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Balancing the minimum error rate and minimum copy consumption in quantum state discrimination

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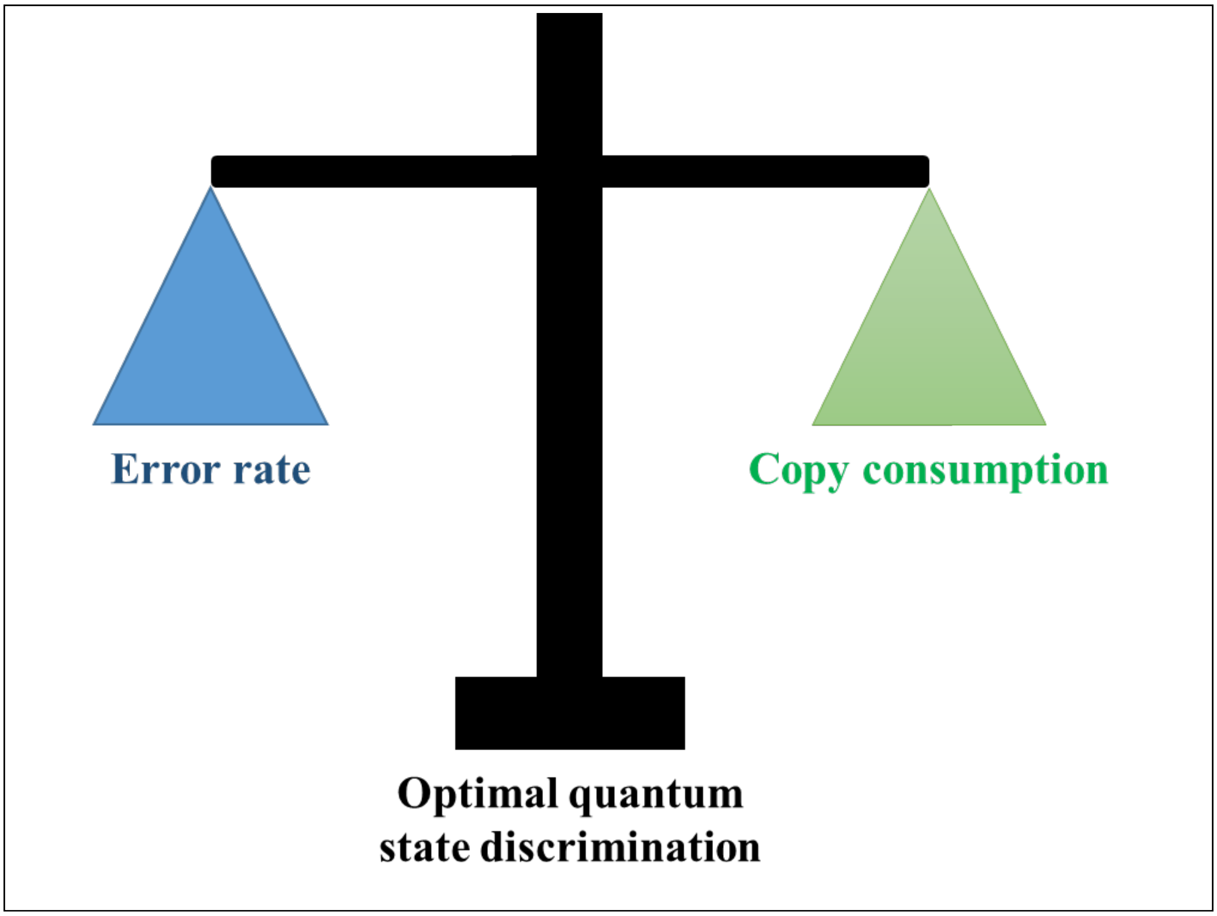
Graphical abstract



Balancing the minimum error rate and copy consumption in optimal quantum state discrimination.

Public summary

■ Introduce a state discrimination model that considers both average error rate and average copy consumption.

■ Reveals the intricate relationship between these two crucial metrics.

■ Showcase the advantages of achieving a balance between error rate and copy consumption in the discrimination task.

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Abstract: Extracting more information and saving quantum resources are two main aims for quantum measurements. However, the optimization of strategies for these two objectives varies when discriminating between quantum states $|\psi_0\rangle$ and $|\psi_1\rangle$ through multiple measurements. In this study, we introduce a novel state discrimination model that reveals the intricate relationship between the average error rate and average copy consumption. By integrating these two crucial metrics and minimizing their weighted sum for any given weight value, our research underscores the infeasibility of simultaneously minimizing these metrics through local measurements with one-way communication. Our findings present a compelling trade-off curve, highlighting the advantages of achieving a balance between error rate and copy consumption in quantum discrimination tasks, offering valuable insights into the optimization of quantum resources while ensuring the accuracy of quantum state discrimination.

Keywords: quantum measurement; quantum control; quantum state discrimination

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1 Introduction

Multiple-copy quantum state discrimination is a fundamental concept with wide-ranging implications for various quantum tasks, including quantum computation[1–3] and quantum communication[4–6]. Perfect discrimination between two non-orthogonal quantum states is unattainable when only a finite number of copies are provided, thus research in this domain predominantly focuses on maximizing efficiency[7–22]. Some studies aim to minimize the error rate probability by measuring a fixed number of copies[7–13, 18, 20, 21], while others seek to minimize the average copy consumption to meet a specific error rate requirement[14–16, 22]. Although it is recognized that optimal strategies differ for these distinct scenarios, the precise constraints between them remain elusive.

In this study, we introduce a state discrimination model to elucidate the inherent relationship between the average error rate and average copy consumption. Consider a scenario where n copies of a quantum state are available. Prior to measurement, it is known that the state is either $|\psi_0\rangle = \cos\frac{x}{2}|0\rangle + \sin\frac{x}{2}|1\rangle$ or $|\psi_1\rangle = \cos\frac{x}{2}|0\rangle - \sin\frac{x}{2}|1\rangle$, with a prior probability q for $|\psi_0\rangle$. Without loss of generality, we consider a specific case with $x = 30^\circ$ and $q = 0.5$ in this work. The task is to measure these copies and discriminate between them, resulting in two possible scenarios. In one scenario, perfect discrimination is achieved (the error rate probability $p_e = 0$) after measuring a subset of the N copies, while the remaining copies can be saved for other tasks. In the other scenario, perfect discrimination is not attained before all copies are measured, resulting in a judgment with an error rate $p_e \geqslant 0$. The central challenge is to determine how to minimize the score, represented as

$$s = w\langle n\rangle + (1-w)\langle p_e\rangle, \tag{1}$$

where $\langle n\rangle$ signifies the expected number of copy consumed when we finish the discrimination process, and $\langle p_e\rangle$ denotes the expected error rate when finishing the discrimination process, with the parameter w ranging from 0 to 1, meaning the requirement to minimize the weighted sum of the two parameters. This work aims to unravel the intricate interplay between these fundamental metrics, offering valuable insights into the optimization of quantum state discrimination in diverse quantum applications.

2 Materials and methods

2.1 Extreme situation

We first consider the extreme situation, if $w = 0$, we need to minimize the average error rate, thus the optimal strategy is local optimal local measurement[8] which can achieve the minimum average error rate

$$s(0) \geqslant \langle p_e\rangle_{\min} = \frac{1-\sqrt{1-\cos^{2n}x}}{2}, \tag{2}$$

where x has been defined in the introduction part, representing the angle of the states. Another extreme situation is $w = 1$

which requires us to minimize the average copy cosumption. In this situation, the optimal strategy is doing three-element POVM (positive operator-valued measure) with three measurement angles 0, $-\frac{x}{2}+\frac{\pi}{2}$, and $\frac{x}{2}+\frac{\pi}{2}$ on each copy[22]. The corresponding minimum average copy consumption is

$$s(1) \geqslant \langle n\rangle_{\min} = \frac{1-\cos^N x}{1-\cos x}. \tag{3}$$

2.2 General situation

In the general context, we begin by introducing the overarching expression for a one-way local strategy. All local adaptive strategies can be effectively described by a POVM with rank-one measurement elements, given as[22]:

$$M(\theta)\mathrm{d}\theta = F_q(\theta)(\cos\theta|0\rangle+\sin\theta|1\rangle)(\cos\theta\langle 0|+\sin\theta\langle 1|)\mathrm{d}\theta, \tag{4}$$

where $\theta\in[0,\pi)$ and is contingent upon the prior probability q for $|\psi_0\rangle$. The function $F_q(\theta)$ uniquely defines the POVM and must adhere to completeness and semidefinite positivity with the following four specific conditions[22]:

$$\int_0^\pi F_q(\theta)\mathrm{d}\theta = 2,\ \int_0^\pi F_q(\theta)\cos 2\theta\mathrm{d}\theta = 0,$$
$$\int_0^\pi F_q(\theta)\sin 2\theta\mathrm{d}\theta = 0,\quad \text{and } F_q(\theta)\geqslant 0. \tag{5}$$

The strategies that satisfy these four constraints form the set denoted as D.

Then we can calculate the minimum value of the average score $\langle s\rangle_{\min}$ by dynamic program to find the globally optimal adaptive local strategy[8, 10, 22]. According to this method, we can define n functions $s_j(q)$, where $j=1,2,3,\cdots,n$. $s_j(q)$ means the minimum average score when we has measured $j-1$ copies and leave a prior probability q for $|\psi_1\rangle$ before we measure the jth copy. $s_j(q)$ will satisfy

$$s_j(q) = \begin{cases} w(j-1); & \text{if } \min(q,1-q)=0, \\ \min\limits_{F\in\mathcal{D}}\int_0^\pi \mathrm{tr}[M(\theta)\rho]s_{j+1}(q_\theta)\mathrm{d}\theta; & \text{otherwise.}\end{cases} \tag{6}$$

where $\rho = q|\psi_0\rangle\langle\psi_0|+(1-q)|\psi_1\rangle\langle\psi_1|$ and $q_\theta = \frac{q\,\mathrm{tr}[M(\theta)|\psi_0\rangle\langle\psi_0|]}{\mathrm{tr}[M(\theta)\rho]}$ denotes the posterior probability. This is the minimum average score since it consider all the possible measurement forms and measurement outcomes. The method to minimize the integral has been shown in Ref. [22] and $s_{n+1}(q) = wn+(1-w)\min\{q,1-q\}$, and Ref. [22] has proven that the measurement which minimize the integral (e.g., the optimal measurement) must be a projective measurement or a three element POVM, and the exact measurement form is determined by the prior probability and the measurement round j (As a comparison, when $w=0$ or 1, the measurement form only depends on the prior probability). Therefore, we can calculate s_n, s_{n-1}, and so on. And $s_1(0.5)$ will be the calculation result.

3 Results and discussion

To unify the unit, here we define a new score $S = w\frac{\langle n\rangle}{\langle n\rangle_{\min}}+(1-w)\frac{\langle p_e\rangle}{\langle p_e\rangle_{\min}}$. This score function is still a linear combination and it satisfies the iterative relationship similar to Eq. (6), only replacing w by $\frac{w}{\langle n\rangle_{\min}}$ and s_j by S_j.

As what we are more concerned about is the constraint between $\langle n\rangle$ and $\langle p_e\rangle$, and $\forall w\in[0,1]$

$$w\frac{\langle n\rangle}{\langle n\rangle_{\min}}+(1-w)\frac{\langle p_e\rangle}{\langle p_e\rangle_{\min}} \geqslant S_{\min}(w). \tag{7}$$

If $\frac{\langle n\rangle}{\langle n\rangle_{\min}}$ has been determined, we can directly calculate the minimum value of the average error rate $\langle p_e\rangle$:

$$\frac{\langle p_e\rangle}{\langle p_e\rangle_{\min}} \geqslant \min\left\{S_{\min}(1),\ \min_{w\in(0,1)}\frac{S_{\min}(w)-w\frac{\langle n\rangle}{\langle n\rangle_{\min}}}{(1-w)}\right\}. \tag{8}$$

Then we can obtain the constraints between the average error rate and average copy consumption.

Through the iterative method described above, we have obtained the minimum average score, denoted as S (as depicted

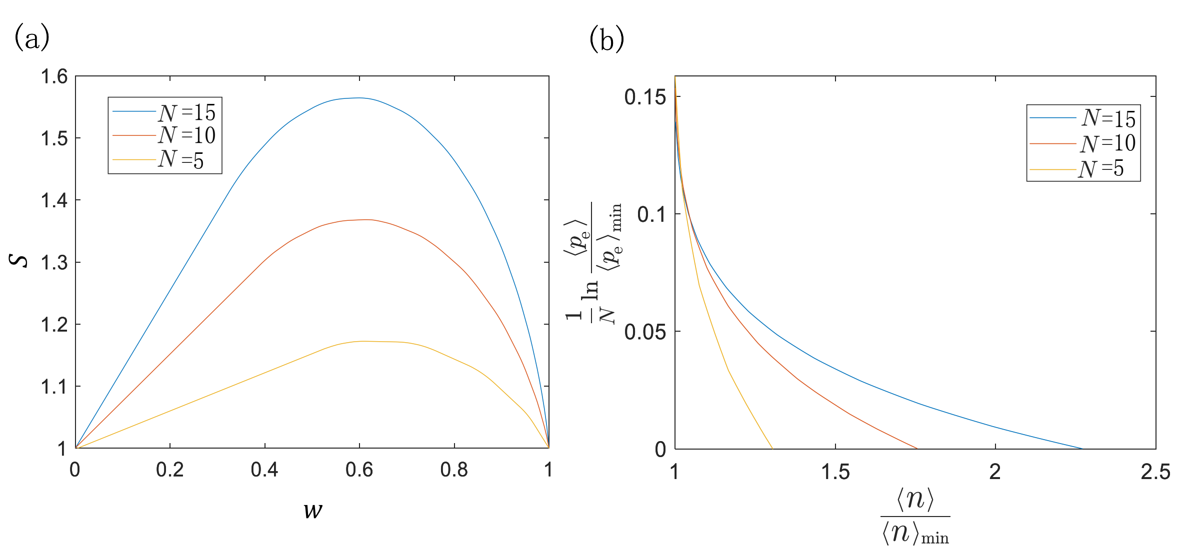

Fig. 1. The constraints between the average copy consumption and average error rate.Here $x=\pi/6$, we calculate the constraint when N=5 or 10 or 15. The average score S is shown in (a) and the corresponding constraint is shown in (b).

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in Fig. 1a), and unveiled the constraints between $\langle n\rangle$ and $\langle p_e\rangle$ (as illustrated in Fig. 1b). Remarkably, in Fig. 1a, S exceeds 1 when the parameter w does not equal 0 or 1, signifying that simultaneous minimization of the average error rate and average copy consumption is unattainable. Moreover, we can find that when $w \in (0,1)$, $S(w)$ is positively correlated with N, showcasing that this trade-off relation will be more strong when the value of N increases.

The constraints depicted in Fig. 1b underscore that optimizing one of the two indices comes at the cost of diminishing the performance of the other. This trade-off reveals valuable insights into the delicate balance required when seeking to optimize quantum state discrimination tasks.

4 Conclusions

Our research uncovers the disparity between minimum-error discrimination and minimum-consumption discrimination, shedding light on the inherent constraints that exist between these two objectives. The crucial observation that minimizing one of the metrics, either $\langle n\rangle$ or $\langle p_e\rangle$, significantly enhances the other, emphasizes the inherent trade-off. In certain instances, a balanced approach that assigns a specific weight to the average cost and the reduction of the error rate may prove to be a more effective strategy, simultaneously achieving the goals of conserving quantum resources and reducing error rates.

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Conflict of interest

The authors declare that they have no conflict of interest.

Biographies

Boxuan Tian was an undergraduate student of University of Scienceand Technoology of China and got this bachlor's degree of Science in 2024. His research interests include quantum information, quantum measurement.

Zhibo Hou is currently an Associate Professor in the School of Physics, University of Science and Technology of China. He obtained his Ph.d. of Optics in 2016. His research interests include quantum metrology, quantum information, and quantum optics.

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