent or otherwise[1-4].

Incidence rate plays a very important role in the research of epidemic models. Incidence rate should be written as $C(N)\frac{S}{N}I$, where N is the total population size. In many epidemic models, the bilinear incidence rate βSI and the standard incidence rate $\beta\frac{S}{N}I$ are frequently used. The bilinear incidence rate is based on the law of mass actions. The incidence rate is proportional to the total population in the environment for the contact rate, that is, it is appropriate when the population is small. However, when the population is large, the standard incidence rate is usually used. The standard incidence rate assumes that the contact rate is a constant. This assumption is in line with reality, because the greater the number of susceptible people, the greater the chance of contact between the infected and susceptible, and the greater the infectious power. That is, it is more suitable when the population is large. But under certain circumstances, such assumptions are also unreasonable. For example, in the actual infection process, no matter how many people are susceptible, the contact rate of a patient with others is also limited per unit time. Therefore, it has been proposed that for the saturation incidence rate, which should generally be

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written as $\frac{aSI}{1+bI}$, where aI measures the infection force of the disease and $\frac{1}{1+bI}$ measures the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the infective individuals. This saturation rate can not only reflect the trend of strengthening control measures as the number of infected people increases, but also reflect the changes in the alertness and self-protection of susceptible people as the number of the infected increases. This incidence can also avoid the unboundedness of an infected person producing new infections per unit time. Therefore, compared with the bilinear incidence rate and the standard incidence rate, saturation incidence rate may be more suitable for our real world. Epidemic models with the saturation incidence rate have been studied by several researchers[5-7].

Infectious diseases have a tremendous influence on human life. Every year, millions of people died of various infectious diseases. In recent years, the control of infectious diseases has become an increasingly complex issue in every country. Three effective strategies of quarantine, vaccination and elimination are usually used to control and prevent the spread of infectious diseases. Quarantine is a common control measure to reduce the transmission of human diseases such as leprosy, plague, cholera, etc. The strategy can also be used to tackle animal diseases, such as rinderpest, foot and mouth disease, psittacosis and so on. It is a very meaningful job to study infectious disease models with quarantine[8-11]. Vaccination is considered to be the most successful intervention policy as well as a cost-effective strategy to reduce the morbidity and mortality of individuals. It has been used to control diseases, such as measles, rubella, diphtheria, influenza, etc. Recently, many researchers have paid great attention to research on infectious models with vaccination strategies[12-16]. Elimination is an important measure to remove the infectious source by sacrificing the discovered infected individuals. This measure has been used to address diseases derived from animals or are spread in animals, such as avian influenza, tuberculosis, tetanus and rotavirus infection and so on. Therefore, some works have studied the infectious disease models that involved elimination strategy[17-18]. However, these models only study a single prevention and control strategy and do not discuss the hybrid case of these strategies.

In this paper, motivated by the work of Refs. [8-18], we are concerned with the combined effects of a saturation incidence rate and quarantine, vaccination and elimination hybrid strategies on the dynamics of infectious disease transmission. To this end, we establish an SEIQR model with the saturation incidence rate and hybrid strategies. We study the stability of the model by means of both theoretical and numerical ways.

2 Model formulation

We assume that the total population is divided into five distinct epidemiological subclasses of individuals which are susceptible, latent, infectious, quarantine, and recovered (removed) with sizes denoted by $S(t)$, $E(t)$, $I(t)$, $Q(t)$ and $R(t)$, respectively. The total population size at time t is denoted by $N(t)$, with $N(t) = S(t) + E(t) + I(t) + Q(t) + R(t)$. We establish the following SEIQR epidemic model of ordinary differential equations:

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \frac{aSI}{1+bI} - (d+p)S, \\ \frac{dE(t)}{dt} = \frac{aSI}{1+bI} - (d+\alpha_1+k_1+\varepsilon)E, \\ \frac{dI(t)}{dt} = \varepsilon E - (d+\alpha_2+k_2+k+\gamma)I, \\ \frac{dQ(t)}{dt} = kI - (d+\alpha_3+\omega)Q, \\ \frac{dR(t)}{dt} = pS + \gamma I + \omega Q - dR. \end{cases} \quad (1)$$

Here Λ is the recruitment rate of the population, $\frac{aSI}{1+bI}$ is the saturation incidence rate of disease, d is the natural death rate of the population, α_1 is the disease-related death rate of the latent class, α_2 is the disease-related death rate of the infectious class, α_3 is the disease-related death rate of the quarantine class, k_1 is the elimination rate of the latent class, k_2 is the elimination rate of the infective class, k is the quarantine rate of the infective class, γ is the natural recovery rate of the infective class, ω is the natural recovery rate of the quarantine class, p is the vaccination rate of the susceptible class, ε is the removed rate from the latent class to the infectious class. Assume that Λ, a, b, d, α_1, α_2, α_3, k_1, k_2, k, γ, ω, p and ε are normal.

Summing up the five equations of system (1) and having

$$N'(t) = \Lambda - dN - (\alpha_1+k_1)E - (\alpha_2+k_2)I - \alpha_3 Q \leqslant \Lambda - dN.$$

By solving the formula of $N'(t)$, we obtain

$$N(t) \leqslant N(0)e^{-dt} + \frac{\Lambda}{d}(1-e^{-dt}),$$

Thus

$$\limsup_{t\to+\infty}(N(t)) = \frac{\Lambda}{d}.$$

From biological considerations, we study system (1) in the following feasible region

$$D = \{(S,E,I,Q,R) \in R_+^5 \mid S \geqslant 0, E \geqslant 0, I \geqslant 0,$$

$$Q \geqslant 0, R \geqslant 0, S + E + I + Q + R \leqslant \frac{\Lambda}{d}\},$$

where R_+^5 denotes the non-negative cone and its lower dimensional faces, D can be shown to be positively invariant with respect to system (1).

3 Stability analysis of the disease-free equilibrium

Set the right sides of system (1) equal zero, that is,

$$\begin{cases} \Lambda - \dfrac{aSI}{1+bI} - (d+p)S = 0, \\ \dfrac{aSI}{1+bI} - (d+\alpha_1+k_1+\varepsilon)E = 0, \\ \varepsilon E - (d+\alpha_2+k_2+k+\gamma)I = 0, \\ kI - (d+\alpha_3+\omega)Q = 0, \\ pS + \gamma I + \omega Q - dR = 0 \end{cases} \tag{2}$$

By calculating Equation(2), system (1) always has a disease-free equilibrium $P_0\left(\dfrac{\Lambda}{d+p}, 0, 0, 0, \dfrac{\Lambda p}{d(d+p)}\right)$. Further, if $\varepsilon a\Lambda > (d+p)(d+\alpha_1+k_1+\varepsilon)(d+\alpha_2+k_2+k+\gamma)$, system (1) admits a unique endemic equilibrium $P^*(S^*, E^*, I^*, Q^*, R^*)$, where

$$I^* = \left(b + \frac{a}{d+p}\right)^{-1}\left(\frac{\varepsilon a\Lambda}{(d+p)(\mu+\alpha_1+k_1+\varepsilon)(\mu+\alpha_2+k_2+k+\gamma)} - 1\right),$$

$$S^* = \frac{\Lambda(1+bI^*)}{d+p+(a+db+pb)I^*},\ E^* = \frac{d+\alpha_2+k_2+k+\gamma}{\varepsilon}I^*,$$

$$Q^* = \frac{\delta}{d+\alpha_3+\omega}I^*,\ R^* = \frac{pS^*+\gamma I^*+\omega Q^*}{d}.$$

Define

$$R_0 = \frac{\varepsilon a\Lambda}{(d+p)(d+\alpha_1+k_1+\varepsilon)(d+\alpha_2+k_2+k+\gamma)}.$$

The R_0 is called the basic reproduction number of system (1). It is easy to obtain the following theorem.

Theorem 3.1 For system (1), there is always a disease-free equilibrium P_0, and there is also a unique endemic equilibrium P^* when $R_0 > 1$.

Theorem 3.2 If $R_0 < 1$, the disease-free equilibrium P_0 of system (1) is locally asymptotically stable. If $R_0 > 1$, the disease-free equilibrium P_0 is unstable.

Proof The Jacobian matrix of system (1) at the disease-free equilibrium $P_0\left(\dfrac{\Lambda}{d+p}, 0, 0, 0, \dfrac{\Lambda p}{d(d+p)}\right)$ is

$$J(P_0) = \begin{pmatrix} -(d+p) & 0 & -\dfrac{a\Lambda}{d+p} & 0 & 0 \\ 0 & -(d+\alpha_1+k_1+\varepsilon) & \dfrac{a\Lambda}{d+p} & 0 & 0 \\ 0 & \varepsilon & -(d+\alpha_2+k_2+k+\gamma) & 0 & 0 \\ 0 & 0 & k & -(d+\alpha_3+\omega) & 0 \\ p & 0 & \gamma & \omega & -d \end{pmatrix}.$$

The three eigenvalues of matrix $J(P_0)$ are

$$\lambda_1 = -d,\ \lambda_2 = -(d+p),\ \lambda_3 = -(d+\alpha_3+\omega).$$

The other two eigenvalues are also the roots of the following equation:

$$\lambda^2 + a_1\lambda + a_2 = 0,$$

where

$$a_1 = (d+\alpha_1+k_1+\varepsilon) + (d+\alpha_2+k_2+k+\gamma),$$
$$a_2 = (d+\alpha_1+k_1+\varepsilon)(a+\alpha_2+k_2+k+\gamma)(1-R_0).$$

Obviously, if $R_0 < 1$, we have the relation $a_2 > 0$. Therefore, all eigenvalues of matrix $J(P_0)$ have negative real parts. Hence, the disease-free equilibrium P_0 is locally asymptotically stable. If $R_0 > 1$, we get the relation $a_2 < 0$. Therefore, the matrix $J(P_0)$ has at least an eigenvalue with positive real part. Thus, the disease-free equilibrium P_0 is

unstable.

Theorem 3.3 If $R_0 < 1$, the disease-free equilibrium P_0 of system (1) is globally asymptotically stable.

Proof Consider the following Lyapunov function:

$$V(t) = E(t) + \frac{d+\alpha_1+k_1+\varepsilon}{\varepsilon} I(t).$$

Calculating the derivative of $V(t)$ along the positive solution of system (1), it follows that

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1)} = \frac{aSI}{1+bI} - \frac{(d+\alpha_1+k_1+\varepsilon)(d+\alpha_2+k_2+k+\gamma)}{\varepsilon} I \leqslant$$

$$\left(\frac{a\Lambda}{d+p} - \frac{(d+\alpha_1+k_1+\varepsilon)(d+\alpha_2+k_2+k+\gamma)}{\varepsilon}\right) I - \frac{a(E+I+Q+R)I}{1+bI} =$$

$$\frac{(d+\alpha_1+k_1+\varepsilon)(d+\alpha_2+k_2+k+\gamma)}{\varepsilon}(R_0-1)I - \frac{a(E+I+Q+R)I}{1+bI} \leqslant 0.$$

Furthermore, $V(t) = 0$ only if $I(t) = 0$. The maximum invariant set in $\{(S,E,I,Q,R) \mid I(t) = 0\}$ is the singleton P_0. The global asymptotical stability of the disease-free equilibrium P_0 follows from LaSalle's invariance when $R_0 < 1$, that is, the disease-free equilibrium P_0 of system (1) is globally asymptotically stable.

4 Local stability analysis of the endemic equilibrium

In this section, we study the local stability of the endemic equilibrium $P^*(S^*,E^*,I^*,Q^*,R^*)$ of system (1) by Routh-Hurwitz criterion theory.

Theorem 4.1 If $R_0 > 1$, the endemic equilibrium P^* of system (1) is locally asymptotically stable.

Proof The Jacobian matrix of system (1) at the endemic equilibrium P^* is

$$J(P^*) = \begin{pmatrix} -(d+p) - \dfrac{aI^*}{1+bI^*} & 0 & -\dfrac{aS^*}{(1+bI^*)^2} & 0 & 0 \\ \dfrac{aI^*}{1+bI^*} & -m & \dfrac{aS^*}{(1+bI^*)^2} & 0 & 0 \\ 0 & \varepsilon & -n & 0 & 0 \\ 0 & 0 & k & -(d+\alpha_3+\omega) & 0 \\ p & 0 & \gamma & \omega & -d \end{pmatrix},$$

where

$$m = d+\alpha_1+k_1+\varepsilon,\ n = d+\alpha_2+k_2+k+\gamma.$$

The two eigenvalues of matrix $J(P^*)$ are

$$\lambda_1 = -d < 0,\ \lambda_2 = -(d+\alpha_3+\omega) < 0.$$

The other three eigenvalues are also the roots of the following equation:

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0,$$

where

$$b_1 = m+n+d+p+\frac{aI^*}{1+bI^*} > 0,$$

$$b_2 = \left(d+p+\frac{aI^*}{1+bI^*}\right)(m+n) + mn\left(1-\frac{1}{1+bI^*}\right) > 0,$$

$$b_3 = (d+p)mn\left(1-\frac{1}{1+bI^*}\right) + \frac{aI^*}{1+bI^*}mn > 0.$$

By calculation, we have

$$b_1b_2 - b_3 = mn\left(1-\frac{1}{1+bI^*}\right)\left(m+n+\frac{aI^*}{1+bI^*}\right) + (m^2+n^2+mn)\left(d+p+\frac{aI^*}{1+bI^*}\right) + (d+p)mn + (m+n)\cdot$$

$$\left(d+p+\frac{aI^*}{1+bI^*}\right)\left(2d+2p+\frac{aI^*}{1+bI^*}\right) > 0.$$

Therefore, all the five eigenvalues have negative real parts. According to the Routh-Hurwitz criterion theory, the endemic equilibrium P^* of system (1) is locally asymptotically stable in D when $R_0 > 1$.

5 The permanence of model and global stability of the endemic equilibrium

In this section, we study the global stability of the endemic equilibrium P^* of system (1) by means of the periodic orbit stability theory and second additive compound matrix.

Since the first three equations of system (1) do not contain Q and R, using the theory of limit differential equations, system (1) is reduced to the following three-dimensional system:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - \dfrac{aSI}{1+bI} - (d+p)S, \\ \dfrac{\mathrm{d}E(t)}{\mathrm{d}t} = \dfrac{aSI}{1+bI} - (d+\alpha_1+k_1+\varepsilon)E, \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \varepsilon E - (d+\alpha_2+k_2+k+\gamma)I \end{cases} \quad (3)$$

Summing up the three equations of system (3) and denoting

$$M = M(t) = S(t) + E(t) + I(t),$$

having

$$M'(t) = \Lambda - dM - pS - (\alpha_1 + k_1)E - (\alpha_2 + k_2 + k + \gamma)I \leqslant \Lambda - dM.$$

From the formula of $M'(t)$, we obtain that the feasible region $T = \{(S,E,I) \in R_+^3 \mid 0 \leqslant S + E + I \leqslant \frac{\Lambda}{d}\}$ is a positive invariant set with respect to system (3).

Lemma 5.1[19,20] Let X be the distance space and $X_1 \subset X$ be the closed positive invariant set of the continuous half-flow Φ, and there is $\alpha > 0$, so that the half-flow Φ is a bit dissipative on $\{X: x \in X, d(x,\partial X_1) \leqslant \alpha\} \cap \overset{0}{X}_1$. Suppose:

① N is the largest closed invariant subset of the half-flow Φ on ∂X_1;

② $\{N_\alpha\}_{\alpha \in A}$ is the non-cyclic coverage of N;

③ $N \subset \partial X_1$ is the union of some isolated closed invariant sets;

④ any compact subset of ∂X_1 contains a finite number of sets of $\{N_\alpha\}_{\alpha \in A}$ at most.

Then, the necessary and sufficient condition for the half-flow Φ to be uniformly permanent is

$$W^+(N_\alpha) \cap \{X: x \in X, d(x,\partial X_1) \leqslant \varepsilon\} \cap \overset{0}{X}_1 = \phi,$$

where $W^+(N_\alpha) = \{y \in X, \omega(y) \subset N_\alpha\}$.

Theorem 5.1 If $R_0 > 1$, system (3) is uniformly permanent.

Proof Because T is the positive invariant set of system (3), there is always the only disease-free equilibrium $\left(\frac{\Lambda}{d+p},0,0\right)$ on the boundary ∂T of T. From Theorem 3.2, the disease-free equilibrium $\left(\frac{\Lambda}{d+p},0,0\right)$ is unstable when $R_0 > 1$. In T, all the trajectories except the S-axis will approach $\left(\frac{\Lambda}{d+p},0,0\right)$ along the S-axis, while the other trajectories starting near $\left(\frac{\Lambda}{d+p},0,0\right)$ will be far away from $\left(\frac{\Lambda}{d+p},0,0\right)$. Let $G = \left\{\left(\frac{\Lambda}{d+p},0,0\right)\right\}$, then G becomes the only largest invariant set $\{N_\alpha\}_{\alpha \in A} = \left\{\left(\frac{\Lambda}{d+p},0,0\right)\right\}$ on ∂T, so

$$W^+(N_\alpha) \cap \{T: x \in T, d(x,\partial T) \leqslant \varepsilon\} = \phi.$$

According to Lemma 5.1, system (3) is uniformly permanent about T when $R_0 > 1$.

In order to study the global asymptotic stability of the endemic equilibrium (S^*,E^*,I^*) of system (3), we first introduce the following lemmas.

Lemma 5.2[19] Let $D \subset R^n$ be an open set, and let $x \to f(x) \in R^n$ be C^1 function defined in D. We consider the autonomous system in R^n.

$$x' = f(x) \tag{4}$$

Assume that the following conditions hold:

① there is a tightly attractive subset $K \subset D$ and the only equilibrium $\bar{x}$ in D;

② $\bar{x}$ is locally asymptotically stable in D;

③ system (4) satisfies a Poincare-Bendixson criterion;

④ a periodic orbit of system (4) is orbitally asymptotically stable.

Then the only equilibrium $\bar{x}$ is globally asymptotically stable in D.

Suppose $x = P(t)$ is a periodic solution of system (4), $O(x) = \{P(t) \mid 0 \leqslant t \leqslant \theta\}$ is a periodic orbit. θ is period of $P(t)$.

Lemma 5.3[19] Consider the following system:

$$Z'(t) = \frac{\partial f^{[2]}}{\partial x}(P(t))\,Z(t) \tag{5}$$

where system (5) is called the second compound equation of system (4) and $\frac{\partial f^{[2]}}{\partial x}$ is the second compound matrix of the Jacobian matrix $\frac{\partial f}{\partial x}$. If the zero solution of system (5) is asymptotically stable, then the periodic orbit $O(x)$ of system (5) is uniformly asymptotically stable.

Lemma 5.4 When $R_0 > 1$, the disease-free equilibrium $\left(\frac{\Lambda}{d+p},0,0\right)$ of system (3) is the unique ω-limit point on ∂T which is not any trajectory from $\overset{0}{T}$, system(3) is uniformly persistent in $\overset{0}{T}$, where ∂T and $\overset{0}{T}$ are the boundary and the interior of T respectively.

Proof It is easy to see that the trajectory of system (3) from ∂T (except the S-axis) must enter $\overset{0}{T}$ or remain on ∂T, while the S-axis is the invariant set of system (3). On the S-axis, there is $\frac{dS}{dt} = \Lambda - (d+p)S$, when $t \to +\infty$, $S(t) \to \frac{\Lambda}{d+p}$, that is, the disease-free equilibrium $\left(\frac{\Lambda}{d+p},0,0\right)$ is the only ω-limit point on ∂T.

Consider the following Lyapunov function:

$$L(t) = \varepsilon E(t) + (d + \alpha_1 + k_1 + \varepsilon)I(t).$$

Calculating the derivative of $L(t)$ along the positive solution of system (3), it follows that

$$L'(t)\big|_{(3)} = \frac{\varepsilon a S I}{1 + bI} - (d + \alpha_1 + k_1 + \varepsilon)(d + \alpha_2 + k_2 + k + \gamma)I =$$

$$\frac{\varepsilon a\Lambda}{d+p}\left(\frac{(d+p)S}{\Lambda}\frac{1}{1+bI}-\frac{1}{R_0}\right)I.$$

Furthermore, when $I\neq 0$, (S,E,I) is sufficiently close to $\left(\frac{\Lambda}{d+p},0,0\right)$, there is $L'(t)>0$, that is, there is a neighborhood U of $\left(\frac{\Lambda}{d+p},0,0\right)$, so that all the trajectories starting from $U\cap\overset{0}{T}$ must run out of U. Thus, the disease-free equilibrium $\left(\frac{\Lambda}{d+p},0,0\right)$ is not the ω-limit point of any trajectory starting from $\overset{0}{T}$.

Lemma 5.5 System (3) is competitive in T, any nonempty compact ω-limit set in $\overset{0}{T}$ is a closed orbit or the equilibrium (S^*,E^*,I^*).

Proof The Jacobian matrix of system (3) at the disease-free equilibrium $P=(S,E,I)\in T$ is

$$J(P)=\begin{pmatrix}-(\Lambda+p)-\frac{aI}{1+bI} & 0 & -\frac{aS}{(1+bI)^2}\\ \frac{aI}{1+bI} & -(\Lambda+\alpha_1+k_1+\varepsilon) & \frac{aS}{(1+bI)^2}\\ 0 & \varepsilon & -(\Lambda+\alpha_2+k_2+k+\gamma)\end{pmatrix}.$$

Select the diagonal matrix $H=\operatorname{diag}(1,-1,1)$, having

$$HJ(P)H=\begin{pmatrix}-(\Lambda+p)-\frac{aI}{1+bI} & 0 & -\frac{aS}{(1+bI)^2}\\ -\frac{aI}{1+bI} & -(\Lambda+\alpha_1+k_1+\varepsilon) & -\frac{aS}{(1+bI)^2}\\ 0 & -\varepsilon & -(\Lambda+\alpha_2+k_2+k+\gamma)\end{pmatrix}.$$

For any $(S,E,I)\in T$, all off-diagonal elements with $HJ(P)H$ are non-positive, so system (3) is competitive in the area T.

Let Ω is a ω-limit set of system (3) in $\overset{0}{T}$. If $(S^*,E^*,I^*)\notin\Omega$, because (S^*,E^*,I^*) is the only balance point in $\overset{0}{T}$, then Ω is a closed trajectory. If $(S^*,E^*,I^*)\in\Omega$, because (S^*,E^*,I^*) is locally asymptotically stable, if any trajectory is close enough to (S^*,E^*,I^*), it must tend to (S^*,E^*,I^*), that is $\Omega=(S^*,E^*,I^*)$.

Lemma 5.6 If $p(t)=(S(t),E(t),I(t))$ is a non-constant periodic solution of system (3), $p(t)$ is asymptotically stable with an asymptotic phase orbit.

Proof Suppose that $T^*(T^*>0)$ is the period of $p(t)$, and calculate the second additive compound matrix of system (3) at $p(t)$:

$$J^{[2]}(P)=\begin{pmatrix}-b_1-\frac{aI}{1+bI} & \frac{aS}{(1+bI)^2} & \frac{aS}{(1+bI)^2}\\ \varepsilon & -b_2-\frac{aI}{1+bI} & 0\\ 0 & \frac{aI}{1+bI} & -b_3\end{pmatrix},$$

where

$$b_1=d+p+m=2d+p+\alpha_1+k_1+\varepsilon,$$
$$b_2=d+p+n=2d+p+\alpha_2+k_2+k+\gamma,$$
$$b_3=m+n=2d+\alpha_1+k_1+\alpha_2+k_2+\varepsilon+k+\gamma.$$

So as to get the second-order composite system of system (3):

$$\begin{cases}\frac{\mathrm{d}X(t)}{\mathrm{d}t}=-\left(b_1+\frac{aI}{1+bI}\right)X+\frac{aS}{(1+bI)^2}(Y+Z),\\ \frac{\mathrm{d}Y(t)}{\mathrm{d}t}=\varepsilon X-\left(b_2+\frac{aI}{1+bI}\right)Y,\\ \frac{\mathrm{d}Z(t)}{\mathrm{d}t}=\frac{aI}{1+bI}Y-b_3Z\end{cases}\tag{6}$$

Next, we prove the asymptotic stability of the zero solution of the system (6).

Define the norm in R^3 $\|(x(t),y(t),z(t))\|=\sup\{|x(t)|,|y(t)|+|z(t)|\}$, consider the following function

$$L(t)=\sup\left\{|X(t)|,\frac{E(t)}{I(t)}(|Y(t)|+|Z(t)|)\right\}.$$

From Theorem 5.1, there is a certain distance between the periodic solution $p(t)=(S(t),E(t),I(t))$ and boundary ∂T, so there must be $k>0$, such that

$$L(t)\geqslant k\sup\{|X(t)|,|Y(t)|+|Z(t)|\}.$$

According to the relevant theory of the lower right derivative of Dini

$$D_+|X(t)|\leqslant-\left(b_1+\frac{aI}{1+bI}\right)|X(t)|+\frac{aS}{(1+bI)^2}(|Y(t)|+|Z(t)|)=-\left(b_1+\frac{aI}{1+bI}\right)|X(t)|+$$

$$\frac{aS}{E(1+bI)^2}\left(\frac{E}{I}(|Y(t)|+|Z(t)|)\right),$$

$$D_+|Y(t)|\leqslant\varepsilon|X(t)|-\left(b_2+\frac{aI}{1+bI}\right)|Y(t)|,$$

$$D_+|Z(t)|\leqslant\frac{aI}{1+bI}|Y(t)|-b_3|Z(t)|.$$

To get

$$D_+\frac{E(t)}{I(t)}(|Y(t)|+|Z(t)|)=$$
$$\left(\frac{E'}{E}-\frac{I'}{I}\right)\frac{E}{I}(|Y(t)|+|Z(t)|)+$$
$$\frac{E}{I}D_+(|Y(t)|+|Z(t)|)\leqslant$$
$$\left(\frac{E'}{E}-\frac{I'}{I}\right)\frac{E}{I}(|Y(t)|+|Z(t)|)+$$
$$\frac{E}{I}\left(\varepsilon|X(t)|-\right.$$
$$\left.\left(b_2+\frac{aI}{1+bI}\right)|Y(t)|+\frac{aI}{1+bI}|Y(t)|-b_3|Z(t)|\right)\leqslant$$
$$\frac{\varepsilon E}{I}|X(t)|+\left(\frac{E'}{E}-\frac{I'}{I}-2d-\alpha_2-k_2-k-\gamma\right)\cdot$$
$$\frac{E}{I}(|Y(t)|+|Z(t)|).$$

Therefore

$$D_+L(t)\leqslant\sup\{g_1(t),g_2(t)\}L(t),$$

where

$$g_1(t)=-\left(b_1+\frac{aI}{1+bI}\right)+\frac{aSI}{E(1+bI)^2},$$

$$g_2(t)=\frac{\varepsilon E}{I}+\left(\frac{E'}{E}-\frac{I'}{I}-2d-\alpha_2-k_2-k-\gamma\right).$$

Obtained by the system (3)

$$\frac{E'}{E}=\frac{aSI}{E(1+bI)}-(d+\alpha_1+k_1+\varepsilon),$$
$$\frac{I'}{I}=\frac{\varepsilon E}{I}-(d+\alpha_2+k_2+k+\gamma).$$

So

$$g_1(t)\leqslant-b_1+\frac{aSI}{E(1+bI)}=$$
$$-b_1+\frac{E'}{E}+(d+\alpha_1+k_1+\varepsilon)=$$
$$\frac{E'}{E}-d-p\leqslant\frac{E'}{E}-d,$$

$$g_2(t)=\frac{\varepsilon E}{I}+\left(\frac{E'}{E}-\frac{\varepsilon E}{I}+(d+\alpha_2+k_2+k+\gamma)-\right.$$
$$\left.2d-\alpha_2-k_2-k-\gamma\right)=\frac{E'}{E}-\mu.$$

That is

$$\sup\{g_1(t),g_2(t)\}\leqslant\frac{E'}{E}-d \tag{7}$$

$$D_+L(t)\leqslant\left(\frac{E'}{E}-d\right)L(t) \tag{8}$$

$$\int_0^{T^*}\sup\{g_1(t),g_2(t)\}\,\mathrm{d}t\leqslant$$
$$\int_0^{T^*}\left(\frac{E'}{E}-d\right)\mathrm{d}t=$$
$$\ln E(t)\Big|_0^{T^*}-\mathrm{d}T^*=-\mathrm{d}T^*<0 \tag{9}$$

From Equations(7), (8) and (9), when $t\to+\infty$, there is $L(t)\to0$. Therefore, the zero solution of system (6) is asymptotically stable. From Lemma 5.3, $p(t)$ is asymptotically stable with an asymptotic phase orbit.

By Lemmas 5.3−5.6, we know that system (3) is satisfied with every condition of Lemma 5.2. According to Lemma 5.2, we can obtain the following theorem.

Theorem 5.2　If $R_0>1$, the endemic equilibrium (S^*,E^*,I^*) of system (3) is globally asymptotically stable.

Theorem 5.3　If $R_0>1$, the endemic equilibrium $P^*(S^*,E^*,I^*,Q^*,R^*)$ of system (1) is globally asymptotically stable.

Proof　By Theorem 5.2, $(S(t),E(t),I(t))\to(S^*,E^*,I^*)$, since the first three equations of system (1) do not contain Q and R, we obtain the following limit system of system (1):

$$\begin{cases}\dfrac{\mathrm{d}Q}{\mathrm{d}t}=kI^*-(d+\alpha_3+\omega)Q,\\[2mm]\dfrac{\mathrm{d}R}{\mathrm{d}t}=\omega Q+\gamma I^*+pS^*-dR.\end{cases}$$

By calculation, having

$$Q(t)=\left(Q(0)+kI^*\int_0^t\mathrm{e}^{(d+\alpha_3+\omega)s}\mathrm{d}s\right)\mathrm{e}^{-(d+\alpha_3+\omega)t},$$

$$R(t)=\left(R(0)+\int_0^t\mathrm{e}^{ds}(\omega Q(s)+\gamma I^*+pS^*)\,\mathrm{d}s\right)\mathrm{e}^{-dt}.$$

When $t\to+\infty$, there is

$$Q(t)\to\frac{\gamma I^*}{d+\alpha_3+\omega}=Q^*,$$

$$R(t)\to\frac{\omega Q^*+\gamma I^*+pS^*}{d}=R^*.$$

Therefore, the endemic equilibrium $P^*(S^*,E^*,I^*,Q^*,R^*)$ of system (1) is globally attractive in the region D [21]. According to Theorem 4.1 , when $R_0>1$, the endemic equilibrium $P^*(S^*,E^*,I^*,Q^*,R^*)$ of system (1) is globally asymptotically stable.

6　Example and numerical simulation

In this section, we illustrate the above-mentioned main theoretical results through numerical simulation.

In system (1), let $\Lambda=0.28$, $d=0.02$, $\varepsilon=0.07$, $\alpha_1=0.08$, $\alpha_2=0.1$, $k_1=0.04$, $k_2=0.08$, $k=0.03$, $\gamma=0.04$, $p=0.03$. When $a=0.08$, $b=0.1$, by computing, we derive $R_0=0.5531<1$ and system (1) has a disease-free equilibrium $P_0(5.6,0,0,0,8.4)$. And we set eight initial conditions (2.6,5,2.6,1.5,2.3), (0.1,0.6,0.9,5.4,7), (2,1,1.8,3.7,5.5), (0.5,3.4,1,2.1,8), (2.5,1.5,5.2,1.8,3), (2.6,3.2,1.5,2.5,4.2), (1.6,0.6,2.1,3.2,6.5), (5,2.7,0.6,4.2,1.5),

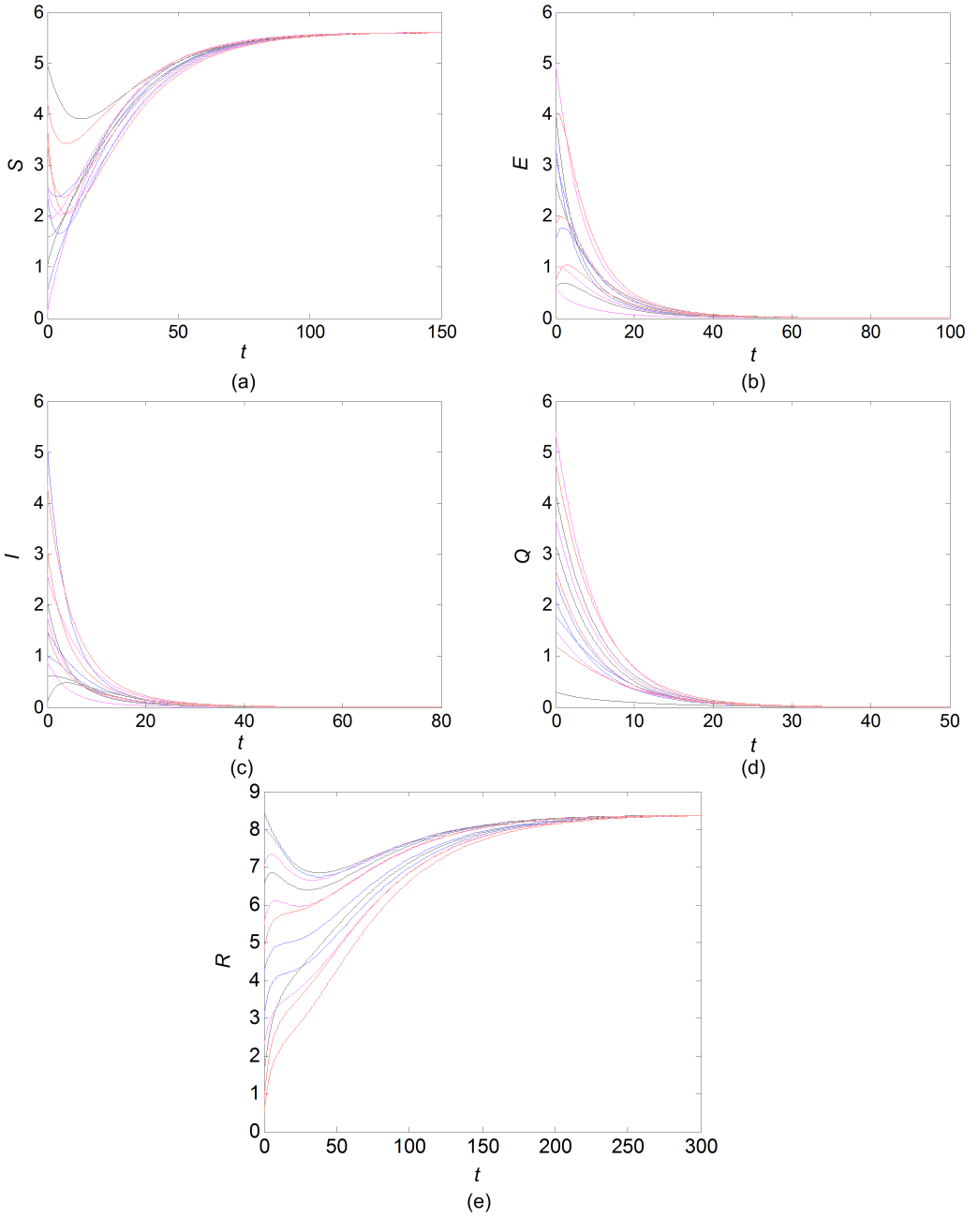

Figure 1. Variational curves of S, E, I, Q and R with time t when $R_0 = 0.5531 < 1$.

$(1, 4.1, 0.1, 0.3, 8.5)$, $(4.3, 0.7, 1.5, 2.7, 4.8)$, $(3.5, 1.8, 3.1, 4.8, 0.8)$ and $(3.9, 4, 4.4, 1.2, 0.5)$, the numerical simulation is shown in Figure 1. From Theorem 3.3, it follows that P_0 is globally asymptotically stable. Figure 1 shows the dynamic behaviors of system (1).

Let $\Lambda = 0.8$, $d = 0.05$, $\varepsilon = 0.07$, $\alpha_1 = 0.08$, $\alpha_2 = 0.1$, $k_1 = 0.04$, $k_2 = 0.08$, $\gamma = 0.04$, $p = 0.03$, $k = 0.03$. When $a = 0.2$, $b = 0.15$, by computing, we derive $R_0 = 2.3334 > 1$ and system (1) has an endemic equilibrium $P^*(5.418, 1.527, 0.358, 0.103, 3.579)$. And we set the same initial conditions as in Figure 1, the numerical simulation is shown in Figure 2. From Theorem 5.3, we notice that P^* is globally asymptotically stable. Numerical simulation illustrates this fact in Figure 2.

From the expression of the basic reproduction number R_0, we see that the R_0 depends on the prevention and control coefficients p, k, k_1 and k_2, and is a monotonically decreasing function of the coefficients p, k, k_1 and k_2 (see Figure 3). Therefore, vaccination, elimination and quarantine strategies are the effective methods to control the prevalence of infectious diseases.

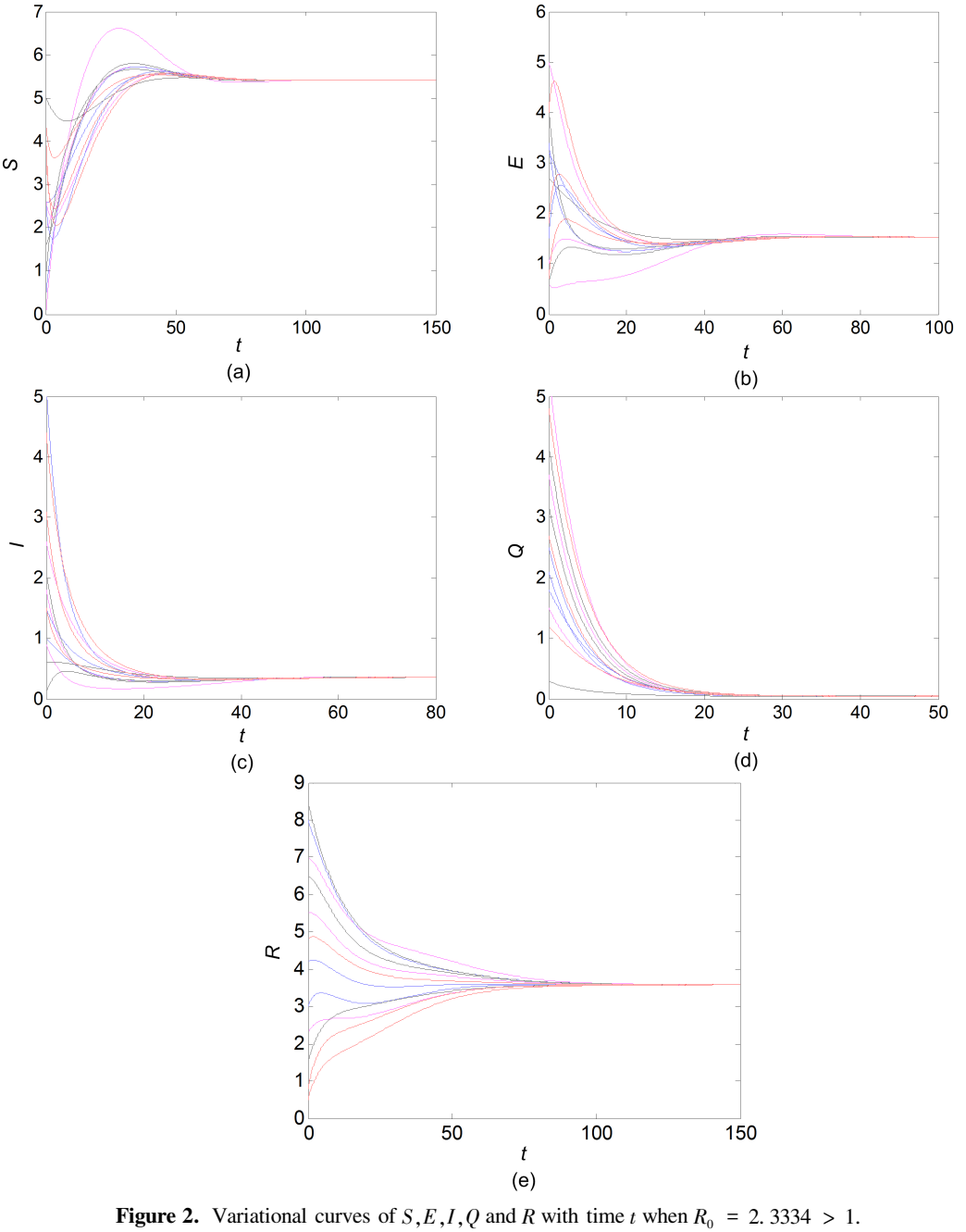

Figure 2. Variational curves of S, E, I, Q and R with time t when $R_0 = 2.3334 > 1$.

From the expression of the R_0, we also see that the R_0 is a monotonically decreasing function of the coefficient γ, but the R_0 is independent of the coefficient ω. Therefore, it is also important to strengthen the non-quarantine treatment to prevent the spread of infectious diseases.

7 Conclusions

In this paper, we formulated an SEIQR model with saturation incidence rate and hybrid strategies, and presented a complete mathematical analysis for the global stability problem at the equilibriums of the model by means of both theoretical and numerical ways. For this model, we defined the basic reproduction number R_0 which represents the average number of secondary infections from a single exposed host and infectious host. When $R_0 < 1$, as is shown in Theorem 3.3, the disease-free equilibrium is globally asymptotically stable by Lyapunov function (see Figure 1), and the disease dies out eventually. When $R_0 > 1$, Theorem 5.3 tells us that the unique endemic equilibrium is globally asymptotically stable by means of the periodic orbit stability theory and second additive compound matrix (see Figure 2), and the disease persists at the endemic equilibrium level if it is initially present. Finally, some

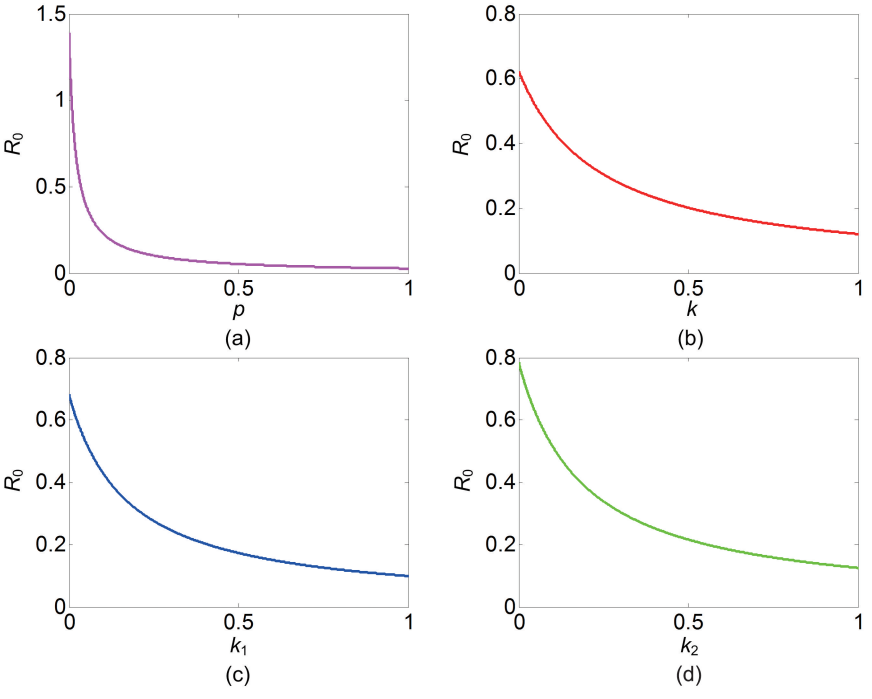

Figure 3. Variational curves of the basic reproduction number R_0 with the prevention and control coefficients p, k, k_1 and k_2, respectively.

numerical simulations were performed to illustrate the results.

Interestingly, the stability of the equilibrium of the model is under the influence of saturation incidence rate and hybrid control strategies. We believe that our study approaches can be applied to solve global stability problems in many other epidemic models. However, the limitation of system (1) is that it does not consider the infectious force in the latent and recovered period. However, for malaria and some other infectious diseases, the latent period and recovered period may be infectious. We leave this concern for future work.

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Conflict of interest

The authors declare no conflict of interest.

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具有饱和发生率与混合控制策略的 SEIQR 模型的全局动力学

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摘要: 建立了一个具有饱和接触率和混合控制策略的 SEIQR 传染病模型,从理论和数值模拟方面分析了模型的稳定性. 首先,得到了疾病灭绝与否的阈值——基本再生数 R_0;其次,当 $R_0<1$ 时,利用 LaSalle 不变集原理证明了无病平衡点是全局渐近稳定的,疾病最终消亡. 当 $R_0>1$ 时,根据 Routh-Hurwitz 判据定理证明了地方病平衡点局部渐近稳定;然后, 当 $R_0>1$ 时,运用周期轨道稳定性理论和第二加性复合矩阵证明了地方病平衡点全局渐近稳定,疾病持续存在;最后,利用计算机仿真,进一步证实理论分析的正确性.

关键词: 基本再生数;平衡点;稳定性;饱和发生率