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Some Partial Orders on Completely Regular Semigroups[X]

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Abstract: The purpose of this paper is to characterize cryptogroups and normal cryptogroups by some partial order relations. It is proved that a completely regular semigroup S is a cryptogroup if and only if S = $\leq$ and S is a normal cryptogroup if and only if C = S.

Key words: completely regular semigroup; partial order; cryptogroup; normal cryptogroup

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0 Introduction and preliminaries

A completely regular semigruop S is called a cryptogroup, if Green's relation H is a congruence on S, and S is called a normal cryptogroup, if H is a congruence on S and $S/$ H is a normal band.

Some partial order relations on semigroups can help us to realize the properties of semigroups and describe the structure of semigroups. For example, we know that $S = (Y; S_\alpha)$ is a normal cryptogroup if and only if for any $\alpha, \beta \in Y, \alpha \geq \beta$ and for any $a \in S_\alpha$, there exists a unique $b \in S_\beta$ such that $a \geq b$. And let $\chi_{\alpha,\beta}$ be the map from S_α to S_β given by $a\chi_{\alpha,\beta} = b$, then S is a strong semilattice of completely simple semigroups S_α and $\chi_{\alpha,\beta}$ are the structure homomorphisms.

Let S be a semigroup. The binary relation Con S which was introduced by Conrad in [1] is de2 fined as following:

$$C = \{ (a, b) \in S \times S \mid \text{for any } s \in S, \quad asa = asb = bsa \}.$$

Burgess and Raphael proved in [2] that C is a partial order relation on S iff S is weakly separative, i.e., if $asa = asb = bsa = bsb$ for any $s \in S$, then $a = b$. In [3] Nambooripad introduced another binary relation N on a semigroup S:

$$N = \{ (a, b) \in S \times S \mid a = axa = axb = bxa, \text{ for some } x \in S \},$$

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and proved that N is a partial order on S iff S is a regular semigroup. In [4], Drazin defined a binary relation S on semigroup S:

$$S = \{(a, b) \in S \times S \mid a^2 = ab = ba\},$$

and proved that if S is a completely regular semigroup, then S is a partial order relation on S. For a regular semigroup S, the natural partial order relation is defined as follows:

$$\leq = \{(a, b) \in S \times S \mid a = eb = bf, \text{ for some } e, f \in E(S)\}.$$

Drazin proved that if S is a completely regular semigroup, then C A S A N. It is easy to prove that N A $\leq$ for any regular semigroup.

A natural question is under what conditions these binary relations coincide. In this paper, we prove that if S is a completely regular semigroup, then S and $\leq$ coincide if and only if S is a cryp2 togroup, and C and S coincide if and only if S is a normal cryptogroup.

In order to establish the main results, we need the following lemmas.

Lemma 1.[5] Let $S = (Y; S_\alpha)$ be a completely regular semigroup and $\alpha, \beta \in Y$ with $\alpha \geq \beta$.

(I) If $a \in S_\alpha$, then there exists $b \in S_\beta$ such that $a \geq b$.

(II) If $a \in E(S)$ and $b \in S$ with $a > b$, then $b \in E(S)$.

(III) If $e \in E(S_\alpha)$, then there exists $f \in E(S_\beta)$ such that $e \geq f$.

Unfortunately, the partial order relation S does not have the property similar to (I) of Lemma 1.

Example 1. Let $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \right\}$ be a matrix semigroup. Obviously, S is a completely regular semigroup, $D_\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$, and $D_\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \right\}$ are two D2classes of S, and $\alpha > \beta$. Denote $a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in D_\alpha$, $b = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \in D_\beta$. Then $b^2 = b$, $ab = c$, $c^2 = c$, $ac = b$. So, (c, a) | S and (b, a) | S.

Lemma 2.[5] Let S be a completely regular semigroup. Then the following statements are e2 quivalent:

(I) S is a normal cryptogroup;

(II) S satisfies D2majorization;

(III) S is a strong semilattice of completely simple semigroups;

(IV) For all $a, x, y \in S$, $(axya)^0 = (ayxa)^0$.

Lemma 3.[5] If S is a completely regular semigroup and μ is the largest idempotent separating congruence on S, then

(I) $a\mu b$ Z $a^0 = b^0$; for any $e \in E(S)$ with $e \leq a^0$, $a^{-1}ea = b^{-1}eb$,

(Ⅱ) $Ker\mu = \{ a \in S \mid$ for any $e \in E(S)$ with $e \leq a^0$, $ea = ae \}$.

1 Main results

Proposition 1. Let S be completely regular semigroup, $a, b \in S$. Then aSb if and only if there exists $e \in E(S)$ such that $a = eb = be$.

Proof. If aS b, then $a^2 = ab = ba$ which implies $a = a^0 b = ba^0$.

Conversely, if there exists $e \in E(S)$ with $a = eb = be$, then $a^2 = aeb = ab = bea = ba$, i.e., $a = a^0 b = ba^0$.

Though the partial order relation S on a completely regular semigroup does not have the property similar to (Ⅰ) of Lemma 1, we have

Proposition 2. Let $S = (Y; S_\alpha)$ be a cryptogroup. Then for any $\alpha, \beta \in Y$, $\alpha \geq \beta$, $a \in S_\alpha$, there exists $b \in S_\beta$ such that bSa.

Proof. By Lemma 1 and Proposition 1, for any $a \in S_\alpha$, $\alpha \geq \beta$, there exists $e \in E(S_\beta)$ such that $e = a^0 ea^0$. Since H is a congruence on S, we have $e = a^0 ea^0$ H$a^{-1} ea$, where $a^{-1} ea \in E(S)$. So $e = a^{-1} ea$. Then $ae = aa^{-1} ea = a^0 ea = ea$. Let $b = ae = ea = eae$. Then $b^0 = (eae)^0 = e$. So $b = ab^0 = b^0 a$ implies $b^2 = ab = ba$, hence bSa.

Since S A $\leq$ for any completely regular semigroup S, by Lemma 1 and Proposition 2, we have

Corollary 1. Let $S = (Y; S_\alpha)$ be a cryptogroup, $\alpha, \beta \in Y$ with $\alpha \geq \beta$, $a \in E(S_\alpha)$, and $b \in S_\beta$. If bSa, then $b \in E(S_\beta)$.

Theorem 1. Let $S = (Y; S_\alpha)$ be a completely regular semigroup. Then the partial order relations S $= \leq$ if and only if S is a cryptogroup.

Proof. Let S be a cryptogroup. Then $\mu_S =$ H.

If $a, b \in S$, $b \leq a$, then $b = af = ea$ for some $e, f \in E(S)$. And $b^0 = a^0 (af)^0 = (ea)^0 a^0$ which implies $b^0 \leq a^0$. By Lemma 3, $b^0 a = ab^0$. Let $b^0 a = ab^0 = c$. Since $b = af = ea$, we have $b = b^0 af = eab^0 = cf = ec$, which implies $b \leq c$. If $b \in S_\beta$ and $a \in S_\alpha$ then $\beta \leq \alpha$, $c = b^0 a \in S_\beta$. Since the natural partial order on any completely simple semigroup is trivial, we conclude $b = c$. So, $b = ab^0 = b^0 a$, and $b^2 = ab = ba$. That is, bSa. Since S A $\leq$ holds for any completely regular semigroups, we get that S $= \leq$.

If $e \in E(S)$, $e \leq a^0$, $b = (ea^{-1} e)^{-1}$, then $b^0 = e(ea^{-1} e)^0 = (ea^{-1} e)^0 e$, i.e., $b^0 = e$ since $b^0 \leq e$ and bDe. So $b = (ea^{-1} e)^{-1} = e(ea^{-1} e)^{-1} = a^0 e(ea^{-1} e)^{-1} = aa^{-1} e(ea^{-1} e)^{-1} e = e(ea^{-1} e)^{-1} ea^0 = e(ea^{-1} e)^{-1} ea^{-1} a$, and $(a^{-1} e(ea^{-1} e)^{-1} e)(a^{-1} e(ea^{-1} e)^{-1} e) = a^{-1} e(ea^{-1} e)^0)(ea^{-1} e)^{-1} e = a^{-1} e(ea^{-1} e)^{-1} e \in E(S)$.

Similarly, $e(ea^{-1} e)^{-1} ea^{-1} \in E(S)$. So, $b \leq a$. Since $\leq =$ S, we have $b^2 = ba = ab$, which implies $b^0 a = ab^0$. Immediately, we have $ea = ae$. By Lemma 3, $Ker\mu = S$. Let aHb, i.e., $a^0 = b^0$. Then $a\mu a^0 = b^0 \mu b$. So, $\mu =$ H and H is a congruence on S.

Theorem 2. Let $S = (Y; S_\alpha)$ be a completely regular semigroup. Then C = S if and only if S

is a normal cryptogroup.

Proof. If S is a normal cryptogroup and $a, b \in S$ with $b\mathrm{S}a$, then $b^2 = ab = ba$. For any $s \in S$, from $b^2 s = abs = bas$, we have $b^2 sb = absb = basb$. Since S/H is a normal band, we have $bsb\mathrm{H}\, b^2 sb = absb\mathrm{H}\, asb^2\, \mathrm{H} asb$. Since $b = b^0 a$, we have $bsb = (bsb)^0 bsb = (bsb)^0 b^0 asb = (bsb)^0 asb = asb$. By $b^2 = ab = ba$, we have $bsb^2 = bsab = bsba$, which induces that $bsb\mathrm{H}bsb^2 = bsba\mathrm{H}b^2 sa\mathrm{H}bsa$. Then $bsa = bsa(bsb)^0 = bsab^0(bsb)^0 = bsb(bsb)^0 = bsb$. Since C A S for any completely regular semigroup[2], we have that C = S.

Conversely, let $S = (Y; S_\alpha)$ be a completely regular semigroup and C = S on S. If $a, x, y \in S$, Then $(axya)^0 = a^0(axya)^0 = (axya)^0 a^0$ which implies $(axya)^0 \mathrm{S} a^0$. By the hypothesis, $(axya)^0 \mathrm{C} a^0$. So, $(axya)^0(ayxa)^0(axya)^0 = a^0(ayxa)^0(axya)^0 = (axya)^0(ayxa)^0 a^0$. Then $(ayxa)^0(axya)^0 = (axya)^0(ayxa)^0$. Since $(ayxa)^0$ and $(axya)^0$ lie in the same D2class, $(ayxa)^0 = (axya)^0$. By Lemma 2, S is a normal cryptogroup.

Proposition 3. Let S be a completely regular semigroup. Then the partial order relation N = $\leq$

Proof. If $a, b \in S$ with $a \leq b$, then there exist $e, f \in E(S)$ such that $a = be = fb$. So $a = ab^0 = ab^{-1}b$ and $a = b^0 a = bb^{-1}a$. Then $ab^{-1}a = fbb^{-1}a = fb^0 a = fa = fb = a$, and $a = ab^{-1}a = ab^{-1}b = bb^{-1}a$. Hence $a\mathrm{N}b$. Since N A $\leq$ for any regular semigroup, the proposition yields.

We supply an example to explain that the condition in Proposition 3 is not necessary.

Example 2. Let $I_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $I_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $I_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $I_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $I_5 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $S = \{ I_1, I_2, I_3, I_4, I_5 \}$ be a matrix semigroup. It is easy to see that S is an inverse semigroup and is not completely regular. The binary relations N $= \{(I_1, I_2), (I_1, I_3), (I_1, I_4), (I_1, I_5)\} = \leq$ in S.

Corollary 2. Let $S = (Y; S_\alpha)$ be a completely regular semigroup. Then S = N if and only if S is a cryptogroup.

Proof. The statement follows from Theorem 1 and Proposition 3.

Corollary 3. Let $S = (Y; S_\alpha)$ be a completely regular semigroup. Then the partial order relations C = S = N = $\leq$ if and only if S is a normal cryptogroup.

Proof. The statement follows from Theorem 1 and Theorem 2.

Let S be a semigroup. The binary relation n should be defined as the following[5]:

$$\mathrm{n} = \{ (a, b) \in S \times S \mid a = xb = by,\ xa = a, \text{ for some } x, y \in S^1. \}$$

It is easy to verify that n is a partial order relation on S.

Proposition 4. Let S be a semigroup. Then the binary relations N = n if and only if S is a reg2 ular semigroup.

Proof. If S is a regular semigroup and $a, b \in S$ with a n b, then there exist $x, y \in S^1$ such that $a = xb = by$, $xa = a$. We can suppose that $x, y \in S$. Let $a' \in V(a)$. Then $ay = xby = xa = a$, $a = aa'a = bya'a$ and $yaa' \cdot yaa' = yaa'aa' = yaa'$. Similarly, $aax' \cdot aax' = aax'$.

So $a = eb = bf$ in which $e = aax'$, $f = ya'a$ are idempotents of S. It is easy to verify that ax', $ya' \in V(a)$. Let $a^3 = ax'$, $a^{33} = ya'$. Then $a = aa^3 b = ba^{33} a$. Let $a = a^{33} aa^3$. Then $aab = aa^{33} aa^3 b = aya'aax'b = aa'aa'a = a$. Similarly, $a = baa = aaa$. So, aNb. That N A n is trivial.

Let N = n on S. Since n is a partial order relation on S, and N is a partial order relation on S if and only if S is a regular semigroup[4], the statement holds.

Since C is always a compatible binary relation on any semigroup and C and n coincide in normal cryprogroup, we induce that the natural partial order is compatible on normal cryptogroups.

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完全正则半群上的一些偏序关系

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摘要:用完全正则半群上的一些偏序关系刻画密码群和正规密码群. 证明了完全正则半群 S 是密码群当且仅当 S = ≤而 S 是正规密码群当且仅当 C = S.

关键词:完全正则半群;偏序;密码群;正规密码群